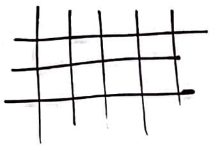
6) EINSTEIN AND BOLTZMANN EQUATIONS

GRAVITY



PARTICLES

e^-, γ, ν
$p, n, \ldots$

sets the metric — define the energy content

Let's review the Einstein equation on the gravity side

- $$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi G T_{\mu\nu} - \Lambda g_{\mu\nu}$$

The Ricci tensors and scalars are obtained from the FLRW metric: $g_{\mu\nu} = (-1, a^2, a^2, a^2)$, such that:

$$R_{\mu\nu} = \Gamma^{\beta}_{\mu\nu,\alpha} - \Gamma^{\alpha}_{\mu\alpha,\nu} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{\mu\nu} - \Gamma^{\alpha}_{\beta\nu}\Gamma^{\beta}_{\mu\alpha}, \quad \Gamma^{\beta}_{\mu\nu,\alpha} = \frac{\partial \Gamma^{\beta}_{\mu\nu}}{\partial x^{\alpha}}$$

Remember that $\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu}\left(\frac{\partial g_{\alpha\nu}}{\partial x^{\beta}} + \frac{\partial g_{\beta\nu}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\nu}}\right)$

and $\Gamma^{0}_{ij} = \dot{a} a \delta_{ij}$, $\Gamma^{i}_{0j} = \Gamma^{i}_{j0} = \frac{\dot{a}}{a}\delta_{ij}$.

Thus the components of $R_{\mu\nu}$ are:

$$\mu=\nu=0: \quad R_{00} = -\Gamma^{i}_{0i,0} - \Gamma^{i}_{j0}\Gamma^{j}_{0i} = -\frac{d}{dt}\Big(\overbrace{\delta_{ii}}^{3}\frac{\dot{a}}{a}\Big) - \left(\frac{\dot{a}}{a}\right)^2 \overbrace{\delta_{ij}\delta_{ij}}^{3}$$

$$= -3\frac{\ddot{a}}{a} + 3\frac{\dot{a}}{a^2}\dot{a} - 3\left(\frac{\dot{a}}{a}\right)^2 = -3\frac{\ddot{a}}{a}.$$

$$\mu=\nu=ij: \quad R_{ij} = \delta_{ij}\left(2\dot{a}^2 + \ddot{a}a\right)$$

The Ricci scalar is then easily obtained as:

$$R = g^{\mu\nu} R_{\mu\nu} = g^{00} R_{00} + g^{ij} R_{ij} = -1\, R_{00} + a^{-2} R_{ii}$$

inverse of $g_{\mu\nu} = (-1, a^2 \delta_{ij}) \Rightarrow g^{\mu\nu} = (-1, a^{-2}\delta_{ij})$

$$= -R_{00} + a^{-2} R_{ii}$$

$$= +3\frac{\ddot{a}}{a} + \frac{3}{a^2}\left(2\dot{a}^2 + \ddot{a}a\right) = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2\right)$$

* $T_\mu^\nu = (-\rho, P, P, P)$

* The CC term can be counted inside of $T_{\mu\nu}$

$$T^\Lambda_{\mu\nu} = -\frac{\Lambda}{8\pi G}\delta_{\mu\nu} \quad \text{or} \quad T^{\Lambda\,\mu}{}_{\nu} = -\frac{\Lambda}{8\pi G}\delta^\mu_\nu$$
$$= -\rho_\Lambda (1,1,1,1)$$

We have $\rho_\Lambda = -P_\Lambda$ or $w = -1$.

Dimensionally: $[\rho] = M^4$ and $[G] = M^{-2} \Rightarrow [\Lambda] = M^2$.

Fixing the indices to 00 and ii, we get the two FRIEDMANN EQUATIONS:

$$00: \quad R_{00} - \frac{1}{2} g_{00} R = -3\cancel{\frac{\ddot{a}}{a}} + \underbrace{\frac{1}{2}\cdot 6\left(\cancel{\frac{\ddot{a}}{a}} + \left(\frac{\dot{a}}{a}\right)^2\right)}_{3H^2,\ H = \frac{\dot{a}}{a}} = 8\pi G\, \overset{\rho}{\overset{\parallel}{T_{00}}}$$

$$H^2 = \frac{8\pi}{3} G\rho \qquad \text{(for } k=0\text{, FLAT universe)}$$

• We introduced $H_0^2 = \frac{8\pi G}{3}\rho_{cr}$, such that:

$$\left(\frac{H}{H_0}\right)^2 = \sum_{\text{species}} \Omega_s \, a^{-3(w+1)} \quad , \; w = \begin{cases} 0 \, , & m \\ \frac{1}{3} \, , & r \\ -1 \, , & \Lambda \end{cases}$$

