3) EXPANDING UNIVERSE AND DISTRIBUTION FUNCTIONS

In the first part we will shortly review the flat FLRW (as motivated by data) and how momentum is defined and conserved.

The conventions we will be using are

$\hbar = c = k_B = 1$ and $\underset{\text{FLAT MINKOWSKI}}{\eta_{\mu\nu}} = \text{diag}(-1, \underbrace{1, 1, 1}_{\text{"mostly positive" metric}})$

• In a homogeneous & isotropic universe, we have

$$g_{\mu\nu} = \text{diag}(-1, a^2, a^2, a^2) \quad , \quad a = a(t)$$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu = -dt^2 + a(t)^2 \left(\frac{dr^2}{1 - \underset{\downarrow\,0}{k} r^2} + r^2 \underbrace{d\Omega^2}_{d\theta^2 + \sin^2\theta \, d\varphi^2} \right)$$

• We will take $k = 0$ from here on.

To understand the behaviour of physical momenta for massive/massless particles, we need to look at the geodesic. In the absence of external force, this equation tells us ~~show~~ how E and p behave.

No forces, Newton : $\vec{F} = 0 \Rightarrow m\vec{a} = 0$, $\frac{d^2x^i}{dt^2} = \ddot{x}^i = 0$.

For general coordinates, this becomes $x^i \to x'^i$

$$\frac{dx^i}{dt} = \frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}, \quad \ddot{x}^i = \frac{d}{dt}\left(\frac{\partial x^i}{\partial x'^j}\frac{dx'^j}{dt}\right)$$

$$= \frac{\partial x^i}{\partial x'^j}\frac{d^2x'^j}{dt^2} + \frac{\partial^2 x^i}{\partial x'^j \partial x'^k}\dot{x}'^i \dot{x}'^k = 0$$

$$\ddot{x}^l + \Gamma^l_{kj}\dot{x}^k\dot{x}^j = 0$$

$$\Gamma^l_{kj} = \left(\frac{\partial x}{\partial x'}\right)^{-1\,l}_{\quad i}\frac{\partial^2 x^i}{\partial x'^j \partial x'^k} = \text{Christoffel symbol}$$

Instead of time we use the affine parameter

$$ds = \sqrt{g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\,d\lambda$$

GEODESIC : $\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0$

$\Downarrow$ FLRW $g_{\mu\nu} = \mathrm{diag}(-1, a^2, a^2, a^2)$

$g^{\mu\nu} = \mathrm{diag}(-1, a^{-2}, a^{-2}, a^{-2})$

$$\underbrace{\Gamma^\mu_{\alpha\beta}}_{\text{sym}} = \frac{1}{2}g^{\mu\nu}\left(\frac{\partial g_{\alpha\nu}}{\partial x^\beta} + \frac{\partial g_{\beta\nu}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\nu}\right) \quad \text{(see Weinberg)}$$

We only get two components : $\Gamma^0_{ij} = \delta_{ij}\dot{a}a$, $\Gamma^i_{0j} = \Gamma^i_{j0} = \delta_{ij}\frac{\dot{a}}{a}$

$\dot{a} = \frac{da}{dt}$

MOMENTUM

comoving momentum

$$P^\mu = \frac{dx^\mu}{d\lambda} = (E, \vec{P}) \quad , \quad P^0 = E = \frac{dx^0}{d\lambda} = \frac{dt}{d\lambda}$$

Let's see how the momentum of a non-interacting particle behaves in an expanding universe. The geodesic equation is given by

$$P^\mu \nabla_\mu P^\nu = 0 \quad \text{or} \quad \frac{dP^\mu}{d\lambda} + \Gamma^\mu_{\alpha\beta} P^\alpha P^\beta = 0$$

The 0-component requires $\mu = 0$ & $\alpha, \beta \neq 0$, $\Gamma^0_{ij} = \delta_{ij} \dot{a} a$

$$\frac{dP^0}{d\lambda} = -\dot{a} a P^{i2} \quad \text{or} \quad E\frac{dE}{dt} = -\dot{a}\cancel{a}\frac{E^2 - m^2}{a^{\cancel{2}}}$$

Here, we took: $\frac{d}{d\lambda} = \frac{\partial x^0}{\partial \lambda}\frac{d}{dx^0} = E\frac{d}{dt}$

and : $g_{\mu\nu} P^\mu P^\nu = -E^2 + \underbrace{a^2 P^{i2}}_{= p^{i2}} = -m^2$

p^i, $p^i = a P^i$

physical   comoving

The on-shell relation for massive particles is preserved: $E^2 - p^2 = m^2$

$$\Rightarrow E\frac{dE}{dt} = -\frac{\dot{a}}{a}(E^2 - m^2).$$

- The energy scaling can be extended smoothly from $m \neq 0$ to $m = 0$, valid for photons and (sometimes) neutrinos.

