Note that entropy is also separately conserved for each species, one can write

$$s \simeq \sum_{s=\text{species}} \left(g_{s=\text{bosons}} \left(\frac{T_s}{T}\right)^3 + \frac{7}{8} g_{s=\text{fermions}} \left(\frac{T_s}{T}\right)^3 \right) T^3$$

Cosmic Budget

- Now that we have explicit expressions for ρ & s, we can evaluate Ω_s for different species with the actual data.
$\searrow$ density parameters

$$\Omega_s = \frac{\rho_s}{\rho_{cr}}, \quad \rho_{cr} = \frac{3H_0^2}{8\pi G} \sim \text{few } \frac{\text{GeV}}{\text{m}^3}, \quad \omega_s = \Omega_s h^2, \; h \sim 0.7$$

-

TYPE	SPECIES	SCALING
MATTER	CDM ... cold dark matter b ... baryons (and e^-), luminous	NR matter $\rho_m \sim mn \propto a^{-3}$
RADIATION	γ ... photons ν ... neutrinos	$\rho_r \propto a^{-4}$, $E \propto a^{-1}$
Λ, DE	Cosmological constant, dark energy	ρ = const.

Ω_γ ... PHOTONS

$k_B \sim 10^{-4}\,eV/K$

Today, we measure: $T_0 = T_\gamma = 2.7K \sim 3\cdot 10^{-4}\,eV$,

Therefore: $$\rho_\gamma = \frac{\pi^2}{15} T_0^4 = 2\cdot 10^{-15}\,eV^4,$$

$$\rho_{cr} = \frac{3H_0^2}{8\pi G} = 10^{-5} h^2 \frac{GeV}{cm^3} \simeq \text{few protons}/m^3,$$

$$\Rightarrow \omega_\gamma = \Omega_\gamma h^2 = \frac{\rho_\gamma h^2}{\rho_{cr}} \simeq 2.5\cdot 10^{-5}.$$

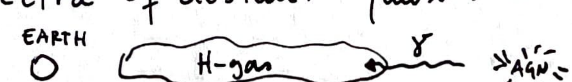
We see that Ω_γ today forms a very small (negligible) part of the total $\Omega_{tot}=1$ energy density budget.

Ω_b ... BARYONS

This refers to **luminous matter** (made of p, n, e^-) and includes stars and intergalactic (H) gas. The contribution from e^- ($m_e \sim 0.5\,MeV$) is negligible wrt. p, n; $m_p \sim m_n \sim GeV$,

$\omega_b = \Omega_b h^2 = 0.022$ from:

* stars and diffuse gas,
* spectra of distant quasars (AGNs),



* CMB spectrum C_ℓ, BAOs,
* BBN input $\eta_B = \frac{n_B}{n_\gamma} = 6\cdot 10^{-10}$

$$\Downarrow$$

$\Omega_b \lesssim 5\%$

Photons – number density

With $T_0 = 2.73$ K, we can evaluate n_γ today.

$$n_\gamma = 2 \int_p f_{BE} = \frac{8\pi}{8\pi^3} T^3 \underbrace{\int_0^\infty \frac{x^2 dx}{e^x - 1}}_{2\zeta(3)} = \underbrace{\frac{2\zeta(3)}{\pi^2}}_{1.2} T^3$$

$$= 410 \text{ cm}^{-3}$$

- For nearly massless fermions (counted as radiation), we get:

$$n_f = g_f \int_p f_{FD} = \frac{g_f \cancel{4\pi} T^3}{2\cancel{8}\pi^{\cancel{3}2}} \int_0^\infty x^2 dx \left(\frac{1}{e^x - 1} - \frac{2}{e^{2x} - 1} \right)$$

$$\frac{2}{e^{2x}-1} \text{ term: } = \frac{1}{4} \int_0^\infty \frac{(2x)^2\, 2dx}{e^{2x} - 1}$$

$$= g_f \frac{\cancel{2}\zeta(3)}{\cancel{2}\pi^2} \left(1 - \frac{1}{4}\right) T^3 = \frac{3}{4} \frac{\zeta(3)}{\pi^2} T^3.$$

Ω_{DM} ... DARK MATTER

DM behaves like a NR = massive component of the universe.
Let's have a look at its number density, energy and pressure, when $p \ll m$:

$$n_m \simeq g \int_p \frac{1}{e^{m/T} e^{p^2/(2mT)} \pm 1} \quad , \quad E = \sqrt{m^2 + p^2} \simeq m\sqrt{1 + \left(\frac{p}{m}\right)^2} \simeq m + \frac{1}{2}\frac{p^2}{m}$$

$$\frac{p^2}{2mT} = x \quad , \quad p\,dp = mT\,dx$$

$$n_m \simeq g\, e^{-m/T} \frac{\cancel{4\pi}}{\cancel{8\pi^3}2} \int_0^\infty dp\, p^2\, e^{-p^2/(2mT)} \qquad \text{(MB limit)}$$

$$= g\, \frac{e^{-m/T}}{2\pi^2} \sqrt{2mT}\, mT \underbrace{\int_0^\infty dx\, x^{1/2} e^{-x}}_{\Gamma(3/2) = \frac{\sqrt{\pi}}{2}}$$

$$= g \left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} .$$

scales as $T^{3/2}$ — Boltzmann suppression in eq.