The 2nd Friedmann equation comes from $\mu\nu = ij$ or equivalently from the 1st and the continuity equation

• $\dot\rho + 3\frac{\dot a}{a}(\rho + P) = 0$ & $\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi}{3}G\rho$, such that:

Namely: $\left(\frac{\dot a}{a}\right)^2 = \frac{8\pi G}{3}\rho \quad /\,\dot{}$

$$\cancel{2}\cancel{\frac{\dot a}{a}}\left(\frac{\ddot a}{a} - \left(\frac{\dot a}{a}\right)^2\right) = \frac{\cancel{8}4\pi G}{3}\left(-3\cancel{\frac{\dot a}{a}}(\rho + P)\right)$$

$$\frac{\ddot a}{a} = \frac{8\pi}{3}G\rho - \frac{4\pi}{3}G(3\rho + 3P) = -\frac{4\pi}{3}G(\rho + 3P) \qquad \boxed{\text{2nd FRIEDMANN}}$$

OR, MORE DIRECTLY:

$$R_{ij} - \frac{1}{2}R g_{ij} = 8\pi G\, T_{ij} \quad , \quad T_{ij} = g_{ik}T^k_j = a^2 P\,\delta_{ij}$$

$$\delta_{ij}\left(2\dot a^2 + \ddot a a - \cancel{\tfrac{1}{2}}\cancel{6}^{\,3}\left(\frac{\ddot a}{a} + \frac{\dot a^2}{a^2}\right)a^2 = 8\pi G a^2 P\right)$$

$$-\cancel{2}\ddot a a - \frac{1}{2}\dot a^2 = \cancel{8}4\pi G a^2 P \quad /: a^2$$

$$\frac{\ddot a}{a} = -\frac{1}{2}H^2 - 4\pi G P = -\frac{4\pi G}{3}(\rho + 3P) \quad \text{it's the same}$$

• The EINSTEIN equations "contain" the continuity eq. $\nabla_\mu T^\mu_0 = 0$.

sum over all sources / species

k=0

Short summary :

1st : $\left(\frac{\dot{a}}{a}\right)^2 = H^2 = \frac{8\pi G}{3}\rho$,

2nd : $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3P)$,

Continuity : $\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + P) = 0$.

LIOUVILLE - BOLTZMANN EQUATION

The central equation that describes particles in the early universe concerns the evolution of the distribution function for each species, i.e. $f_s(\vec{x}, \vec{p}, t)$.

In the absence of sources, or **drains**, and for **stable** species, the total number in a given phase space volume is: $dN = f(x,p,t)\, d^3x \,\frac{d^3p}{(2\pi)^3}$,

$$\frac{dN}{dt} = 0 = \frac{\partial N}{\partial t} + \frac{\partial N}{\partial x}\dot{x} + \frac{\partial N}{\partial p}\dot{p} = 0,$$

OR : $\frac{\partial f}{\partial t} - \frac{\partial f}{\partial x}\dot{x} + \frac{\partial f}{\partial p}\dot{p} = 0$.

FOR EXAMPLE, the Harmonic Oscillator gives:

$$H = \frac{p^2}{2m} + \frac{1}{2} k x^2$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m}, \quad \dot{p} = -\frac{\partial H}{\partial x} = -kx \qquad \begin{aligned}(F &= -\nabla V \\ &= -kx \\ &= ma \\ &= m\dot{p})\end{aligned}$$

Then the Liouville operator becomes

$$\partial_t f + \partial_x f \frac{p}{m} + \partial_p f (-kx) = 0.$$

For a much more in-depth reading, see Binney, Tremaine: Galactic Dynamics (2008), p. 561. (or Weinberg's: Astrophysics (2021))

A <u>static</u> equilibrium distribution satisfies $\frac{\partial f}{\partial t} = 0$, this is true if $f = f(E)$ only, Jeans theorem.

H.O.

$$E = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2} k x^2 \Rightarrow \frac{\partial E}{\partial x} = kx, \quad \frac{\partial E}{\partial p} = \frac{p}{m}$$

$$0 = \frac{df}{dt} = \partial_t f + \frac{\partial f}{\partial E}\frac{\partial E}{\partial x}\frac{p}{m} - \frac{\partial f}{\partial E}\frac{\partial E}{\partial p} kx =$$

$$\partial_t f + \frac{\partial f}{\partial E}\left(kx\frac{p}{m} - \frac{p}{m} kx\right) = \partial_t f = 0.$$

Once we add sources / drains or particle-changing interactions, we have: $\frac{df}{dt} = C[f]$ → collision term

The collision terms will encode the microscopic physics, detailed p-dependence and statistics.

