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BIG BANG NUCLEOSYNTHESIS

Prediction of the abundances of the light elements

D ^{3}He ^{4}He ^{7}Li

synthesized at the end of the first three minutes, based on well-understood Standard Model physics.

Prediction in good agreement with observational data.

REFERENCES:

- "The first three minutes" S. Weinberg (popular book)
- "Nucleosynthesis without a computer" - Mukhanov astro-ph/0303073
- "Primordial nucleosynthesis without a computer"
  Esmailzadeh, Starkman, Dimopoulos
  Astrophys. J. 378 (1991) 504
- "Cosmological Helium production simplified"
  Bernstein, Brown, Feinberg
  Rev. Mod. Phys. 61 (1989) 25

I will follow the last reference and use the same equation numbering.

• OVERVIEW

• At temperatures $T > \text{MeV}$ (but below $\Lambda_{QCD} \sim 200$ MeV)

protons and neutrons in thermal equilibrium

n_p $\quad$ n_n (number densities)

• At $T \sim \text{MeV}$ neutrons freeze out, and

$$
(1.15) \qquad \left.\begin{aligned}
&n + p \leftrightarrow D + \gamma \\
&D + D \leftrightarrow T + p \\
&T + D \leftrightarrow {}^4He + n
\end{aligned}\right\} \text{ thermal and chemical equilibrium}
$$

• At $T \sim \frac{1}{25} \mathcal{E}_D$ $\qquad$ $\mathcal{E}_D = 2.23$ MeV deuteron binding energy

the reactions (1.15) proceed almost entirely to the right.

All the neutrons present at freeze-out are captured in ^4He

$\mathcal{E}_{He} = 28.3$ MeV $\quad$ binding energy of ^4He.

ROUGH TREATMENT

$T \simeq 100$ MeV $\qquad$ Muons [redacted]

Total energy density dominated by relativistic species

$e^\pm, \nu_e, \nu_\mu, \nu_\tau, \gamma$

$m_n, m_p \simeq 1$ GeV $\quad$ non-relativistic $\quad$ $n_n, n_p \sim e^{-m/T}$ suppressed

Particles in thermal equilibrium through

(1.1 a) $\nu_e + n \longleftrightarrow p + e^-$

(1.1 b) $e^+ + n \longleftrightarrow p + \bar{\nu}_e$

(1.1 c) $n \longleftrightarrow p + e^- + \bar{\nu}_e$

Can compute $\dfrac{n_n(T)}{n_p(T)} \simeq e^{-\frac{\Delta m}{T}}$

$\Delta m = m_n - m_p \simeq 1.29$ MeV

@ $T = 100$ MeV $\qquad e^{-\frac{\Delta m}{T}} = 0.987$

Define relative number of neutrons

$$X(T) \equiv \frac{n_n(T)}{n_n(T) + n_p(T)}$$

$$X_{eq}(T) = \frac{1}{1 + e^{\Delta m/T}}$$ in equilibrium

Should we worry about neutron decay?

We shall see that ^4He production occurs when the Universe is 180 s (three minutes) old. Once captured in ^4He, neutrons do not decay anymore. For free neutrons the lifetime is

$\tau \simeq 881$ s

$\Rightarrow$ as a first approximation <u>neglect neutron decay</u> (1.1 c)

Follow reactions (1.1 a) and (1.1 b) in the expanding universe :

$$\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi\rho}{3M_p^2} \qquad M_p = 1.22 \times 10^{19}\ \mathrm{GeV}$$

The energy density is dominated by relativistic particles

$$\rho = g_* \frac{\pi^2}{30} T^4 \qquad g_* = \underset{\substack{\uparrow \\ \text{photons}}}{2} + \underset{\substack{\uparrow \\ \text{electron}}}{\frac{7}{4}} + \underset{\substack{\uparrow \\ \text{positron}}}{\frac{7}{4}} + 3\underset{\substack{\uparrow \\ \text{neutrinos}}}{\frac{7}{4}} = \frac{43}{4}$$

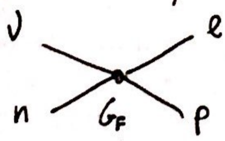
$\Gamma(T) \simeq n_\nu(T) \langle \sigma v \rangle_T$ rate of reactions (1.1a-b)

