3) EXPANDING UNIVERSE

- GR gives us the metric, the Friedmann equation and the expansion of the universe
- Statistichal Mechanics (Stat Mech) gives us the equation of state needed for equilibrium (and some out of eq.) physics.
- I'm a particle physicist so $\hbar = c = k_B = 1$

but a reasonable one so: $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$

FLAT MINKOWSKI METRIC

- Assuming a homogeneous and spatially isotropic universe

$$g_{\mu\nu} = \begin{pmatrix} -1 & & & \emptyset \\ & a^2 & & \\ & & a^2 & \\ & & & a^2 \end{pmatrix}$$ FLRW metric ($k=0$)

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = dt^2 - a^2(t) \left(\frac{dr^2}{1-\underset{\downarrow \atop 0}{k}r^2} + r^2 d\Omega^2 \right)$$

•) In order to understand the behavior of x^μ and momentum P^μ we need the geodesic equation in FLRW

Newton, no external force $\ddot{x}^i = 0$

In GR instead of time, there is the affine parameter λ: $\int ds = \int \sqrt{g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}}\, d\lambda$

For general coordinates $x^i \to x'^i$

$\ddot{x}^i = 0$ becomes

$$\frac{d}{dt}\left(\frac{\partial x^i}{\partial x'^j} \frac{\partial x'^j}{\partial t} \right) = \frac{\partial x^i}{\partial x'^j} \frac{d^2 x'^j}{dt^2} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k} \dot{x}'^j \dot{x}'^k = 0$$

GR: $\ddot{x}^l + \Gamma^l_{kj} \dot{x}^k \dot{x}^j = 0$

$$\Gamma^l_{kj} = \left(\frac{\partial x'}{\partial x}\right)^{-1\,l}_{\ \ i} \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}$$ ... Christoffel symbol

GR: $$\frac{d^2 x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

(dot ~~is~~ derivative will be reserved for t ~~over λ here~~ $\dot{a} = \frac{da}{dt}, \dots$) ~~$\dot{x}^\mu = \frac{dx^\mu}{d\lambda}$~~)

·) The Christoffels can be written conveniently directly from the metric $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$.

$$\underbrace{\Gamma^{\mu}_{\alpha\beta}}_{\text{symm.}} = \frac{1}{2} g^{\mu\nu} \left(\underbrace{\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}}}_{\alpha \leftrightarrow \beta \text{ symm.}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}} \right)$$

$$\text{FLRW} \Downarrow \quad g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2), \quad g^{\mu\nu} = \mathrm{diag}(-1, a^{-2}, a^{-2}, a^{-2})$$

There are only two non-zero components

$$\Gamma^{0}_{ij} = \delta_{ij}\, \dot{a} a, \quad \Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \delta_{ij} \frac{\dot{a}}{a}$$

Where $\dot{a} = \dfrac{da}{dt}$.

MOMENTUM

·) A proper definition of momentum is crucial for understanding the dynamics in an expanding backround.

$$P^{\mu} = \frac{dx^{\mu}}{d\lambda} = (E, \vec{P})$$

comoving momentum

•) With this definition of energy-momentum

$$\frac{d}{d\lambda} = \frac{\partial x^0}{\partial \lambda}\frac{d}{dx^0} = E\frac{d}{dt}$$

$$\underset{p^0 = E}{\overset{\shortparallel}{}} \quad \Rightarrow \quad \frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta = 0$$

•) In an expanding FLRW we have the geodesic

$$\Gamma^0_{ij} = \delta_{ij}\,\dot{a}a$$

$$\frac{dP^0}{d\lambda} + \Gamma^0_{ij} P^i P^j = 0 \qquad \text{or: } P^\mu \nabla_\mu P^\nu = 0$$

$$E\frac{dE}{dt} = -\dot{a}a\frac{E^2 - m^2}{a^2} = -\frac{\dot{a}}{a}\left(E^2 - m^2\right)$$

where we took: $g_{\mu\nu}P^\mu P^\nu = -E^2 + \underbrace{a^2 P^{i2}}_{p^i \ldots \text{physical momentum}} = -m^2$

The on-shell relation: $E^2 - p^2 = m^2$ is preserved.

For $m=0$ like photons or sometimes neutrinos

$$\frac{dE}{dt} = -\frac{\dot{a}}{a}E \qquad \text{or: } \frac{dE}{E} = -\frac{da}{a} \Rightarrow E \propto a^{-1}.$$

The energy of photons diminishes with the scale factor, λ's get stretched.

We will revisit this again in the Boltzmann equation.