Once all the components of the plasma are in equilibrium, this term vanishes:

$$C[f_{eq}] = 0, \quad \text{or} \quad \frac{df_{eq}}{dt} = 0.$$

We will now move from Minkowski to FLRW to derive the Liouville operator in an expanding universe, using the proper on-shell relations.

$$P^2 = P_\mu P^\mu = -E^2 + a^2 P^{i2} = -m^2$$

$$= g_{\mu\nu} P^\mu P^\nu = g_{00} P^{02} + g_{ii} P^{i2} = -E^2 + p^{i2}$$

on shell: $$E^2 - p^2 = E^2 - a^2 P^2 = m^2$$

$$p = aP$$

We will separate the radial parts $p = \sqrt{\Sigma p^{i2}}$ from the directional parts: $\hat{p}^i = \frac{p^i}{p}$, such that:

$$\frac{df}{dt} = \partial_t f + \partial_{x^i} f \, \dot{x}^i + \partial_p f \, \dot{p} + \partial_{\hat{p}^i} f \, \dot{\hat{p}}^i.$$

Now, a homogeneous & isotropic universe has

Homogeneous & isotropic : $\partial_{x^i} f\, \dot{x}^i \to 0$, $\partial_{\hat{p}^i} f\, \dot{\hat{p}}^i \to 0$.

This is true at leading order, without any metric perturbations. If we turn on ϕ & Ψ potentials, these terms would have to be included. Without them:

$$\boxed{\frac{df}{dt} \simeq \partial_t f + \partial_p f\, \dot{p}.}$$

- $\dot{p}$ is obtained from the geodesic (no interactions here, so we're ~free-falling), $P^\mu = \dfrac{dx^\mu}{d\lambda}$

$$\frac{dP^0}{d\lambda} = \frac{dP^0}{dt}\frac{dt}{d\lambda} = P^0 \frac{dP^0}{dt} = -\Gamma^0_{ij} P^i P^j = -\delta_{ij}\, \dot{a} a P^{i2}$$

$$\frac{dt}{d\lambda} = \frac{dx^0}{d\lambda}, \quad P^0 = E, \quad \frac{dP^0}{dt} = \frac{dE}{dt}$$

$$= -\frac{\dot{a}}{a} \underbrace{a^2 P^{i2}}_{p^{i2} = p^2}$$

- so : $E \dfrac{dE}{dt} = -Hp^2$

$$E\frac{dE}{dt} = \frac{1}{2}\frac{d}{dt}(E^2) = \frac{1}{2}\frac{d}{dt}(p^2 - m^2) = p\frac{dp}{dt} = -Hp^2,$$

FINALLY : $\dfrac{dp}{dt} = -Hp$,

$$\Rightarrow \boxed{\frac{\partial f}{\partial t} - Hp\frac{\partial f}{\partial p} = C[f]}.$$

- This immediatelly tells us how n behaves:

$$dN = f\, d^3x \frac{d^3p}{(2\pi)^3} = n\, d^3x \quad , \quad \int_p = \int \frac{d^3p}{(2\pi)^3} \, ,$$

$$\frac{dn}{dt} = \int_p \frac{df}{dt}$$

$$= \int_p \partial_t f - H \int_p p \frac{\partial f}{\partial p} = \int_p C[f]$$

$$\star : \quad \partial_t n - H \int_0^\infty \frac{4\pi}{(2\pi)^3} p^2 dp \; p \frac{\partial f}{\partial p} = \int_p C[f] \, .$$

- We will integrate the 2^{nd} term by parts, taking into account that a regular distribution vanishes with $p^n f \to 0$ for $n=3$: $p^3 f(p) \xrightarrow[p\to\infty]{p\to 0} 0$.