When $\Gamma(T) \simeq H$ the neutron-to-baryon ratio freeze out

$$X(T \simeq 0) = X_{eq}(T_F)$$

$n_\nu(T) \sim T^3$ neutrino number density

$\sigma \sim G_F^2 T^2$



$v \sim 1$

Fermi constant

$G_F = 10^{-5}\ \mathrm{GeV}^{-2}$

$$\Gamma(T) \simeq G_F^2 T^5$$

T_F defined by $\Gamma(T_F) \sim H \Rightarrow T_F^3 \sim \frac{\sqrt{g_*}}{G_F^2 M_p}$

$$\boxed{T_F \sim g_*^{1/6} \times 1\ \mathrm{MeV}}$$

At the end of reactions (1.15), once $T < \frac{1}{25} \varepsilon_D$,

all the available neutrons will be in ^4He

$$\text{mass fraction of } ^4He = Y_4 \equiv \frac{m_{He}\, n_{He}}{m_p n_p + m_n n_n}$$

$m_n \simeq m_p$ $\quad m_{He} \simeq 4\, m_n$ $\quad n_{He} = \frac{n_n}{2}$ $\quad$ ^4He contains 2 neutrons

$$\boxed{Y_4 = \frac{4 \frac{1}{2} n_n}{n_p + n_n} = 2\, X(T \simeq 0)}$$

The freeze-out ratio $X(T \simeq 0)$ of neutron-to-baryon determines the amount of primordial ^4He production.

We are after a more accurate determination of $X(T \simeq 0)$.

NEUTRON-PROTON RATIO (detailed)

In this section we follow time t, rather than temperature T.

Recall that in a radiation dominated expanding Universe $t \sim \frac{1}{T^2}$

($T = 100$ MeV $\leftrightarrow t = 10^{-4}$ s , $T = 1$ MeV $\leftrightarrow t \simeq 1$ s)

$\lambda_{pn}(t)$: rate of processes to convert $p \to n$

$\lambda_{np}(t)$: " " " " " $n \to p$

Boltzmann equation

$$\frac{dX(t)}{dt} = \lambda_{pn}(t)\left[1 - X(t)\right] - \lambda_{np}(t)\,X(t) \tag{2.1}$$

Solution

$$X(t) = \int_{t_0}^{t} dt'\; I(t,t')\,\lambda_{pn}(t') + I(t,t_0)\,X(t_0) \tag{2.2}$$

$$I(t,t') = \exp\left[-\int_{t'}^{t} dt''\,\Gamma(t'')\right] \tag{2.3}$$

$$\Gamma(t) = \lambda_{pn}(t) + \lambda_{np}(t) \tag{2.4}$$

Let t_0 be the time when $T \simeq 100$ MeV. At t_0 λ_{pn} and λ_{np} are very large $\Rightarrow$ $\Gamma(t_0)$ very large $\underset{(2.3)}{\Longrightarrow}$ $I(t,t_0)$ very small $\Rightarrow$ neglect $I(t,t_0)X(t_0)$ in (2.2).

Also, we can set $t_0 = 0$ in the integral in (2.2). We have

$$X(t) = \int_{0}^{t} dt'\; I(t,t')\,\lambda_{pn}(t') \tag{2.5}$$

Note that

$$I(t,t') = \frac{1}{\Gamma(t')}\,\frac{d}{dt'}\,I(t,t') \tag{2.6}$$

Integrate (2.5) by parts

$$X(t) = \frac{\lambda_{pn}(t)}{\Gamma(t)} - \int_{0}^{t} dt'\; I(t,t')\,\frac{d}{dt'}\left[\frac{\lambda_{pn}(t')}{\Gamma(t')}\right] \tag{2.7}$$

Use (2.6) again and integrate by parts

(2.8) $$X(t) \simeq \frac{\lambda_{pn}(t)}{\Gamma(t)} - \frac{1}{\Gamma(t)} \frac{d}{dt}\left[\frac{\lambda_{pn}(t)}{\Gamma(t)}\right] + O\left(\left(\frac{d\lambda_{pn}}{dt}\right)^2, \frac{d^2\lambda_{pn}}{dt^2}\right)$$

Principle of detailed balance (will get back to it)

