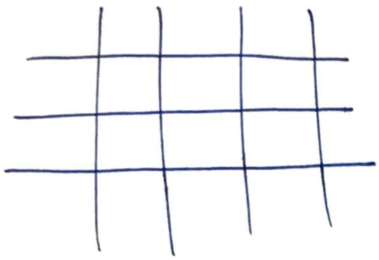
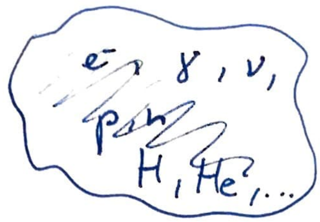
4.) EARLY UNIVERSE STATISTICS





Gravity sets the metric

Plasma sets the matter content

EINSTEIN vs. BOLTZMANN

•) Let's start on the gravity side with the Einstein equation

$$\text{Einstein tensor} \rightarrow G_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G\, T_{\mu\nu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu}$$

•) Ricci tensor & scalar $\left(\Gamma^{\beta}_{\mu\nu,\alpha} = \dfrac{\partial \Gamma^{\beta}_{\mu\nu}}{\partial x^{\alpha}} \right)$

$$R_{\mu\nu} = \Gamma^{\alpha}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}$$

FLRW : $\Gamma^{0}_{ij} = \delta_{ij}\dot{a}a \,, \quad \Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \delta_{ij}\dfrac{\dot{a}}{a}$

$$R_{00} = -\Gamma^{i}_{0i,0} - \Gamma^{i}_{j0}\Gamma^{j}_{0i} = -\frac{d}{dt}\Big(\underbrace{\delta_{ii}}_{=3}\frac{\dot{a}}{a}\Big) - \left(\frac{\dot{a}}{a}\right)^2 \delta_{ij}\delta_{ij}$$

$$R_{00} = -3\frac{\ddot{a}}{a} + 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a}$$

$\mu = \nu = i$: $R_{ij} = \delta_{ij}\left(2\dot{a}^2 + \ddot{a}a\right)$

•) Ricci scalar $R = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{ij} R_{ij}$

$$= -R_{00} + a^{-2} R_{ii}$$

$$= 3\frac{\ddot{a}}{a} + \frac{3}{a^2}\left(2\dot{a}^2 + \ddot{a}a\right)$$

$$= 6\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{\ddot{a}}{a}\right).$$

•) The Λ term can be moved on the ~~RHS~~ inside T^μ_ν and counted as the energy content of the universe

$$T^\mu_\nu = -\frac{\Lambda}{8\pi G}\delta^\mu_\nu = -\rho_\Lambda \, \mathrm{diag}(1,1,1,1)$$

$$\rho_\Lambda = -P_\Lambda \quad \text{or} \quad w = -1$$

•) Note that $[\rho] = T^4 = M^4$, $[G] = M^{-2} \Rightarrow [\Lambda] = M^2$

•) Plugging in $R_{\mu\nu}$ and R and $T_{\mu\nu}$ we get the two FRIEDMANN EQUATIONS.

•) The 0-0 component of the tensor equation gives the 1^{st} Friedmann equation

$$R_{00} - \frac{1}{2} g_{00} R = -3\frac{\ddot{a}}{a} - \frac{1}{2}(-1)\,6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)$$

$$= 8\pi G\, T_{00} = 8\pi G \rho$$

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \qquad \text{for } k=0 \text{ (FLAT)}$$

•) Introducing $H_0^2 = \frac{8\pi G}{3}\rho_{cr}$ and dividing, gives

$$\left(\frac{H}{H_0}\right)^2 = \sum_s \Omega_s\, a^{-3(w+1)}$$

•) Note that the 2^{nd} FRIEDMANN equation follows from $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$ and from the 1^{st} one.