You can convince yourself easily by expanding the FD or BE functions for $p \to 0$ ~~and~~ $p \to \infty$.

$$p^3 f \Big|_0^\infty = 0 = \int dp\, \partial_p (p^3 f) = \int dp \left(p^3 \partial_p f + 3p^2 f \right)$$

$$\text{OR} : \quad \int p^3 \partial_p f\, dp = -3 \int p^2 dp\, f$$

$$\star \quad \Rightarrow \quad \partial_t n + 3H \int_p f = \partial_t n + 3H\, n = \int_p C[f]$$

$$\boxed{\partial_t n + 3H n = \int_p C[f]}$$

- In the absence of interactions with $C = 0$, we get:

$$\partial_t n = -3Hn$$

$$\text{or} \quad \frac{dn}{dt} = -3\frac{\dot{a}}{a}n = -3\frac{n}{a}\frac{da}{dt}$$

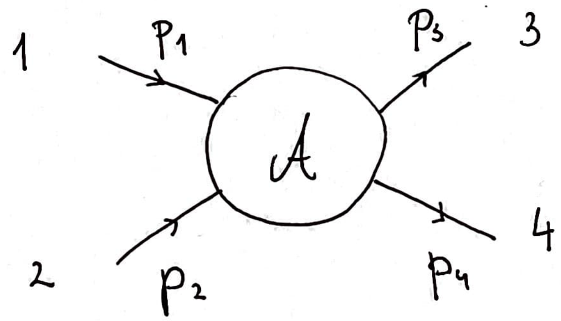
$$\text{or:} \quad \frac{dn}{n} = -3\frac{da}{a}, \quad \ln n = -3\ln a, \quad \underline{\underline{n \propto a^{-3}}}$$

as expected.

- COLLISION TERMS and THERMALLY AVERAGED RATES $\langle\sigma v\rangle$

These rates are defined and calculated for any type of process, such as scatterings (elastic, inelastic), annihilations and (inverse) decay rates.

We will consider a very generic 2-2 process: $1+2 \to 3+4$



$$\Rightarrow C[f]$$

$$\text{and } C[f_{eq}] = 0.$$

These interactions are local, they happen at a fixed point in space-time, (x, t). Locally, space-time is Minkowski and four-momenta are conserved in the usual way : $\quad p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu ,$

or : $\quad E_1 + E_2 = E_3 + E_4 ,$

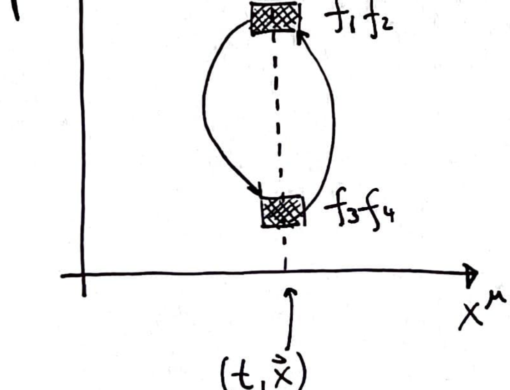
$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4 .$$

Moreover, all particles need to be on-shell

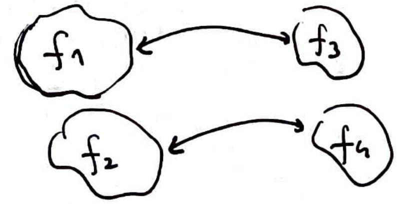
$$E_i^2 - p_i^2 = m_i^2 \qquad \forall i = 1, \ldots, 4$$

The amplitude $\mathcal{A}$ contains all the details of the microscopic interaction, such as the strength of interactions and momentum dependence (usually, the spins are averaged and we integrate over all the angles).

When these interactions are "strong enough", the momenta can be transferred efficiently (kinetic eq.) and type of species exchanged (chemical eq.).



• phase space gets "reshuffled"



$\boxed{\text{LOCAL!}}$ ~ Minkowski, the usual QFT.

- Let us focus on f_1 and the related $n_1 = \int_p f_1$:

$$\dot{n}_1 + 3H n_1 = \int_{p_1} \underbrace{C[f(p_1)]}_{\text{collision term}}$$

"1" comes from the collision of "3" & "4", hence $\oplus$

$$C[f_1(p_1)] = \underbrace{\sum_{p_{2,3,4}}}_{\substack{\text{integration over momenta}\\ \text{w. conservation \& invariance}}}^{\vec{p}_1+\vec{p}_2=\vec{p}_3+\vec{p}_4} \underbrace{\delta(E_1+E_2-E_3-E_4)}_{\substack{\text{usual energy}\\ \text{conservation}}} \underbrace{\overline{|A|^2}}_{\substack{\text{averaged amplitude}\\ \text{squared (as in the usual QFT)}}} (\tilde{f}_3\tilde{f}_4 - \tilde{f}_1\tilde{f}_2)$$

n_1 drains when collided w. "2"

- The tildes stand for the statistical blocking/enhancing

$$\tilde{f}_i = f_i(1 \pm f_j)$$

↳ final state suppression

$+$ for bosons (BE enhancement),

$-$ for fermions (FD blocking).