(2.9) $$\lambda_{pn}(t) = e^{-\frac{\Delta m}{T}} \lambda_{np}(t)$$

(2.10) $$\frac{\lambda_{pn}(t)}{\Gamma(t)} = \frac{\lambda_{pn}(t)}{\lambda_{pn}(t) + \lambda_{np}(t)} = \frac{1}{1 + e^{\Delta m/T}} = X_{eq}(T)$$

Consider $X_{eq}(T(t-t_i)) = X_{eq}(T) - \frac{1}{\Gamma(t)} \frac{d}{dt} X_{eq}(T) + \cdots$

$\frac{1}{\Gamma(t)}$

We can interpret (2.8) as the Taylor expansion of

(2.11) $$X(t) \simeq X_{eq}\left(T\left(t - \frac{1}{\Gamma(t)}\right)\right)$$

The population has an equilibrium value, but at an earlier time, a time retarded by the reaction time $\frac{1}{\Gamma(t)}$.

Another way of putting this: call the earlier time temperature

$T\left(t - \frac{1}{\Gamma(t)}\right) \equiv T_{eff}(t)$

Conservation of entropy in the expanding Universe implies

$a(t)\,T(t) = \text{constant} \implies \frac{\dot{T}}{T} = -\frac{\dot{a}}{a} = -H$

$\dot{T} = \frac{\delta T}{\delta t} \implies \delta T = -H\,T\,\delta t$

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With $\delta t = -\frac{1}{\Gamma(t)}$ we have

$$T_{eff}(t) = T(t) + \delta T = T\left(1 + \frac{H}{\Gamma}\right)$$

We can reinterpret (2.11) as

$$X(T) \simeq X_{eq}\left(T_{eff}(t)\right)$$

As long as the reactions are fast, $\Gamma \gg H$, we have $T_{eff} \simeq T$.

Close to neutron freeze-out, $\Gamma \sim H$, we have $T_{eff} > T$.

However, for $\Gamma \sim H$ the result is not quantitatively correct, and we need more details.

$$\lambda_{np} = \lambda(\nu + n \to p + e^-) + \lambda(e^+ + n \to p + \bar{\nu}) + \lambda(n \to p + e^- + \bar{\nu})$$

$$\lambda(\nu + n \to p + e^-) = A \int_0^\infty dp_\nu\, p_\nu^2\, p_e\, E_e\, (1 - f_e)\, f_\nu \qquad (2.16\ a)$$

$$\lambda(e^+ + n \to p + \bar{\nu}) = A \int_0^\infty dp_e\, p_e^2\, p_\nu\, E_\nu\, (1 - f_\nu)\, f_e \qquad (2.16\ b)$$

$$\lambda(n \to p + \bar{\nu} + e^-) = A \int_0^\infty dp_e\, p_e^2\, p_\nu\, E_\nu\, (1 - f_\nu)(1 - f_e)$$

Here, $A \sim G_F^2$ but we don't care about its value for now.

$$E_e = \sqrt{p_e^2 + m_e^2} \qquad E_\nu = p_\nu$$

$$f_e = \frac{1}{e^{E_e/T_e} + 1} \qquad f_\nu = \frac{1}{e^{E_\nu/T_\nu} + 1}$$

In our temperature range of interest, $T_\nu \neq T_e$. However, the difference between T_e and T_ν is small, and we neglect it for now.

$$e^- + \gamma \leftrightarrow e^- + \gamma \quad \Rightarrow \quad T_e = T_\gamma$$

So for now we take $\quad T_\nu = T_e = T_\gamma = T$

Energy conservation implies: (neglecting nucleon recoil)

(2.16 a) $\longrightarrow \quad E_e = E_\nu + \Delta m$

(2.16 b) $\longrightarrow \quad E_\nu = E_e + \Delta m$

Consider the inverse reaction of (2.16 a)

$$\lambda(e^- + p \to n + \nu) = A \int_{p_0}^{\infty} dp_e \, p_e^2 \, p_\nu \, E_\nu (1 - f_\nu) f_e$$

$$\left[\int_{p_0}^{\infty} dp_e \, p_e^2 p_\nu E_\nu = \int_{E_0}^{\infty} dE_e \, E_e \, p_e \, p_\nu \underset{p_\nu}{\overset{\shortparallel}{E_\nu}} = \int_0^\infty dE_\nu \, E_e \, p_e \, p_\nu^2 \right]$$