$$\rho_m = g\int_p E f \simeq m g \int_p f = m n_m \Rightarrow \boxed{\rho_m \simeq m n_m}$$

- Finally, let's see how P_m comes out from statistical mechanics.

DM Pressure

$$P_m = g\int_p \frac{p^2}{3E} f \simeq g e^{-m/T} \frac{4\pi}{8\pi^3}\int_0^\infty \frac{p^4}{3m} e^{-\frac{p^2}{2mT}} dp$$

$$= \frac{g}{2\pi^2} e^{-m/T} (2mT)^{3/2} T \underbrace{\int_0^\infty x^{3/2} e^{-x} dx}_{\Gamma(5/2) = \frac{3}{2}\Gamma(\frac{3}{2})}$$

$$= T g \frac{(mT)^{3/2}}{\sqrt{2}\pi^2} \Gamma(3/2) = T n_m .$$

- This is suppressed wrt. $\rho_m \simeq m n_m > P_m = T n_m$, because $m > T$. [in order to be cold (NR).]

DM needs to be stable $\tau > \tau_u$, dark (non-luminous) with suppressed EM couplings and massive (gravitates).

- PDG '21

$\omega_{DM} = \Omega_{DM} h^2 \sim 0.12$,

$\Omega_{DM} = 0.265 \sim 1/4$.

Required by:

- CMB, explains C_ℓ spectrum,
- BBN, gives enough ω_m,
- standard candles,
- rotational curves.

Ω_ν ... NEUTRINOS $\qquad s = \frac{1}{2}$, FERMI-DIRAC

- In the plasma of the early universe above GeV, neutrinos are coupled to electrons and hadrons. At a certain $T_\nu^{dec} \simeq 1.5\,\mathrm{MeV}$ they decouple (freeze-out) and start to cool down with the expanding universe. Today they form the cosmic neutrino background CνB with a temperature $T_{\nu 0}$, which we will calculate shortly. CνB is yet to be observed.

- We know from colliders LEP/LHC (by measuring the total decay width of Z, $\Gamma_Z^{tot.}$) that there are three light generations of neutrinos. Because $Q(\nu)=0$, they <u>only couple chirally</u>, with <u>one</u> - left handed component. The RH component does not exist in the SM $\Rightarrow m_\nu = 0$.

IN SUMMARY: $\quad \nu_L, \bar{\nu}_L \times 3$ generations:

$$g_\nu = 2 \cdot n_g = 6.$$

- NEUTRINO OSCILLATION experiments have measured:

$$\Delta m^2_{12} \simeq 10^{-3}\,eV^2 \quad , \quad \Delta m^2_{23} \sim 10^{-5}\,eV^2 .$$

- We do not know the overall mass scale, or $m_{\nu_{min}}$, but nuclear β decays, the endpoint part of the tritium spectrum, measured by KATRIN, set a limit $m_{\nu_e} < 0.8\,eV$

- Also, if neutrinos are Majorana particles, $0\nu 2\beta$ measurements limit: $m_\nu^{ee} = \sum_{i=1}^{3} V_{ei}^2\, m_{\nu_i} \lesssim 10^{-2}\,eV$. KamLAND-Zen

- Therefore, for the most part of thermal history of the universe, neutrinos behave relativistically

$$T \gg m_\nu \quad , \quad E_\nu \sim p_\nu \gg m_\nu .$$

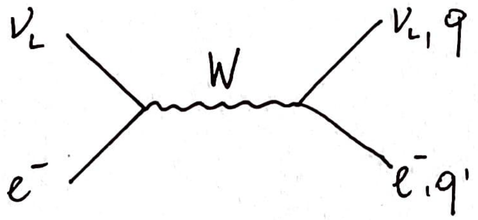
We can calculate their energy density: fermions

$$\rho_\nu = g_\nu \int_p \frac{p}{e^{p/T}+1} = 6 \cdot \underbrace{\frac{7}{8}}_{=1-\frac{1}{2^3}} \cdot \frac{\pi^2}{30}\, T_\nu^4 = 3 \cdot \frac{7}{8}\left(\frac{T_\nu}{T_\gamma}\right)^4 \rho_\gamma$$