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$$

1^{st}: $\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho \;\Big|^{\cdot}$

$$\cancel{2}\cancel{\frac{\dot{a}}{a}}\left(\frac{\ddot{a}}{a} - \left(\frac{\dot{a}}{a}\right)^2\right) = -\frac{48\pi G}{3}\, 3\,\cancel{\frac{\dot{a}}{a}}(\rho + P) \Rightarrow \frac{\ddot{a}}{a} = \frac{8\pi G}{3}\rho - \frac{4\pi G}{3}(\rho + 3P)$$

$$= -\frac{4\pi G}{3}(\rho + 3P).$$

LIOUVILLE - BOLTZMANN EQUATION

•) The second piece of the theoretical framework concerns the statistical part in an expanding universe. The question is how do the distribution functions (in general $f = f(\vec{x}, \vec{p}, t)$) evolve in time.

•) In depth discussion: Binney, Tremaine : Galactic dynamics '08

Remember $dN(x, p, t) = f(x, p, t)\, d^3x \frac{d^3p}{(2\pi)^3}$

Without sources or drains $\frac{dN}{dt} = 0$

$$\Rightarrow \quad \frac{\partial N}{\partial t} + \frac{\partial N}{\partial x}\dot{x} + \frac{\partial N}{\partial p}\dot{p} = 0$$

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial p}\dot{p} = 0 \qquad \dot{x} = \frac{p}{m}, \; \dot{p} = ma$$

•) For example HO : $H = \frac{p^2}{2m} + \frac{1}{2}kx^2$

$$F = -\nabla V = -kx = \dot{p} = -\frac{\partial H}{\partial q_{=x}}$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

•) In the case of HO, we have:

$$\partial_t f + \partial_x f \frac{p}{m} - \frac{\partial f}{\partial p} kx = 0$$

•) A <u>static</u> equilibrium distribution satisfies $\partial_t f = 0$

This is true if $f = f(E)$ only. (Jeans theorem)

$$\partial_t f + \frac{\partial f}{\partial E} \cdot \underbrace{\frac{\partial E}{\partial x}}_{= kx} \frac{p}{m} - \frac{\partial f}{\partial E} \underbrace{\frac{\partial E}{\partial p}}_{= \frac{p}{m}} kx = 0$$

•) All of this concerns particles in a TD system without sources or drains. When we add those, we have

$$\frac{df}{dt} = C[f]$$

collision term, may change particle types, transfers between f_s for different species s.

•) When all the components of the plasma equilibrate, the $C[f_{eq}] = 0$ as well.

$$\frac{df_{eq}}{dt} = 0 = C[f_{eq}]$$

BOLTZMANN EQUATION IN FLRW

•) We go from Minkowski $g_{\mu\nu} = (-1,1,1,1)$, $p = m\dot{x}$

to FLRW $g_{\mu\nu} = (-1, a^2, a^2, a^2)$, $P^\mu = \frac{dx^\mu}{d\lambda}$

on-shell condition

$$P^2 = P_\mu P^\mu = g_{\mu\nu} P^\mu P^\nu = -E^2 + a^2 P^{i2} = -E^2 + p^{i2} = -m^2$$

valid for mas

• It is useful to separate the momentum dependence into size $p = \sqrt{p^2}$ and direction $\hat{p}^i = \frac{p^i}{p}$

$$\text{Then: } \frac{df}{dt} = \partial_t f + \partial_{x^i} f \, \dot{x}^i + \partial_p f \, \dot{p} + \partial_{\hat{p}^i} f \, \dot{\hat{p}}^i$$

$$= \partial_t f + \partial_p f \, \dot{p}$$ in a homogeneous isotropic universe

• The last two terms $\partial_{x^i} f \, \dot{x}^i$ and $\partial_{\hat{p}^i} f \, \dot{\hat{p}}^i$ are relevant for perturbations.