$$p_e \, dp_e = E_e \, dE_e \qquad \underbrace{dE_e = dE_\nu}_{\text{energy conservation}} = dp_\nu$$

$$1 - f_\nu = e^{E_\nu/T} f_\nu \qquad f_e = (1 - f_e)\, e^{-E_e/T}$$

$$\Rightarrow (1 - f_\nu) f_e = f_\nu (1 - f_e)\, e^{(E_\nu - E_e)/T} \qquad E_e - E_\nu = \Delta m$$

$$\Rightarrow \lambda(e^- + p \to n + \nu) = e^{-\frac{\Delta m}{T}} \lambda(\nu + n \to p + e^-)$$

<u>principle of detailed balance</u>

We have already used it..

Next approximation:

during freeze-out T is low compared to the energies E_e, E_ν which dominate the integrals in λ.

$$\Rightarrow \quad f_{e,\nu} \simeq \exp\left(-\frac{E_{e,\nu}}{T}\right) \qquad \text{Boltzmann}$$

$$\Rightarrow \quad 1 - f_{e,\nu} \simeq 1$$

So

$$\lambda(\nu + n \to p + e^-) = A \int_0^\infty dp_\nu \, p_\nu^2 \, p_e \, E_e \, e^{-E_\nu/T} \qquad (2.21\,a)$$

$$\lambda(e^+ + n \to p + \bar{\nu}) = A \int_0^\infty dp_e \, p_e^2 \, p_\nu \, E_\nu \, e^{-E_e/T} \qquad (2.21\,b)$$

$$\lambda(n \to p + \bar{\nu} + e^-) = A \int_0^{p_0} dp_e \, p_e^2 \, p_\nu \, E_\nu \qquad (2.21\,c)$$

Neglect m_e in a and b:

$$p_e = \overbrace{E_e = \Delta m + E_\nu}^{a} \qquad (p_\nu = E_\nu)$$

$$\underbrace{E_\nu = \Delta m + E_e}_{b}$$

$\Rightarrow$ (2.21 a) = (2.21 b)

$$\lambda(\nu + n \to p + e^-) = A\, T^3 \left(4!\, T^2 + 2 \cdot 3!\, T \Delta m + 2!\, \Delta m^2\right) \qquad (2.22)$$

$$= \lambda(e^+ + n \to p + \bar{\nu})$$

We have neglected effects of order $\left(\frac{m_e}{T}\right)^2$. The approximation breaks down for $T < m_e$, but there the λ's are small.

We want to trade A for τ (neutron lifetime):

$$\frac{1}{\tau} = \lambda(n \to p + \bar{\nu} + e^-) = \frac{A}{5}\left(\Delta m^2 - m_e^2\right)^{\frac{1}{2}}\left(\frac{1}{6}\Delta m^4 - \frac{3}{4}\Delta m^2 m_e^2 - \frac{2}{3} m_e^4\right)$$

$$+ \frac{A}{4} m_e^4 \, \Delta m \cosh^{-1}\left(\frac{\Delta m}{m_e}\right)$$

$\Delta m = 1.29$ MeV $\qquad m_e = 0.511$ MeV

$$\frac{1}{\tau} = 0.0157\ A\ \Delta m^5 \qquad \text{or}$$

$$A = \frac{1}{4}\ \frac{a}{\tau}\ \Delta m^{-5} \qquad a = 255$$

In this section we neglect free-neutron decay, so the total λ_{np} is twice that given in (2.22).

$$y = \frac{\Delta m}{T} \qquad \lambda_{np}(t) = \frac{a}{\tau y^5}\left(12 + 6y + y^2\right) \qquad (2.27)$$

Recall (2.7); change variable $t \to y$

$$(2.28) \qquad X(y) = \frac{\lambda_{pn}(y)}{\Gamma(y)} - \int_0^y dy'\, I(y,y')\, \frac{d}{dy'}\left(\frac{\lambda_{pn}(y')}{\Gamma(y')}\right)$$

The detailed balance (2.9) gives

$$\Gamma(y) = \left(1 + e^{-y}\right)\lambda_{np}(y)$$

and the integrating factor is

$$I(y,y') = \exp\left[-\int_{y'}^{y} dy''\left(\frac{dt''}{dy''}\right)\Gamma(y'')\right]$$

$$\frac{dt}{dy} = \frac{dt}{dT}\,\frac{dT}{dy} \qquad \frac{dT}{dy} = -\frac{\Delta m}{y^2}$$