EXAMPLE : $e_1^+ e_2^- \to e_3^+ e_4^-$: $\tilde{f}_1\tilde{f}_2 = f_1(1-f_3)\, f_2(1-f_4)$,

$e_1 e_2 \to \gamma_3 \gamma_4$: $\tilde{f}_1\tilde{f}_2 = f_1(1+f_3)\, f_2(1+f_4)$.

- The sum over momenta of 2,3,4 is done in the usual way :

$$\int_{OS} d^4p = \underbrace{\int \frac{d^3p}{(2\pi)^3}}_{\int_p} \int_0^\infty dE\, \delta(\underbrace{E^2-p^2-m^2}_{f(E)})$$

on-shell

$$= \int_p \int_0^\infty dE\, \frac{\delta(f=0)}{|f'|} = \int_p \int_0^\infty dE\, \frac{\delta(E-\sqrt{p^2-m^2})}{|2E|}$$

• Combining all this, we get:

$$C[f_1] = \frac{1}{2E_1}\left(\prod_{i=2}^{4}\int_{\Pi_i}\right)(2\pi)^4\delta^{(4)}(p_1^\mu+p_2^\mu-p_3^\mu-p_4^\mu)\,\overline{|\mathcal{A}|^2}$$
$$\times\left(\tilde{f}_3\tilde{f}_4-\tilde{f}_1\tilde{f}_2\right),$$

where: $\displaystyle\int_{\Pi_i} = \int\frac{d^3p_i}{(2\pi)^3\,2E_i(p_i)} = \int_{p_i}\frac{1}{2E_i(p_i)}$,

and: $\delta^{(4)}(p_1+p_2-p_3-p_4) = \delta(E_1+E_2-E_3-E_4)\,\delta^{(3)}(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)$.

GENERAL BOLTZMANN

The final form of the Boltzmann equation is

"1": $$\dot{n}_1 + 3Hn_1 = \left(\prod_{i=1}^{4}\int_{\Pi_i}\right)(2\pi)^4\delta^{(4)}(p_1+p_2-p_3-p_4)\,\overline{|\mathcal{A}|^2}\times$$
$$\| $$
$$a^{-3}\frac{\partial}{\partial t}(n_1a^3) \qquad \left[f_1f_2(1\pm f_3)(1\pm f_4)-f_3f_4(1\pm f_1)(1\pm f_2)\right]$$

This is the central equation that we will use to keep track of the number densities n_1:

$$\boxed{a^{-3}\frac{\partial}{\partial t}(n_1a^3) = \left(\prod_{i=1}^{4}\int_{\Pi_i}\right)(2\pi)^4\delta^4(\Sigma p)\,\overline{|\mathcal{A}|^2}\left(\tilde{f}_1\tilde{f}_2-\tilde{f}_3\tilde{f}_4\right)}$$

The Boltzmann equation that we derived is very general and holds for in- and out-of-equilibrium processes. However, for most practical purposes, we can simplify it further.

$\boxed{\text{SIMPLIFIED BOLTZMANN}}$ in three steps:

i) Kinetic
ii) no blocking
iii) $\mu = 0$ normalization

i) Kinetic equilibrium with rapid exchange of (E, p) guarantees that the distribution functions f_s depend only on E_s, T, μ_s

for each species

FERMIONS : $$f_s = \frac{1}{e^{(E_s - \mu_s)/T} + 1},$$

BOSONS : $$f_s = \frac{1}{e^{(E_s - \mu_s)/T} - 1}.$$

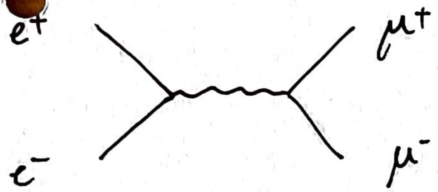
We assumed that all species s are equilibrated, so $T_s = T$ for all s. Each species then has its own $E_s = \sqrt{p_s^2 + m_s^2}$ and μ_s, which in general depends on T and needs to be calculated $\boxed{\mu_s \leftrightarrow n_s}$.

ib) In <u>chemical</u> equilibrium, we also have:

$$\sum_{i \in \text{initial}} \mu_i = \sum_{i \in \text{final}} \mu_i ,$$

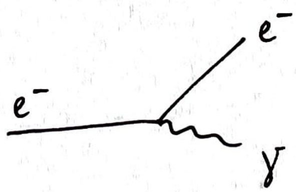
which follows from conserved charges, like Qem, $\Delta B = 0$, $\Delta L = 0$, etc. (baryon #, lepton #, flavour, ..)