MASSLESS
$m = 0$

$$\frac{dE}{dt} = -\frac{\dot a}{a} E \quad \Rightarrow \quad \frac{dE}{E} = -\frac{da}{a}$$

$E \propto a^{-1}$ as expected.

- Next, we move on to the energy density ρ and pressure P. We will define these two microscopically later on, but here we focus on their inter-related behavior in an expanding universe. It follows from

$$\nabla_\mu T^\mu{}_\nu = 0 \, , \qquad T^\mu{}_\nu = \begin{pmatrix} -\rho & & & \\ & P & & \\ & & P & \\ & & & P \end{pmatrix}$$

$$= \frac{\partial T^\mu{}_\nu}{\partial x^\mu} + \Gamma^\mu_{\alpha\mu} T^\alpha{}_\nu - \Gamma^\alpha_{\nu\mu} T^\mu{}_\alpha$$

off-diagonal $= 0$, $= -\rho$

$$\nu = 0 : \quad \nabla_\mu T^\mu{}_0 = \frac{\partial T^0{}_0}{\partial x^0} + \Gamma^{\overset{i}{\mu}}_{0\underset{i}{\mu}} T^0{}_0 - \Gamma^{\overset{j}{\alpha}}_{0\underset{i}{\mu}} T^i{}_\alpha = 0$$

α becomes j

$$-\frac{d\rho}{dt} = \delta^i_i \frac{\dot a}{a} \rho - \delta^i_i \frac{\dot a}{a} P = 0$$

OR : $$\dot\rho + 3 \frac{\dot a}{a} (\rho + P) = 0.$$

$$\dot\rho + 3\frac{\dot a}{a}(\rho + P) = 0$$

$$\Rightarrow \frac{1}{a^3}\frac{d}{dt}(\rho\, a^3) = \dot\rho + 3\frac{\dot a}{a}\rho = -3\frac{\dot a}{a}P.$$

- We will quantify the relation (microscopically) between P and ρ a bit later, but the result is

MATTER : $P_m \simeq 0$ is pressureless

RADIATION : $P_\gamma = \frac{1}{3}\rho_\gamma$

MATTER : $\frac{d}{dt}(\rho_m a^3) = 0 \Rightarrow \rho_m \propto a^{-3}$

RADIATION : $\dot\rho_\gamma + 3\frac{\dot a}{a}\rho_\gamma = -\cancel{3}\frac{\dot a}{a}\frac{1}{\cancel{3}}\rho_\gamma$ OR : $\dot\rho_\gamma + 4\frac{\dot a}{a}\rho_\gamma = 0$

which implies $\frac{\dot\rho_\gamma}{\rho_\gamma} = -4\frac{\dot a}{a}$, $\frac{d\rho_\gamma}{\rho_\gamma} = -4\frac{da}{a}$,

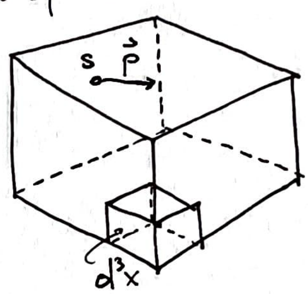
$\rho_\gamma \propto a^{-4}$, which holds for light (GWs) and very light particles (relativistic species with $m \ll T$).

In general, one defines an equation of state $P = w\rho$

$w = (0, 1/3, -1)$ for (m, γ, Λ): $\frac{\dot\rho}{\rho} = -\frac{\dot a}{a}3(1+w)$, $\rho \propto a^{-3(1+w)}$.

DISTRIBUTION FUNCTIONS

A given species s of particles in a TD system is described by a distribution function f_s. In broad generality, f_s is space, momentum and time-dependent. Here, we will define the number density $n_s(t)$ as $\frac{dN}{d^3x} = \frac{dN}{dV}$



For homogeneous and isotropic systems $n(x,t) = n(t)$

NUMBER DENSITY

$$\boxed{n_s(t) = g_s \int \frac{d^3p}{(2\pi)^3} f_s(p,t)}$$

- Normalization includes $(2\pi)^3$: $d^3x\, d^3p \leq (2\pi\hbar)^3$
- g_s counts all the internal d.o.f.s, such as the number of spins, color, generations, etc..
- Once we have the distribution functions, we can derive ρ, P, the entropy density s and perturbations of the energy momentum tensor T_μ^ν (beyond $\delta\rho$ δP).