Entropy conservation $\Rightarrow \dfrac{\dot T}{T} = -\dfrac{\dot a}{a} \Rightarrow \dfrac{dT}{dt} = -HT$

$$H = \left(\frac{4\pi^3}{45 M_p^2}\, g_*\right)^{\frac{1}{2}} T^2 \Rightarrow \frac{dT}{dt} = -\left(\frac{4\pi^3}{45 M_p^2}\, g_*\right)^{\frac{1}{2}} T^3$$

$$T^3 = \left(\frac{\Delta m}{y}\right)^3 \Rightarrow \frac{dt}{dT} = +\left(\frac{45}{4\pi^3 g_*}\right)^{\frac{1}{2}} \frac{M_p\, y}{\Delta m^2}$$

So we have

$$I(y,y') = \exp\left[K(y) - K(y')\right] \qquad (2.33)$$

$$K(y) = -b \int dy' \left(\frac{12}{y'^4} + \frac{6}{y'^3} + \frac{1}{y'^2}\right)(1+e^{-y'}) \qquad (2.34)$$

$$b = a \left(\frac{45}{4\pi^3 g_*}\right)^{\frac{1}{2}} \frac{M_P}{\tau \Delta m^2} \qquad (2.35)$$

Can integrate (2.34)

$$K(y) = b\left[\left(\frac{4}{y^3} + \frac{3}{y^2} + \frac{1}{y}\right) + \left(\frac{4}{y^3} + \frac{1}{y^2}\right)e^{-y}\right] \qquad (2.36)$$

Define again

$$X_{eq}(y) = \frac{\lambda_{pn}(y)}{\Gamma(y)} = \frac{1}{1+e^y}$$

$$\Rightarrow \quad X(y) = X_{eq}(y) + \overbrace{\int_0^y dy' \, e^{y'} X_{eq}^2(y') \exp\left[K(y) - K(y')\right]}^{\text{"freeze-out" correction}} \qquad (2.38)$$

We can evaluate this last integral numerically and plot $X(y)$ (see fig. 1).

$$X(y \to \infty) = X(T=0) = 0.151$$

SUMMARY SO FAR:

We have computed the freeze-out value of the neutron fraction $\frac{n}{p}$. More precisely we have dealt with the quantity $X = \frac{n_n}{n_n + n_p}$

To solve the Boltzmann equation for X we have worked out the reaction rates for

(a) $\nu + n \longleftrightarrow p + e^-$

(b) $e^+ + n \longleftrightarrow p + \bar{\nu}$

These are the important processes to determine whether the neutrons are in thermal equilibrium. The process

$n \longleftrightarrow p + e^- + \bar{\nu}$

can be neglected to a first approximation.

We have also made the following approximations:

- $T_e = T_\nu$
- Boltzmann instead of Fermi-Dirac distributions
- $m_e = 0$

With these we found an analytic expression for the rate (2.27), and then for the neutron abundance ratio (2.38).

In the end, $X(T \simeq 0) = 0.151$, with these approximations.

• Dependence on number of neutrino species.

Our result for $X(y)$ (2.38) depends on the parameter b (2.35):

$$b = \frac{0.823}{\sqrt{g_*}} \qquad (2.39)$$

If we vary the number of neutrino types, then g_* changes and

$$\frac{\delta b}{b} = -\frac{1}{2}\frac{\delta g_*}{g_*}$$

Such a variation affects $K(y)$ and produces

$$\frac{\delta X(y=\infty)}{X(y=\infty)} = -c\,\frac{\delta b}{b}$$

where

$$c = \frac{1}{X(y=\infty)}\int_0^\infty dy'\, e^{y'} X_{eq}^2(y')\, K(y')\, \exp(-K(y')) = 0.52$$

(the integral is easily evaluated numerically)

Suppose we add one more neutrino species: $\delta g_* = +\frac{7}{4}$

$$\Rightarrow \quad \frac{\delta X(y=\infty)}{X(y=\infty)} = 0.042$$

The variation δX is positive:

more relativistic neutrinos $\rightarrow$ more energy density

$\rightarrow$ faster expansion $\rightarrow$ neutrons come out of equilibrium sooner, at a higher temperature, with a larger population.