EXAMPLE #1 QED in equilibrium



$$\mu_{e^+} + \mu_{e^-} = \mu_{\mu^+} + \mu_{\mu^-} \Rightarrow \cancel{2}\mu_{e^-} = \cancel{2}\mu_{\mu^-}$$
(see below)

We can always radiate a photon and charge is conserved, so:

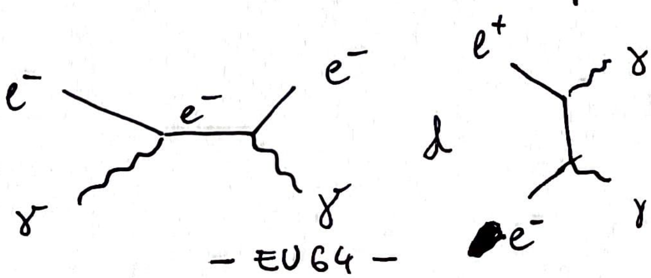


: $\mu_{e^-} = \mu_{e^-} + \mu_\gamma$,

$$\Rightarrow \mu_\gamma = 0 .$$

Another way to say the same thing is that γ's are their own anti-particles and μ measures the difference between n_s and $n_{\bar{s}}$, where s = species, $\bar{s}$ = anti-species.

EXAMPLE #2:



: $e^+ + e^- \to 2\gamma$

$\mu_{e^+} + \mu_{e^-} = 2\mu_\gamma = 0$

$\mu_{e^+} = -\mu_{e^-}$.

ii). The second step towards simplifying the Boltzmann equation is to focus on $T < E-\mu$ part of the distribution. There, the distinction between BE & FD goes away and we go to MB (Maxwell-Boltzmann)

$$f_{FD/BE} = \frac{1}{e^{(E-\mu)/T} \pm 1} \longrightarrow e^{-(E-\mu)/T} = f_{MB}.$$

Furtermore, we have: $e^{-\frac{E-\mu}{T}} \ll 1$, because $E-\mu > T$, so we can neglect the statistical factors.

$$\tilde{f}_3 \tilde{f}_4 - \tilde{f}_1 \tilde{f}_2 \longrightarrow f_3 f_4 - f_1 f_2$$

all with the same T (kinetic eq.)

$$\longrightarrow e^{-(E_3+E_4-\mu_3-\mu_4)/T} - e^{-(E_1+E_2-\mu_1-\mu_2)/T}$$

- Energy is always conserved, so: $E_3 + E_4 = E_1 + E_2$, which implies:

$$\longrightarrow e^{-(E_1+E_2)/T} \left(e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} \right).$$

Now we see how $C[f_{eq}] = 0$. When <u>full</u> chemical equilibrium is reached, $\mu_1 + \mu_2 = \mu_3 + \mu_4$ and the term in the bracket goes to zero, i.e. $C[f_{eq}] = 0$.

• We mentioned that one can always discuss either $n(t)$ or $\mu(t)$ interchangeably. This is particularly transparent in the MB limit:

$$n_s = g_s \int_p f_s^{MB} = g_s e^{\mu_s/T} \int_p e^{-E_s/T},$$

(dropping the species index)

$$n = g\, e^{\mu/T} \underbrace{\int \frac{d^3p}{(2\pi)^3} e^{-\frac{\sqrt{p^2+m^2}}{T}}}_{n^{(0)}} .$$

The (0) superscripts are the $\mu=0$ number densities. It turns out that it's very convenient to normalize to this quantity, when studying how processes go out of equilibrium. $\boxed{n = g\, e^{\mu/T} n^{(0)}}$

$\boxed{\mu=0}$ <u>number densities</u> can be evaluated easily:

a) massless limit: $m=0$, $E=p$ (γ's, ν's, any "radiation")

$$n^{(0)}_{m=0} = g \int_0^\infty \frac{4\pi p^2\, dp}{(2\pi)^3} e^{-p/T}, \quad x = \frac{p}{T},\ dx = \frac{1}{T} dp$$

$$= g \frac{T^3}{2\pi^2} \underbrace{\int_0^\infty x^2 e^{-x} dx}_{2!} = \frac{g T^3}{\pi^2}.$$

low T photons, MB approx.
(γ's, $g=2$, $\mu_\gamma=0$
$n_\gamma^{(0)} = \frac{2}{\pi^2} T^3$)