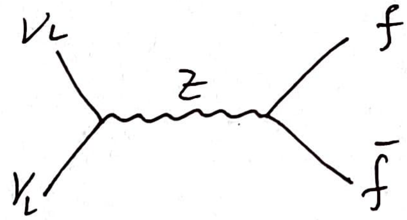
no. of generations

$$\rho_{\nu_0} \simeq 3 \cdot \frac{7}{8}\left(\frac{T_{\nu_0}}{T_0}\right)^4 \rho_{\gamma_0} . \quad \text{(if they were massless)}$$

- As we discussed previously, weak interactions are in equilibrium when $T \gtrsim \alpha_w^2 M_{pe}$ via:



and



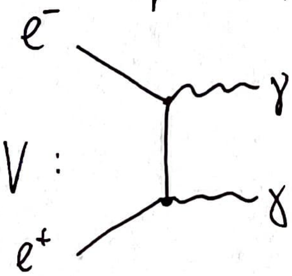
So, at high T, weak interactions are in equil., but as we shall see, the interaction rate drops faster than H in radiation domination, therefore:

$$\frac{\Gamma}{H} \sim 1 \quad \text{at} \quad T_{dec} \sim 1.5 \text{ MeV}.$$

NEUTRINO FREEZE-OUT

- After freeze-out, neutrinos free-stream and cool down due to the expansion and remain relativistic until $T \lesssim 0.1\,eV$ or so, until today with $T_\gamma \sim 5 \cdot 10^{-4} eV$

- Between freeze-out and today, we have ν_L, e^+, e^-, γ however at one point ($@\, T_\gamma \simeq m_e$), e^+e^- start to annihilate (and not recuperate) such that the photons heat up via:



$T_{ann} \sim$ MeV :

$T_\gamma \sim m_e \sim 0.5\,$MeV

This heats up γ, but not ν, thus: $T_{\gamma 0} > T_{\nu 0}$.

today

NEUTRINO COOLING

Let us calculate by how much ν's are colder than γ's

ENTROPY CONSERVATION : $\frac{d}{dt}(sa^3) = 0,$

BOSONS / FERMIONS : $s_s = \frac{2\pi^2}{45} T^3 \begin{cases} g_B, & \text{BOSONS}, \\ \frac{7}{8} g_F, & \text{FERMIONS}, \end{cases}$

Now we simply look at s_1 before and s_2 after annihilation and require:

$$s_1 a_1^3 = s_2 a_2^3,$$

before : $s_1 = s_{e^\pm} + s_\nu + s_\gamma \Big|_{T_1} = \frac{2\pi^2}{45} T_1^3 \Big(\underbrace{\tfrac{7}{8} \cdot 4}_{e^\pm,\ 2\text{ helicities}} + \tfrac{7}{8} \cdot \underbrace{6}_{2\text{ helicities},\ 3\text{ generations}} + 2 \Big)$

$$= \frac{43}{90} \pi^2 T_1^3 .$$

After decoupling, neutrinos have done nothing but cooled down, as they continue to do:

$$a_1 T_1 = a_2 T_\nu .$$

($N = \text{const}$, $n \propto a^{-3}$, $T_\nu \propto a^{-1}$)
$\parallel$
$na^3 = \text{const}$

But photons heat up to T_γ

$$s_2 = \frac{2\pi^2}{45} \left(2 T_\gamma^3 + \frac{7}{8} 6 T_\nu^3 \right).$$

Putting it all together, we have from $s_1 a_1^3 = s_2 a_2^3$

$$\frac{43}{2\,\cancel{90}}\cancel{\pi^2}(a_1 T_1)^3 = \frac{2\cancel{\pi^2}}{\cancel{45}}\left(2(a_2 T_\gamma)^3 + \frac{21}{4}(a_2 T_\nu)^3\right)$$

$$(a_1 T_1)^3 = (a_2 T_\nu)^3 \quad ; \quad a_2 \text{ cancels out !}$$

$$22\,\cancel{43}\,T_\nu^3 = 8\,T_\gamma^3 + \cancel{21\,T_\nu^3}$$

$$\boxed{T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma}$$

We have calculated that T_ν is reduced by $\left(\frac{4}{11}\right)^{1/3}$ wrt. the photon T_γ. After $e^\pm$ cooling the neutrinos, nothing much happens and both T_ν and T_γ scale as a^{-1} until today: $a_2 T_\gamma = a_0 T_{\gamma 0}$, $a_2 T_\nu = a_0 T_{\nu 0}$, with $a_0 = 1$,

so even today: $T_{\nu 0} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma 0} \lesssim 2$ K.