NEUTRON DECAY CORRECTION

Change of notation : $X(y) \to \bar{X}(y)$ in (2.38)

Including the effect of neutron decay

$$X(t) = e^{-t/\tau}\,\bar{X}(y(t))$$

↳ does not vary much during the period in which neutrons decay.

When the temperature drops below the deuteron binding energy, the neutrons are captured in deuterons, and the deuterons collide to place all of the ~~xxx~~ neutrons present at this time t_c into ^4He :

$$X(T=0) = X(t=\infty) = X(t=t_c) = e^{-t_c/\tau}\,\bar{X}(y=\infty)$$

Approximate (but long) calculation of t_c

As we will see, at t_c we have $T \ll m_e$. At this point the neutrinos have decoupled from the thermal bath, and the electrons and positrons have annihilated, heating the photon gas but not altering the neutrino distribution.

Therefore, at times close to t_c we have $T_\gamma \neq T_\nu$. More precisely

$$T_\gamma(t) = \left(\frac{11}{4}\right)^{\frac{1}{3}} T_\nu(t) \qquad \text{for } t \text{ such that } T_\gamma \ll m_e$$

Translate T into t

$$a(t)\, T_\nu(t) = \text{constant} \quad \Rightarrow \quad \frac{\dot{T}_\nu}{T} = -\frac{\dot{a}}{a}$$

$$\frac{\dot{a}}{a} \equiv H = \left(\frac{8\pi\rho}{3M_P^2}\right)^{\frac{1}{2}}$$

(3.7) $$t = \int_{T_\nu}^{\infty} \frac{dT'_\nu}{T'_\nu} \left(\frac{3M_P^2}{8\pi\rho}\right)^{\frac{1}{2}}$$

At high temperature $(T \gg m_e)$ $\quad T_\nu = T_\gamma$

$$\rho = g_* \frac{\pi^2}{30} T_\nu^4 \qquad g_* = \frac{43}{4}$$

At low temperature $(T \ll m_e)$

$$\rho_0 = g_{*\nu} \frac{\pi^2}{30} T_\nu^4 + g_{*\gamma} \frac{\pi^2}{30} T_\gamma^4 = g_{*eff} \frac{\pi^2}{30} T_\nu^4$$

$$g_{*eff} = g_{*\nu} + \left(\frac{11}{4}\right)^{\frac{4}{3}} g_{*\gamma} = 3 \cdot \frac{7}{4} + \left(\frac{11}{4}\right)^{\frac{4}{3}} \cdot 2 \simeq 13$$

The low-temperature form is relevant for times close to t_c:

(3.10) $$t = \left(\frac{45}{16\pi^3 g_{*eff}}\right)^{\frac{1}{2}} \frac{M_P}{T_\nu^2} + t_0$$

Here $t_0 \simeq 2\,s$ (see Appendix for derivation)

Recall the equilibrium distributions:

$$n_a = g_a\, e^{-(\mu_a + m_a)/T} \left(\frac{m_a T_\gamma}{2\pi}\right)^{3/2} \qquad a = n, p, D$$

At early times the free gases of neutrons, protons, and deuterons are in chemical equilibrium

$$\mu_D = \mu_n + \mu_p$$

$$(3.14) \qquad \frac{n_n n_p}{n_D} = \frac{g_p g_n}{g_D} \left(\frac{m_p m_n}{m_D}\right)^{3/2} \left(\frac{T_\gamma}{2\pi}\right)^{\frac{3}{2}} e^{-\mathcal{E}_D/T}$$

(Saha eq)

$\mathcal{E}_D = m_p + m_n - m_D$ $\qquad g_p = g_n = 2 \qquad g_D = 3$ statistical spin weights

As we will see (3.14) gives a very small fraction of deuterons until T_γ is well below $\mathcal{E}_D$: DEUTERON BOTTLENECK
that inhibits the formation of ^4He through the reactions

$n + p \to D + \gamma$ (3.16 a)

$D + D \to T + p$ (3.16 b)

$D + T \to {}^4He + n$ (3.16 c)