•) The $\frac{dp}{dt}$ term in an expanding universe can be obtained from the geodesic

$$\frac{dP^0}{d\lambda} = \frac{dP^0}{dt} \underbrace{\frac{dt}{d\lambda}}_{= P^0}^{= x^0} = P^0 \frac{dP^0}{dt} = - \Gamma^0_{ij} P^i P^j = - \delta_{ij} \frac{\dot{a}}{\cancel{a}} a P^{i2}$$

the rest are zero

$$\overset{E}{\overset{\|}{p^0}} \, \overset{E}{\overset{\|}{\frac{dp^0}{dt}}} = -\frac{\dot{a} a p^{i2}}{a^4} = -\frac{\dot{a}}{a} p^2 \qquad a^2 p^{i2} = p^{i2} = p^2$$

$$/\!/$$

•) $\frac{1}{2} \frac{d}{dt} (E^2) = \frac{1}{2} \frac{d}{dt} (p^2 - m^2) = p \frac{dp}{dt}$

Finally: $\cancel{p} \frac{dp}{dt} = -H p^{\cancel{2}}$

The Boltzmann equation thus becomes

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial p} \frac{dp}{dt} = \frac{\partial f}{\partial t} - Hp \frac{\partial f}{\partial p} = C[f]$$

•) A similar derivation includes also the NH piece

$$\partial_t f + \frac{p}{E} \frac{\hat{p}^i}{a} \partial_{x^i} f - Hp \, \partial_p f = C[f] \qquad (\star)$$

•) From the definition of number density

$$dN = n \, d^3x = \int f \frac{d^3p}{(2\pi)^3} d^3x$$

$$\frac{dn}{dt} = \int \frac{df}{dt} \frac{d^3p}{(2\pi)^3} \qquad \int_p \equiv \int \frac{d^3p}{(2\pi)^3}$$

Integrating $\star$ over $\int_p$, we will get an eq. for n.

Let us plug the definition of n into the Boltzmann equation integrated over p

$$\int_p \frac{\partial f}{\partial t} - H \int_p p \frac{\partial f}{\partial p} = \int_p C[f] \quad , \quad \int_p = \int \frac{d^3p}{(2\pi)^3}$$

$$\frac{dn}{dt} - H \int \frac{4\pi}{(2\pi)^3} p^3 \frac{\partial f}{\partial p} dp = \int_p C[f]$$

•) Regular distributions vanish $p^3 f \xrightarrow[p\to\infty]{p\to 0} 0$ (CHECK on FD & BE)

$$p^3 f \Big|_0^\infty = \int dp\, \partial_p (p^3 f) = \int (p^3 \partial_p f + 3p^2 f)\, dp = 0$$

$$\Rightarrow \int p^3 \frac{\partial f}{\partial p} dp = -3 \int p^2 f dp$$

We get: $\frac{dn}{dt} + 3H \int \frac{4\pi p^2}{(2\pi)^3} f dp = \frac{dn}{dt} + 3Hn = \int_p C[f]$

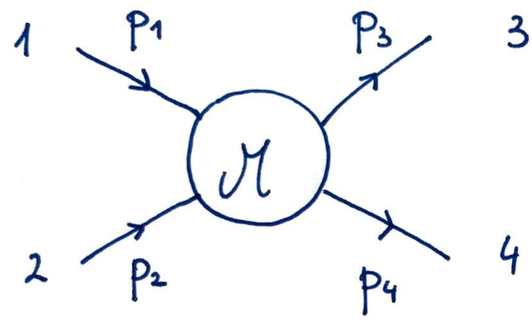
$$\boxed{\frac{dn}{dt} + 3Hn = \int_p C[f]}$$

• Of course when $C=0$ $\quad \frac{dn}{dt} = -3 \frac{da}{dt} \frac{n}{a}$ or

$$\frac{dn}{n} = -3 \frac{da}{a} \quad \text{or} \quad n \propto a^{-3} \checkmark$$

COLLISIONS AND THERMALLY AVERAGED $\langle \sigma v \rangle$

•) Let's consider a generic 2-2 scattering process with 1,2 the incoming and 3,4 outgoing.



$\Rightarrow C[f]$

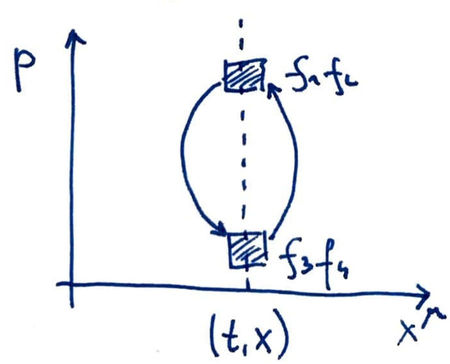
•) Four-momentum conservation implies

$$E_1 + E_2 = E_3 + E_4 \quad , \quad \vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 \ ,$$

all of the particles are on-shell : $E_i^2 - p_i^2 = m_i^2$

The microscopic interactions inside $\mathcal{M}$ will transfer momenta (kinetic) and type of species (chemical) and if $C[f]$ is 'strong enough' these interactions will maintain particles in equilibrium.