b) massive case, zero μ, MB limit:

$$E = \sqrt{m^2+p^2} = m\sqrt{1+\left(\frac{p}{m}\right)^2} \simeq m + \frac{p^2}{2m}$$

$$n^{(0)}_{m>p} = g\int_0^\infty \frac{p^2\,dp}{2\pi^2}\, e^{-m/T}\, e^{-p^2/(2mT)} \quad , \; x = \frac{p^2}{2mT} \in [0,\infty]$$

$$= \frac{g}{2\pi^2} e^{-m/T}\, mT \int_0^\infty dx\, \sqrt{2mT}\, x^{1/2} e^{-x} \qquad dx = \frac{p\,dp}{mT}$$

$$= \frac{g}{\sqrt{2}\pi^2} e^{-m/T} (mT)^{3/2} \underbrace{\int_0^\infty x^{1/2} e^{-x}\, dx}_{\Gamma(3/2) = \frac{1}{2}! = \frac{\sqrt{\pi}}{2}} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} .$$

A short summary

$$n^{(0)}_{m=0} = \frac{gT^3}{\pi^2} ,$$

$$n^{(0)}_{m>p} = g\left(\frac{mT}{2\pi}\right)^{3/2} e^{-m/T} .$$

How should we interpret this? Massless particles will scale as $n \propto T^3 \propto a^{-3}$, as is the case for photons. Massive particles with $\mu = 0$ have equal amount of particles and anti-particles. When the temperature T drops below m, annihilations can still take place efficiently: $f\bar{f} \to 2\gamma$ or $f_1\bar{f}_1 \to f_2\bar{f}_2$, but the reverse process is energetically disfavored by the $e^{-m/T}$ factor. We say that the equilibrium $n_m^{(0)}$ becomes Boltzmann suppressed at low T, below the m threshold.

Using $n_s^{(0)}$, we can rewrite the part of the Boltzmann equation as:

$$e^{(\mu_3+\mu_4)/T} - e^{(\mu_1+\mu_2)/T} = \frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}.$$

Thus we simplified: $\tilde{f}_3\tilde{f}_4 - \tilde{f}_1\tilde{f}_2 \simeq e^{-(E_1+E_2)/T}\left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right)$.

The simplified Boltzmann equation is given by:

$$\dot{n}_1 + 3Hn_1 = \underbrace{\left(\prod_{i=1}^{4}\int_{\Pi_i}\right)(2\pi)^4\delta^{(4)}(\Sigma p)\overline{|\mathcal{A}|^2}\,\frac{e^{-(E_1+E_2)/T}}{n_1^{(0)} n_2^{(0)}}}_{\langle\sigma v\rangle}\left(\frac{n_1^{(0)} n_2^{(0)}}{n_3^{(0)} n_4^{(0)}}\, n_3 n_4 - n_1 n_2\right)$$

SIMPLIFIED BOLTZMANN EQUATION

$$a^{-3}\frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)}\langle\sigma v\rangle\left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}\right),$$

THERMALLY AVERAGED CROSS-SECTION $\langle\sigma v\rangle$

$$\langle\sigma v\rangle = \frac{1}{n_1^{(0)} n_2^{(0)}}\left(\prod_{i=1}^{4}\int_{\Pi_i}\right)\delta^{(4)}(\Sigma p)\overline{|\mathcal{A}|^2}\, e^{-(E_1+E_2)/T}, \quad f_i^{(0)} = e^{-E_i/T}$$

$$= \frac{1}{n_1^{(0)} n_2^{(0)}}\int_{p_{1,2}} f_1^{(0)} f_2^{(0)} \int_{\Pi_{3,4}}(2\pi)^4\delta^{(4)}(\Sigma p)\frac{\overline{|\mathcal{A}|^2}}{4E_1E_2}. \quad \int_{\Pi_i} = \int_{p_i}\frac{1}{2E_i}$$

- First, let's see why this is called a thermally averaged cross-section, then we will do a dimensional analysis of the simplified Boltzmann.