•) Another important aspect is the measure of distance and luminosity in an expanding universe.

received time (today)

$$dx = dt/a \quad \text{for light } ds^2 = 0 = -dt^2 + a^2 dx^2$$

$$\chi(t) = \int_{t}^{t_0} \frac{dt}{a}$$

t ... emitted time

$$= \int_{a}^{1} \frac{da'}{a'^2 H} \quad \text{... in terms of the scale factor}$$

$$\frac{\dot{a}}{a} = H = \frac{da}{a\,dt} \Rightarrow dt = \frac{da}{aH}$$

$$= \int_{z}^{0} \frac{dz'}{H} \quad \text{... in terms of the redshift}$$

$$z + 1 = \frac{1}{a} \Rightarrow dz = -\frac{1}{a^2} da$$

for $z \sim 0$ $\quad \chi \sim \frac{z}{H_0}$, $H \sim$ const. for a small interval

•) Another way to interpret χ is to think of it as the distance travelled by light

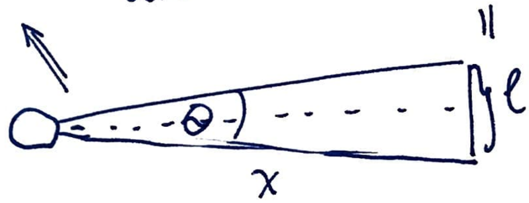
$$\eta = \int_{0}^{t} \frac{dt'}{a'} \quad \text{or} \quad d\eta = \frac{dt}{a} \quad \eta \text{... conformal time}$$

matter : $\eta \propto \sqrt{a}$, radiation : $\eta \propto a$, Λ : $\eta \propto a^{-1}$.

•) The physical significance of η is that it defines the horizon beyond which light could not have travelled since the beginning of times. This defines causally disconnected regions of the universe.

•) For objects nearby we can measure the paralax and define an angular diameter distance

$$d_A = \frac{l}{\theta} \, , \qquad \theta = \frac{l}{a\chi} \qquad \frac{l}{a}$$



$$= a\chi$$

$$= \frac{\chi}{1+z}$$

•) Usually stars galaxies are too far away to resolve their size but we may infer their luminosity L by other means (standard candles)

$$F = \frac{L}{4\pi d_L^2}$$

flux

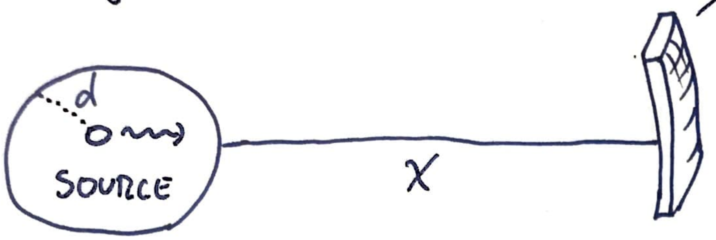
$$F = \frac{L a^2}{4\pi \chi^2} \Rightarrow d_L = \frac{\chi}{a} = \frac{d_A}{a^2}$$

•) Why $L(x) = La^2$

energy is reduced by a

$$\int_{E_{emit}}^{E_{tod}} \frac{dE}{E} = -\int_{a_{emit}}^{1} \frac{da}{a} \Rightarrow E_{tod} = E_{emit}\, a_{emit}$$

$$a_{emit} < 1$$



near : $F = \frac{L}{4\pi d^2}$ far : $F = \frac{La^2}{4\pi x^2}$

today $a = 1$, $x \to d$

back : $a \ll 1$ so $L \to La^2$

ENERGY CONTENT VS. EXPANSION

•) Energy density ρ and pressure P are combined in a energy-momentum tensor

$$T^{\mu}{}_{\nu} = \begin{pmatrix} -\rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}.$$

In the absence of gravity (a trivial flat metric) the conservation of ρ and P is expressed by the continuity $\dot{\rho} = 0$ and Euler $\partial_i P = 0$ equations. These are generalized to covariant derivative of $T^{\mu}{}_{\nu}$ for each ν : $\nabla_\mu T^{\mu}{}_{\nu} = 0$

•) The conservation of T^μ_ν is defined by

$$\nabla_\mu T^\mu_\nu = \frac{\partial T^\mu_\nu}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha_\nu - \Gamma^\alpha_{\nu\mu} T^\mu_\alpha = 0$$