ENERGY DENSITY

$$\boxed{\rho_s = g_s \int \frac{d^3p}{(2\pi)^3} f_s(p)\, E \,, \quad E^2 - p^2 = m^2}$$

For example, photons have: $\rho_\gamma = 2 \int \frac{d^3p}{(2\pi)^3} f_\gamma\, p$

$g_s = 2$ (two polarizations) $\qquad$ $E = p$ because $m_\gamma = 0$

For a NR particle: $E \sim m \Rightarrow \rho_m = g_s\, m \int \frac{d^3p}{(2\pi)^3} f_s = m\, n_s$.

PRESSURE

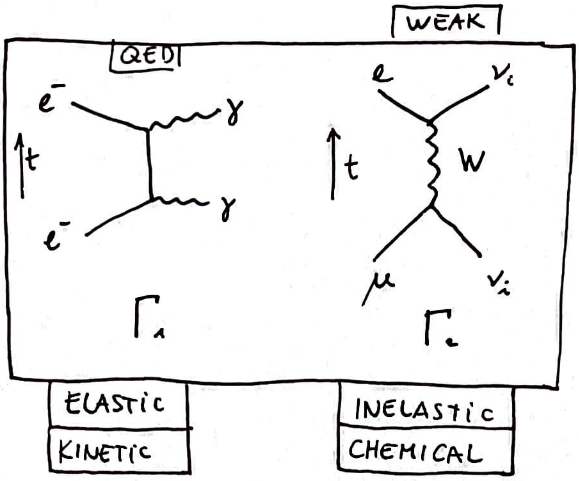
$$\boxed{P = g_s \int \frac{d^3p}{(2\pi)^3} f_s \frac{p^2}{3E}}$$

$$= g_s \int \frac{d^3p}{(2\pi)^3} f \frac{p^2}{3\sqrt{p^2+m^2}} \xrightarrow{m=0} \frac{1}{3} \underbrace{g_s \int \frac{d^3p}{(2\pi)^3} f_s\, p}_{\rho_s}$$

We see how $P = \frac{1}{3}\rho$ for relativistic species. For NR (matter) components; $p \ll m \Rightarrow \frac{p^2}{3m} < E \sim m$, i.e. $p^2 < 3m^2$, therefore $P_m \sim g_s \int_p f_s \frac{p^2}{3m} \ll \rho_m \sim g_s \int_p f_s m$, which justifies $P_m \simeq 0$.

- Before moving on to the entropy density, we need to define the notion of equilibrium and the associated temperature.

Equilibria and Temperature



When $\Gamma \gtrsim H$, species will exchange momenta in Γ_1 and efficiently match their momenta

$\Downarrow$

kinetic equilibrium

We will use the photon temperature as a proxy for $T = T_\gamma$.

In a homogeneous, isotropic system without a prefered direction, f_s will depend only on p^2 and T. We will see how in some cases different species can have $T_s \neq T_\gamma$.

Chemical equilibrium

Some interactions act between different species, such as Γ_2. When these are "active", i.e. $\Gamma_2 \gtrsim H$, relative distributions will start to match and we reach chemical eq.

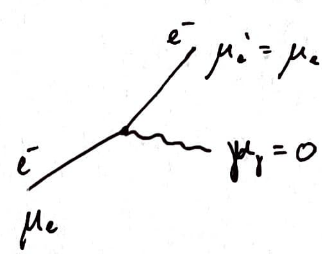
From TD: $\quad dE = -p\,dV + T\,dS + \mu\,dN \quad / : dV$

$$\rho = -P + T s + \mu n$$

energy density (ρ), pressure (P), temperature (T), entropy density (s), chemical potential (μ), number density (n)

We will discuss μ in more detail when we'll talk about Baryogenesis. Here we state that photons carry no μ, i.e.

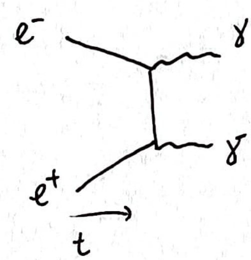
$$\mu_\gamma = 0$$



In chemical equilibrium, $\sum \mu_i = \text{const.}$ so $\mu_{\text{INITIAL}} = \mu_{\text{FINAL}}$.

When we radiate a photon, $\mu_e = \mu_e + \mu_\gamma \Rightarrow \mu_\gamma = 0$, in other words we don't have to pay an extra price in $dE = \mu\, dN$, which is exactly what μ measures.