Define $\quad X_a = \dfrac{n_a}{n_B}$

n_B : total baryon number density

$$X_p + X_n + 2X_D + 3X_T + 4X_{4He} + \cdots = 1$$

$n_\gamma = \dfrac{2\zeta(3)}{\pi^2} T_\gamma^3$ photon number density $\qquad \zeta(3) \simeq 1.202$

$$(3.18) \qquad \boxed{\eta = \frac{n_B}{n_\gamma}} \sim 10^{-10}$$

Rewrite (3.14) : $\quad \dfrac{X_n X_p}{X_D} = G_{np}$

$$(3.20) \qquad G_{np} = \frac{\sqrt{\pi}}{12\zeta(3)} \frac{1}{\eta} \left(\frac{m_p}{T_\gamma}\right)^{\frac{3}{2}} e^{-\mathcal{E}_D/T_\gamma}$$

@ $T \simeq 0.1$ MeV $\qquad G_{np} \simeq 10^5$

Since $X_n \simeq 0.1$, at temperatures above 0.1 MeV the deuteron fraction is less than 10^{-4}, an insufficient population of D to produce much ^4He by (3.16 b and c) BOTTLENECK

Go beyond Saha equilibrium equation and study rate equations.

Define $\quad z = \dfrac{\varepsilon_D}{T_\gamma} \qquad R = \dfrac{dt}{dz} \underbrace{\langle \sigma v \rangle_T}_{\text{thermally averaged cross section}} n_B$

~~Use~~ Trade n_B for η

Use (3-10) to compute $\dfrac{dt}{dz}$

$$R = \frac{\eta}{z^2}\left(\frac{45}{\pi^7 g_{*eff}}\right)^{\frac{1}{2}}\left(\frac{11}{4}\right)^{\frac{2}{3}} \zeta(3)\, \varepsilon_D M_p \langle \sigma v \rangle_T \tag{3.23}$$

Since $\dfrac{dt}{dz} \sim \dfrac{1}{H} \quad \Rightarrow \quad R \sim \dfrac{\Gamma}{H}$

For n and p we have

$$\frac{dX_n}{dz} = -R_{np}\left(X_p X_n - G_{np} X_D\right) + \cdots \tag{3.24a}$$

$$\frac{dX_p}{dz} = -R_{np}\left(X_p X_n - G_{np} X_D\right) + \cdots \tag{3.24b}$$

With $\langle \sigma_{np} v \rangle_T = 4.55 \times 10^{-20} \dfrac{cm^3}{s}$ (Peebles 1966)

$$R_{np} \simeq 5 \left(\frac{29}{z}\right)^2 \left(\frac{\eta}{\eta_0}\right) \tag{3.25}$$

$\eta_0 = 5 \times 10^{-10}$

Before neutron capture, $z < 29 \Rightarrow R_{np} \gtrsim 5 \Rightarrow \Gamma_{np} > H$

If the deuterons are not depleted by other reactions (3.16 b and c), they are kept in equilibrium with protons and neutrons:

$$X_p + X_n + 2X_D = 1 \qquad\qquad X_D = G_{np}^{-1} X_n X_p$$

$$G_{np}^{-1} \ll 1 \qquad \Rightarrow \quad X_D \ll X_n \sim X_p$$

$$\Rightarrow \quad X_p^{(0)} + X_n^{(0)} = 1 \qquad\qquad X_D^{(1)} = G_{np}^{-1} X_n^{(0)} X_p^{(0)} \qquad\qquad (3.27)$$

$$X_D^{(0)} = 0$$

With this first approximation

$$X_p + X_n \simeq 1 - 2\, G_{np}^{-1} X_p^{(0)} X_n^{(0)}$$

For $z \sim 30$, from (3.20) we have $G_{np}^{-1} \sim e^{z} \Rightarrow \frac{d}{dz} G_{np}^{-1} \approx G_{np}^{-1}$

$$\Rightarrow \quad \frac{d}{dz}(X_p + X_n) \simeq -2\, G_{np}^{-1} X_p^{(0)} X_n^{(0)}$$

Add (3.24 a + b) and use (3.27)

$$R_{np}\,(X_p X_n - G_{np} X_D) \simeq X_D^{(1)} \qquad\qquad (3.30)$$

The analysis so far only considered the reaction (3.16 a).