$\nu = 0$:
$$\nabla_\mu T^\mu_0 = \frac{\partial T^0_0}{\partial x^0} + \Gamma^{\mu = j}_{0\mu = j} T^0_0 - \Gamma^\alpha_{0\mu = i} T^i_\alpha = 0$$

we have
$$\dot{\rho} + \frac{\dot{a}}{a}(3\rho + 3P) = 0 \quad / a^3$$

$$\frac{1}{a^3}\frac{d(\rho a^3)}{dt} = \dot{\rho} + 3\frac{\dot{a}}{a}\rho = -3\frac{\dot{a}}{a}P$$

•) We will quantify the relation between P and ρ in more detail (microscopically) a bit later but for matter $P \sim 0$ and for light $P = \frac{\rho}{3}$.

matter :
$$\frac{1}{a^3}\frac{d(\rho_m a^3)}{dt} = 0 \quad \Rightarrow \quad \rho_m \propto a^{-3}$$

radiation :
$$\dot{\rho} + \frac{\dot{a}}{a} 4\rho = 0 \quad \text{or} \quad \frac{d\rho}{\rho} = -4\frac{da}{a}$$

thus for light : $\rho_\gamma \propto a^{-4}$.

(or ~~non~~rel. particles)

•) We can define a more general relation

$$P = w \rho \quad , \quad w = \begin{cases} 0 & \text{matter} \\ 1/3 & \text{radiation} \\ -1 & \text{C.C. } \Lambda \end{cases}$$

Then the conservation of $T^{\mu}_{\ 0}$ implies

$$\frac{\dot{\rho}}{\rho} = - \frac{\dot{a}}{a} 3 (1+w)$$

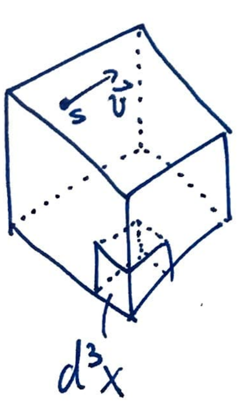
or $$\frac{d\rho}{\rho} = -3(1+w) \frac{da}{a} \Rightarrow \rho \propto a^{-3(1+w)}$$

even if $w(a)$: $$\rho = \rho_0 e^{-3 \int^a \frac{da}{a} (1+w(a))}$$

DISTRIBUTION FUNCTIONS

•) In a general TD system we describe species of particles in some macrostate by a distribution function $f_s(\vec{x}, \vec{p}, t)$

species



•) the number of particles at any given time within a volume d^3x

$$\frac{dN}{d^3x} = n(t) \simeq \int d^3p \, f_s = g_s \int \frac{d^3p}{(2\pi)^3} f_s$$

•) The usual normalization includes the $(2\pi)^3$ such that $d^3x\, d^3p \leq (2\pi\hbar)^3$

•) Internal degrees of freedom, independent of momentum (spin, number of colors, # of gauge bosons) are all counted inside of g_s

$$n_s = g_s \int \frac{d^3p}{(2\pi)^3} f_s$$

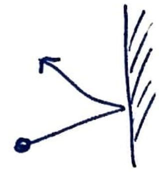
With the distribution function at hand we can derive the macroscopic properties, such as energy density ρ, the pressure P, entropy density s and perturb. of $T_\mu^{\ \nu}$.

ENERGY DENSITY : $\rho = g \int \frac{d^3p}{(2\pi)^3} f\, E(p)$

where : $E^2 - p^2 = m^2$, $E = p$ for γ's

•) As we mentioned earlier, we will define P microscopically and $P = \frac{1}{3}\rho$ for $m=0$.

Pressure : in elastic collisions



$$F = \frac{\Delta p}{t} = \frac{2mv}{2L/v}$$

$$P_x = \frac{F_x}{S} = \frac{mv^2}{V}$$

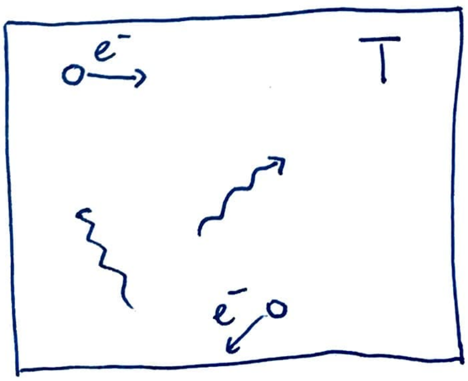
$$P = \frac{N m v_x^2}{V} = \frac{1}{3}\,\frac{N}{V}\, m \langle |v| \rangle^2 = \frac{1}{3}\,\frac{p^2}{E}$$

Definition of pressure : $P \equiv g \int \frac{d^3p}{(2\pi)^3}\, f\, \frac{p^2}{3E}$

$$= g \int \frac{d^3p}{(2\pi)^3}\, f\, \frac{p^2}{3\sqrt{p^2+m^2}} \xrightarrow{m=0} g \int \frac{d^3p}{(2\pi)^3}\, f\, \frac{p}{3}$$

- Note that for $m=0$ we get $P = \frac{1}{3}\rho$, this is the limit that any highly relativistic particle will approach.