- Let us demonstrate why g_* is very useful and repeat the above calculation, focusing only on photons:

$$a_1 T_1 = a_2 T_\gamma \,, \quad g_*(T_1)(a_1 T_1)^3 = g_*(T_2)(a_2 T_\gamma)^3, \quad (a_1 T_1)^3 = (a_2 T_\nu)^3$$

$$T_\nu = \left(\frac{g_*(T_2)}{g_*(T_1)}\right)^{1/3} T_\gamma$$

$$\left.\begin{aligned} g_*(T_1) &= 2 + 4\cdot\frac{7}{8} = \frac{11}{2}, \\ g_*(T_2) &= 2. \end{aligned}\right\} \longrightarrow T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma \qquad \text{VOILÁ.}$$

NEUTRINO n_ν and ρ_ν TODAY

- For massless fermions, we would get

$$\rho_\nu = \underset{\#\text{ gen}}{3} \cdot \frac{7}{8} \cdot \frac{\pi^2}{15} T_\nu^4 = \frac{21}{8}\left(\frac{T_\nu}{T_\gamma}\right)^4 \rho_\gamma = \frac{21}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma$$

$$\simeq 0.7\, \rho_\gamma \quad \text{or} \quad \Omega_\nu \sim 1.7 \cdot 10^{-5} = \text{tiny}$$

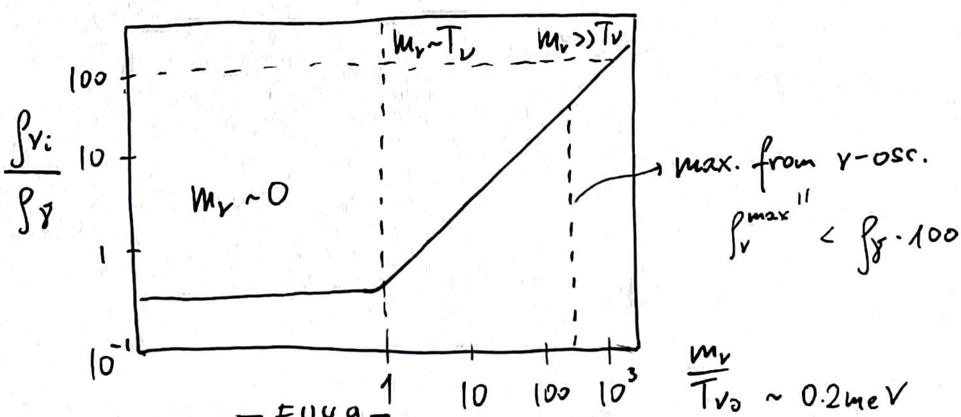
- However, from neutrino oscillations, at least two - if not all three $m_{\nu_i} > T_{\nu 0}$ and hence non-relativistic NR.

E.G. $m_1 = 0$, $m_2 \sim \sqrt{10^{-3}}\,eV \sim 30\,meV$, $m_3 \sim \sqrt{10^{-5}}\,eV \simeq 3\,meV$

WHILE $T_{\nu 0} = \left(\frac{4}{11}\right)^{1/3} T_{\gamma 0} \simeq 0.2\,meV$

In this case, we have: $\rho_{\nu_i} = 2 \int_p \frac{\sqrt{p^2 + m_i^2}}{e^{p/T_\nu} + 1}$

The distribution of ν's when they froze-out at $T_{dec} \sim 1.5\,MeV$ was relativistic. After that they simply free-stream.



The $\rho_{\nu i} / \rho_\gamma$ vs. m_ν / T_ν plot can be easily understood in the two limits

- $m_\nu \simeq 0$: $\rho_\nu = \frac{21}{8}\left(\frac{4}{11}\right)^{4/3} \rho_\gamma \sim 0.7 \rho_\gamma \rightarrow \frac{\rho_\nu}{\rho_\gamma} \simeq 0.7$

- $m_\nu > T_{\nu 0}$: Here, νs behave as non-relativistic matter

$$\rho_{\nu_i} = m_i n_{\nu_i} \quad \text{with} \quad n_{\nu_i} = 2 \int_p \frac{1}{e^{p/T_\nu}+1} \simeq \frac{3}{11} n_\gamma$$

$$= \frac{6\zeta(3)}{11\pi^2} T_\gamma^3$$

If all three are 'massive' $m_{\nu_i} \gtrsim T_{\nu 0}$ $\forall i = 1,2,3$, then

$$\rho_\nu = \sum_{i=1}^{3} m_{\nu_i} n_{\nu_i} = 3 \frac{6\zeta(3)}{11\pi^2} T_\gamma^3 \cdot \sum_{i=1}^{3} m_{\nu_i}$$

$$\Omega_\nu h^2 = \frac{\sum m_{\nu_i}}{94\,eV}$$

As we mentioned, KATRIN has $m_{\nu_e} \lesssim 0.8\,eV$, while constraints from large scale structures : $\sum_i m_{\nu_i} \lesssim 0.04\,eV$