This also implies that $\mu_f = \mu_{\bar f}$, from annihilation, say



$$\mu_{e^-} + \mu_{e^+} = 2\mu_\gamma = 0 \Rightarrow \mu_{e^-} = -\mu_{e^+}.$$

As we shall see $\mu \neq 0$ measures or implies $n_f \neq n_{\bar f}$, which also confirms $\mu = 0$ for particles that don't have anti-particles, such as photons, Z, Majorana states.

For each conserved quantity, we get a sum rule on μ_i that will be enforced by interactions in equilibrium. E.g. in QED, charge is conserved $Q_{tot} = 0 \Rightarrow \gamma^* \to e^+ e^- \Rightarrow \mu_\gamma = 0 = \mu_{\bar e} + \mu_e$ or $\mu_{e^+} + \mu_{e^-} = 0$.

EQUILIBRIUM DISTRIBUTIONS

We had $f_s(x^i, p^i, t) \xrightarrow{\text{hom.}} f_s(p^2, t)$; now we will further narrow down f to two possibilities, which apply <u>only</u> in kinetic (and chemical) equilibrium. Depending on the spin, we have ($k_B = 1$, $\hbar = c = 1$)

FERMI-DIRAC: f^{eq}_{FD}

EQUILIBRIUM DISTRIBUTIONS

$$f_s^{eq} = \frac{1}{e^{(E-\mu_s)/T} \pm 1}, \quad \begin{array}{l} + \ldots \text{FERMIONS} \\ - \ldots \text{BOSONS} \end{array}$$

BOSE-EINSTEIN: f^{eq}_{BE}

T is the common temperature (e.g. all e^- will have the same T and μ_s is the common chemical potential for a given species in the thermal bath). The distinction between fermions and bosons is relevant at low energies $E-\mu \lesssim T$ (think Bose-Einstein condensation vs. Fermi levels & Pauli blocking). Conversely, when $E-\mu \gg T$, the particles are moving fast and in this part of the distribution,

$$f_s^{eq} \sim e^{-(E-\mu_s)/T} = f^{eq}_{MB}$$

↳ MAXWELL-BOLTZMANN universal distribution

For example, photons in equilibrium will follow

$$f_\gamma^{eq} = \frac{1}{e^{p/T}-1}, \quad \text{because } \mu_\gamma = 0,\ E = p \text{ and } T_\gamma = T.$$

The high-energy tail of the distribution $p \gg T$ will be "Boltzmann" suppressed, because we transfer to $f_\gamma^{eq} \to f_{\gamma MB}^{eq} \sim e^{-p/T}$ and fewer events will happen there.

In the absence of μ (like for photons), we can derive a very simple expression for the entropy density s, which is given by

$$s = \frac{\partial P}{\partial T} = \frac{\rho + P}{T}. \qquad [\text{HOMEWORK}]$$

This relation will be very useful to track the entropy density, as well as temperature of the photons throghout the evolution (expansion) of the universe.

ENTROPY CONSERVATION

We will now combine the TD relation for s with the continuity equation from FLRW's $\nabla T = 0$ to show that sa^3 is conserved.

$$\frac{\partial P}{\partial T} = \frac{\rho + P}{T} \qquad \& \qquad \dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$$

$$\frac{1}{a^3}\frac{d}{dt}\left((\rho+P)a^3\right) = \dot{\rho} + \dot{P} + (\rho+P)3\frac{\dot{a}}{a} = \dot{P} = \frac{dP}{dt}$$

$$= \frac{\partial P}{\partial T}\frac{dT}{dt} = \frac{\rho+P}{T}\frac{dT}{dt} = \frac{\rho+P}{T}\dot{T}$$

Ok, now we have:

$$a^{-3}T\frac{d}{dt}(sa^3) = \frac{d}{dt}\left(\frac{\rho+P}{T}a^3\right) = a^{-3}\cancel{T}\frac{d}{dt}\left((\rho+P)a^3\right)\cancel{\frac{1}{T}}$$

$$+ \cancel{a^{-3}}T(\rho+P)\cancel{a^3}\cdot\left(-\frac{1}{T^2}\right)\dot{T}$$

$$= \frac{\rho+P}{T}\dot{T} - \frac{1}{T}(\rho+P)\dot{T} = \underline{\underline{0}}$$

$$\Downarrow$$

$$\boxed{\frac{d}{dt}(sa^3) = 0}, \quad sa^3 = \text{const.}$$

ENTROPY DENSITY scales as a^{-3} and sa^3 is conserved

With these generalities, we can evaluate s an ρ.