- Before moving on to entropy we should define the notion ~~and~~ of temperature and equilibrium.



If the interaction rate Γ is large enough for the mean free path v/Γ to be large enough, the system will equilibrate.

-) To reach <u>kinetic</u> equilibrium the particles have to exchange their momenta and energies in inelastic collisions.

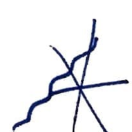
-) For a homogeneous system f will only depend on p^2 (or E) and T.

remember: $dE = -p\,dV + T\,dS + \mu\,dN \quad /:dV$

$$\rho = -P + Ts + \mu n$$

-) When the number-changing interactions proceed quickly enough the relative energy distribution between species is established and we also reach the <u>chemical</u> equilibrium.

* $\sum \mu_i = \text{const.}$ in eq.

* for bosons $\mu_\gamma = 0$  not for fermions:  breaks Lorentz

* anti particles $\mu_{\bar{f}} = -\mu_f$

•) If all the particles equilibrate kin & chemically they reach a common temperature T that define their distribution function

$$f(p) = \frac{1}{e^{(E-\mu)/T} \pm 1} \qquad \begin{array}{l} + \ldots \text{ fermions} \\ - \ldots \text{ bosons} \end{array}$$

•) The distinction is only relevant when

$e^{(E-\mu)/T} \sim 1$ so $E-\mu \lesssim T$ so the particles are not very energetic ~cold

Conversely when $E-\mu > T$ the particles are moving fast and $(E-\mu)/T \gg 1$, thus we go to the Maxwell-Boltzmann

$$f \sim e^{-(E-\mu)/T} .$$

•) In the absence of μ (like for γ's) we have

$$\frac{\partial P}{\partial T} = \frac{\rho + P}{T} = s \qquad \text{(HW)}$$

•) The entropy density is then conserved, scales as a^{-3}.

From $$\frac{\partial P}{\partial T} = \frac{\rho + P}{T}$$

Continuity: $$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$$

$$\Rightarrow \frac{1}{a^3}\frac{d}{dt}\left((\rho+P)a^3\right) = \dot{\rho} + \dot{P} + 3\frac{\dot{a}}{a}(\rho + P) = \dot{P}$$

$$= \frac{\partial P}{\partial T}\cdot \dot{T} = \frac{\rho + P}{T}\cdot\dot{T}$$

$$\Rightarrow a^{-3}T\left(\frac{d}{dt}\Big(\underbrace{\frac{\rho+P}{T}}_{s}a^3\Big)\right) = 0$$

$$\Rightarrow \frac{d}{dt}(sa^3) = 0 \Rightarrow sa^3 = \text{const.}$$

•) We see that the entropy is conserved and scales like the a^{-3}.

•) Radiation (γ's, ν's) $m \sim 0$ $\quad P \approx \frac{\rho}{3}$

$$s = \frac{\rho + P}{T} = \frac{\rho + \frac{1}{3}\rho}{T} = \frac{4}{3}\frac{\rho}{T}$$

•) For $m \neq 0$

$$s = \frac{1}{T}\int \frac{d^3p}{(2\pi)^3}\Big(\underbrace{E}_{\sqrt{p^2+m^2}} + \frac{1}{3}\frac{p^2}{\underbrace{\sqrt{p^2+m^2}}_{E}}\Big) f(p,T)$$

•) Let's evaluate some of these integrals for the photons with $g_\gamma = 2$ for the two polarizations and $m_\gamma = 0$.

$$\rho_\gamma = 2 \cdot \int \frac{d^3p}{(2\pi)^3} E\, f_{BE}$$

$$= \cancel{2}\,\frac{\cancel{4\pi}\ 1}{\cancel{8}\pi^{\cancel{3}2}} \int_0^\infty p^2\, dp\ p\ \frac{1}{e^{p/T}-1}$$

$$= \frac{1}{\pi^2}\int_0^\infty \frac{p^3\, dp}{e^{p/T}-1} \qquad p/T = x\ ,\ dp = T dx$$

$$= \frac{T^4}{\pi^2}\underbrace{\int_0^\infty \frac{x^3\, dx}{e^x-1}}_{\frac{\pi^4}{15}} = \frac{\pi^2}{15}T^4$$

and also: $s_\gamma = \frac{4}{3}\frac{\rho}{T} = \frac{4\pi^2}{45}T^3$

•) In general, we define $g_{*s}(T)$, such that

$$s = \frac{2\pi^2}{45}\, g_{*s}(T)\, T^3 = \sum_i g_i \int \frac{d^3p}{(2\pi)^3}\left(E + \frac{p^2}{E}\right) f$$