KINETIC RE-SHUFFLING of phase space



•) We assume local interactions preserving causality, they happen at (t,x).

•) Let's concentrate on the number density of $n_1(t)$

$$\dot{n}_1 + 3Hn_1 = \int_{p_1} C[f(p_1)]$$

$$C[f_1(p_1)] = \sum_{p_{2,3,4}} \underbrace{\delta(E_1+E_2-E_3-E_4)}_{\text{energy conservation}} \underbrace{|\mathcal{M}|^2}_{\substack{\text{matrix}\\ \text{element}}} \underbrace{(\tilde{f}_3\tilde{f}_4 - \tilde{f}_1\tilde{f}_2)}_{\substack{\text{redistribution}\\ \text{of phase space}}}$$

•) The statistical factors $\tilde{f}_i = f_i(1 \pm f_j)$

↳ FINAL state suppression

$+$ for bosons

$-$ for fermions due to Fermi blocking

•) The sum over momenta is the usual integral in QFT

$$f'(E) = \frac{df}{dE} = 2E$$

$$\int d^3p \int dE\, \delta(\underbrace{E^2-p^2-m^2}_{f(E)})$$

$$= \int_p \int_0^\infty dE \frac{\delta(E-\sqrt{p^2-m^2})}{|2E|} \quad \text{for all } i=2,3,4$$

$$\Rightarrow \boxed{\begin{aligned} C[f_1] = \frac{1}{2E_1} \prod_{i=2}^{4} \int_{\pi_i} (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4) \overline{|\mathcal{M}|^2} \times \\ [f_3 f_4 (1\pm f_1)(1\pm f_2) - f_1 f_2 (1\pm f_3)(1\pm f_4)] \end{aligned}}$$

•) The final form of the Boltzmann equation is :

$$\dot{n}_1 + 3Hn_1 = \prod_{i=1}^{4} \int_{\pi_i} (2\pi)^4 \delta^{(4)}(p_1+p_2-p_3-p_4) |\mathcal{M}|^2 \times \left[f_1 f_2 (1 \pm f_3)(1 \pm f_4) - f_3 f_4 (1 \pm f_1)(1 \pm f_2) \right]$$

We will use it extensively to keep track of number densities for various physical processes.

<u>ADDENDUM</u>: <u>Perturbations</u>

If we are interested in fluctuations, we can modify the metric by

$$g_{00} = -(1+2\psi)$$

$$g_{ij} = a^2 \delta_{ij} (1+2\phi) \qquad (\psi, \phi)(x^\mu)$$

$\Downarrow$ geodesic equation modifies to

$$p^\mu = \Big(\underbrace{E}_{\sqrt{p^2+m^2}}(1-\psi), \frac{p^i}{a}(1-\phi) \Big)$$

$$\Rightarrow \frac{dp^i}{dt} = -(H+\dot{\phi})p^i - \frac{E}{a}\psi_{,i} - \frac{1}{a}\frac{p^i}{E} p^k \phi_{,k} - \frac{p^2}{aE}\phi_{,i}$$

$$\frac{df}{dt} = \partial_t f + \frac{p}{E}\frac{\hat{p}^i}{a}\partial_{x^i} f - \left(H + \dot{\phi} + \frac{E}{ap}\hat{p}^i \psi_{,i}\right) p \, \partial_p f$$

·) After solving for f, we can get the usual ρ, s, n, P, as well as perturbed

$$T^0_0 = -g \int_p E f = -\rho \,, \quad T^0_i = g\, a\,(1+\phi-\psi) \int_{p_i} p_i f$$

$$T^i_j = g \int_p \frac{p^i p_j}{E} f$$