- The usual cross-section $\sigma_{12\to34}$ is defined as (see e.g. PDG review Kinematics)

$$\sigma = \int_{\Pi_{3,4}} (2\pi)^4 \delta^{(4)}(\Sigma p) \frac{\overline{|A|^2}}{4E_1E_2v_{12}} = \int_{\Pi_{3,4}} d\sigma \,,$$

where : $v_{12} = \dfrac{\sqrt{(p_1 p_2)^2-(m_1m_2)^2}}{E_1E_2}$ = $\begin{cases} 2, & m_{1,2}\to 0 \text{ \& } p_{1,2}=E_{1,2}, \text{ for a central collision} \\ \dfrac{\cancel{m_1}m_2\sqrt{(1+v_1v_2)^2-1}}{m_1\cancel{m_2}} \simeq \sqrt{2v_1v_2}\,, \end{cases}$

4D: $p_1^\mu p_{2\mu}$

NB: $[v_{12}]=0$.

central collision:

[$p_1\cdot p_2 = p_1^\mu\cdot p_{2\mu}$; $m_{1,2}=0$ $\quad p_1=E_1(1,1)$, $p_2=E_2(1,-1)$ $\Rightarrow$ $p_1\cdot p_2 = E_1E_2(1+1)$

$m_{1,2}>|\vec{p}_{1,2}|$: $p_{1,2}\simeq m_{1,2}(1, \pm v_{1,2})$ $\Rightarrow$ $p_1\cdot p_2 = m_1m_2(1-v_1(-v_2))$]

- Using this notation, $\langle\sigma v\rangle$ is rewritten as :

$$\langle\sigma v\rangle = \frac{1}{n_1^{(0)}n_2^{(0)}} \int_{p_{1,2}} f_1^{(0)} f_2^{(0)} \int_{\Pi_{3,4}} d\sigma\, v_{12} \,,$$

which is really a thermal average of $\int_{\Pi_{3,4}} d\sigma\, v_{12}$, since :

$$\frac{1}{n_1^{(0)}n_2^{(0)}} \underbrace{\int_{p_1} f_1^{(0)}}_{n_1^{(0)}} \underbrace{\int_{p_2} f_2^{(0)}}_{n_2^{(0)}} = 1, \text{ \& } \langle\sigma\rangle = \frac{1}{n_1^{(0)}n_2^{(0)}} \int_{p_{12}} f_1^{(0)} f_2^{(0)} \sigma.$$

• What does $\langle \sigma v \rangle$ do? The microscopic theory, encoded inside of $\mathcal{M}$ will produce $|\mathcal{M}|^2$, which is a function of masses and momenta. Thermal averaging will then ~~reshuffle~~ pick out the statistically favored momenta at a given T. For example, when $m_i \to 0$, all that is left are masses of intermediate mediators and momenta.

think $\langle p^2 \rangle \sim (k_B T)^2$ or $\langle |p| \rangle \sim T$

E.g. QED $e^+ e^- \to \gamma\gamma$, $m_\gamma = 0$, $T \gg m_\gamma$, only p_i remain and thermal averaging replaces $p \to T$.

On the other hand, if the temperature drops below a given mass, the p is replaced by m (on dimensional grounds and in the calc. of $\overline{|\mathcal{M}|^2}$ when $p_i \ll m_i$), such that $p \to m$ instead of T. (as in σ_T, the Thomson σ)

DIMENSIONAL ANALYSIS of the SIMPLIFIED BOLTZMANN

$$a^{-3}\frac{d(n_1 a^3)}{dt} = n_1^{(0)} n_2^{(0)} \langle \sigma v \rangle \overbrace{\underbrace{\left(\frac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} - \frac{n_1 n_2}{n_1^{(0)} n_2^{(0)}} \right)}_{O(1),\ \text{dimensionless}}}^{\text{SAHA}}$$

$[a]$ cancels out

$[t] \sim H^{-1}$

$n_1^{(0)} \wr n_1$

$$\sim \cancel{n_1} H = \cancel{n_1} \Gamma , \quad \Gamma = n_2^{(0)} \langle \sigma v \rangle$$

Γ = interaction rate.

- The LHS is essentially set by the Hubble rate. The universe is expanding and H grows with T, when we go back in time.
- The RHS depends on Γ and the $\overset{\text{SAHA}}{(\dots)}$ factor.

When $\Gamma \simeq H$ the interactions are on the verge of equilibration. When $\Gamma \gg H$, they are fully equilibrated but for the $\overset{\text{Boltzmann}}{\text{equality}}$ to hold, the

SAHA APPROXIMATION : $\dfrac{n_3 n_4}{n_3^{(0)} n_4^{(0)}} = \dfrac{n_1 n_2}{n_1^{(0)} n_2^{(0)}}$, needs to be valid then

- go back to f_{MB}, the Saha equation and energy conservation implies : $e^{(\mu_1+\mu_2)/T} = e^{(\mu_3+\mu_4)/T}$

CHEMICAL EQUILIBRIUM OR : $\mu_1 + \mu_2 = \mu_3 + \mu_4$.