Thus $\quad g_{*s} T^3 a^3 = \text{const.}$ and if g_* doesn't change : $T \propto a^{-1}$.

•) When $p << m$ and $E \sim m$, the f becomes Boltzmann supresed by $e^{-m/T}$.

•) For $E > m > T$ we can just count g_i for bosons and $\frac{7}{8} g_i$ for fermions because

$$\int_0^\infty \frac{dx\, x^3}{e^x+1} = \int_0^\infty \left(\frac{x^3}{e^x-1} - \frac{2x^3}{e^{2x}-1} \right) dx$$

$$= \frac{\pi^4}{15} - \frac{1}{2^3} \frac{\pi^4}{15} = \frac{7}{8} \int_0^\infty \frac{x^3}{e^x-1} dx$$

$$s \simeq \left(\sum g_b \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} g_f \left(\frac{T_i}{T}\right)^3 \right) T^3$$

Cosmic Budget

With these preliminaries established, we can survey the energy density contributions from various components.

$$\Omega_s = \frac{\rho_s}{\rho_{cr}} \quad , \quad \rho_{cr} = \frac{3H_0^2}{8\pi G} \sim \frac{\text{GeV}}{m^3}$$

density parameters

or $\omega_s = \Omega_s h^2$, $h \simeq 0.7$

	species	type	scaling	
m matter	CDM	cold dark matter	$\rho_m \sim mn \propto a^{-3}$	non-relativistic matter, baryons include e^-
	b	baryons (luminous)		
r radiation	γ	photons	$\rho_r \propto a^{-4}$	(nearly) massless particles whose $E \propto 1/a$
	ν	neutrinos		
	Λ, DE	cosmological constant, dark energy	$\rho \cong$ const.	

PHOTONS

•) Energy density $\quad \rho_\gamma = \frac{\pi^2}{15} T_0^4 \quad , T_0 = 2.7\,K$

$$\sim 3 \cdot 10^{-4}\,eV$$

$$\simeq 2 \cdot 10^{-15}\,eV^4$$

$$\rho_{cr} = \frac{3H_0^2}{8\pi G} = 10^{-5} h^2 \frac{GeV}{cm^3}$$

$$\Rightarrow \quad \Omega_\gamma = \frac{\rho_\gamma}{\rho_{cr}} = 2.47 \cdot 10^{-5}$$

$$\boxed{\Omega_\gamma h^2 \simeq 2.5 \cdot 10^{-5}}$$

radiation energy density today $\ll 1$

·) We can also calculate the number density of photons

$$n_\gamma = 2\int \frac{d^3p}{(2\pi)^3} f_{BE}$$

$$= \frac{\cancel{8\pi} T^3}{\cancel{8}\pi^{\cancel{3}2}} \underbrace{\int_0^\infty \frac{x^2 dx}{e^x - 1}}_{2\xi(3)} = \overset{1.2}{\overset{\parallel}{\frac{2\xi(3)}{\pi^2}}} T^3$$

Again with $T_0 = 2.7\,K$ $\quad n_\gamma \cong 410 .\, cm^{-3}$

For nearly massless fermions (radiation), we have

$$n_f = g \int \frac{d^3p}{(2\pi)^3} f_{FD}$$

$$= g \frac{\cancel{4\pi}}{\cancel{2}\cancel{8}\pi^{\cancel{3}2}} \int p^2 dp \frac{1}{e^{p/T} + 1}$$

$$= \frac{gT^3}{2\pi^2} \int x^2 dx \left(\underbrace{\frac{1}{e^x - 1}}_{2\xi(3)} - \underbrace{\frac{2}{e^{2x} - 1}}_{\frac{1}{4}\int_0^\infty \frac{(2x)^2 2dx}{e^{2x} - 1}} \right)$$

$$= \frac{gT^3\xi(3)}{\pi^2} \cdot \left(1 - \frac{1}{4}\right)$$

$$= \frac{3}{4} \frac{\xi(3)}{\pi^2} g T^3 .$$