Next, consider (3.16 b)

$$\frac{dX_D}{dz} = R_{np}\,(X_p X_n - G_{np} X_D) - R_{DD}\,(2X_D^2 - G_{DD} X_T X_p) + \dots \qquad (3.31)$$

Similarly to (3.20) we can derive the equilibrium distributions

$$G_{DD} = \frac{9}{4}\left(\frac{m_D^2}{m_T m_P}\right)^{3/2} e^{-B/T_\gamma}$$

$$B = 2m_D - m_T - m_P \simeq 4.02 \text{ MeV}$$

G_{DD} is always a small number, neglect it in (3.31).

To compute R_{DD} we need σ_{DD} for $D + D \to T + p$. The result is

$$(3.37) \qquad R_{DD} = 2.4 \cdot 10^7 \left(\frac{\eta}{\eta_0}\right) z^{-4/3} e^{-1.44 z^{1/3}}$$

Here there is a typo in the paper

Once (3.16 b) is efficient, the D's are converted into T + p and the $D + \gamma \to n + p$ to Keep the ~~produce~~ free neutrons around is no longer efficient. This happens when

$$\left.\frac{dX_D}{dz}\right|_{z=z_c} \triangleq R_{np}(X_p X_n - G_{np} X_D) - R_{DD}\, 2X_D^2 \simeq 0 \qquad (3.38)$$

We take (3.38) as the definition of t_c, as at that point the neutrons quickly end up in ^{4}He.

Use (3.30)

$$\left.\frac{dX}{dz}\right|_{z=z_c} \simeq X_D^{(1)} - 2R_{DD} X_D^{(1)\,2} = 0$$

$$2R_{DD} X_D^{(1)} = 1 \qquad (3.39)$$

Use $X_D^{(1)} = G_{np}^{-1} X_p^{(0)} X_n^{(0)}$ with $X_n^{(0)} \simeq 0.15$ from previous section

$$\Rightarrow X_p^{(0)} X_n^{(0)} \simeq 0.13$$

and G_{np} from (3.20)

Then, from (3.39) we get

$$2.9 \cdot 10^{-6} \left(\frac{\eta}{\eta_0}\right)^2 z_c^{-17/6}\, e^{-1.44\, z_c^{1/3}}\, e^{z_c} \simeq 1 \qquad (3.40)$$

Solution: $\quad z_c = 26 \quad$ for $\eta = \eta_0$

$$\Rightarrow T_{\gamma,c} = \frac{\varepsilon_D}{26} = 0.086 \text{ MeV} \qquad (3.41)$$

From (3.10)

$$t = \left(\frac{45}{16\pi^3 g_{*eff}}\right)^{\frac{1}{2}} \left(\frac{11}{4}\right)^{\frac{2}{3}} \frac{M_P}{T_\gamma^2} + \underbrace{t_0}_{2s}$$

using (3.41) we get

$$t_c = 180 \text{ s} \qquad (3.42)$$

$$\Rightarrow \quad X(T \simeq 0) = X(t = t_c) = \exp\left(-\frac{180}{896}\right) \bar{X}(y = \infty)$$

$$= 0.818 \cdot 0.151 = 0.123$$

The ^4He mass fraction

$$Y_4 = 2X(t = t_c) = 0.247$$

With some simplifying assumptions, we have computed Y_4, the ^4He mass fraction.

BBN also produces D, ^3He, ^7Li with much lower abundances.

I recommend taking a look at astro-ph/0303073 by Mukhanov for the analytic estimates of these less abundant elements.

(see fig. 2 in Mukhanov and fig. 1 in PDG review)

COMPARISON OF THEORY WITH OBSERVATIONS

Challenge: the Universe is quite different now compared to BBN time; galaxies and stars have formed. Primordial matter is processed in stellar thermonuclear reactions occurring in the recent Universe: some of the primordial nuclei are transformed into heavier elements, others are destroyed by hard photons emitted in star formation.

Key: look at astrophysical sites with low metallicity (low abundance of heavy elements = "metals").

For Helium 4 a good target is metal-poor extragalactic regions, like blue compact galaxies, generally found at low redshift.

The observational value quoted by the PDG is

$$Y_p = 0.245 \pm 0.003$$

primordial

to be compared to Y_4 calculated analytically (or numerically).

For considerations on the other elements refer to the PDG review.