ENTROPY AND ENERGY DENSITIES

- For relativistic species $m \sim 0$ and $P \sim \frac{\rho}{3}$, we have $s = \frac{\rho + P}{T} = \frac{4}{3}\frac{\rho}{T}$.

- For NR particles with $p \ll m$, we have

$$s = \frac{1}{T}\int_p \left(E + \frac{p^2}{3E}\right) f_s(p,T) \quad , \quad \int_p = \int \frac{d^3p}{(2\pi)^3}$$

PHOTONS

Let us evaluate these integrals for the case of photons with $m_\gamma = 0$, $g_\gamma = 2$, $\mu_\gamma = 0$ (BOSE-EINSTEIN)

$$\rho_\gamma = g_\gamma \int_p E f_{BE} = 2\int \frac{d^3p}{(2\pi)^3}\, p\, \frac{1}{e^{p/T}-1} \quad , \quad d^3p = 4\pi p^2 dp$$

$$= \frac{\cancel{8\pi}}{\cancel{8}\pi^{\cancel{3}2}} \int_0^\infty dp \frac{p^3}{e^{p/T}-1} = \frac{T^4}{\pi^2} \underbrace{\int_0^\infty \frac{x^3 dx}{e^x - 1}}_{\frac{\pi^4}{15}} \quad , \quad x = \frac{p}{T}, \quad dx = \frac{dp}{T}$$

$$= \frac{\pi^2}{15} T^4 .$$

$$s_\gamma = \frac{4}{3}\frac{\rho}{T} = \frac{4\pi^2}{45} T^3 .$$

$$\boxed{\begin{array}{c} \rho_\gamma \propto T^4 \\ s_\gamma \propto T^3 \end{array}}$$

Thermal bath ρ & s

In general, we define g_{*s} and $g_*(T)$, such that

$$s = \frac{2\pi^2}{45} g_{*s}(T)\, T^3 = \frac{1}{T}\sum_i g_i \int_p \left(E + \frac{p^2}{3E}\right) f_i$$

sum over all species (fermions, bosons) in equilibrium.

Then $g_{*s} T^3 a^3 = \text{const.}$

and if g_{*s} does not change, $T \propto a^{-1}$.

Equivalently, we have:

$$\rho = \frac{\pi^2}{30} g_*(T)\, T^4 = \sum_i g_i \int_p E f_i$$

DEFINITIONS $g_*(T)$, $g_{*s}(T)$

$$g_{*s} = \frac{45}{2\pi^2 T^4} \sum_i g_i \int_p \left(E + \frac{p^2}{3E}\right) f_i$$

$$g_* = \frac{30}{\pi^2 T^4} \sum_i g_i \int_p E f_i .$$

Note that when $T \gg m_i$, the two g_{*s} and g_* agree.

When T drops, $p \ll m$, $E \sim m$ (NR limit), the f_i becomes Boltzmann suppressed and g_* drops.

In the relativistic limit $E > m$, we can simply count $g_* = g_i$ for bosons and $g_* = \frac{7}{8} g_i$ for fermions, because:

$$\underbrace{\int_0^\infty \frac{x^3}{e^x+1}}_{\text{from } f_{FD}} = \int_0^\infty \left(\frac{x^3}{e^x-1} - \frac{2x^3}{e^{2x}-1}\right) dx = \frac{\pi^4}{15}\left(1 - \frac{1}{2^3}\right) = \frac{7}{8}\frac{\pi^4}{15}$$

Note that entropy is also separately conserved for each species, one can write

$$s \simeq \sum_{s=\text{species}} \left(g_{s=\text{bosons}} \left(\frac{T_s}{T}\right)^3 + \frac{7}{8} g_{s=\text{fermions}} \left(\frac{T_s}{T}\right)^3 \right) T^3$$

Cosmic Budget

- Now that we have explicit expressions for ρ & s, we can evaluate Ω_s for different species with the actual data.
  $\searrow$ density parameters

$$\Omega_s = \frac{\rho_s}{\rho_{cr}}, \quad \rho_{cr} = \frac{3H_0^2}{8\pi G} \sim \text{few } \frac{\text{GeV}}{m^3}, \quad \omega_s = \Omega_s h^2, \; h \sim 0.7$$

-

TYPE	SPECIES	SCALING
MATTER	CDM ... cold dark matter b ... baryons (and e^-), luminous	NR matter $\rho_m \sim mn \propto a^{-3}$
RADIATION	γ ... photons ν ... neutrinos	$\rho_r \propto a^{-4}$, $E \propto a^{-1}$
Λ, DE	Cosmological constant, dark energy	ρ = const.