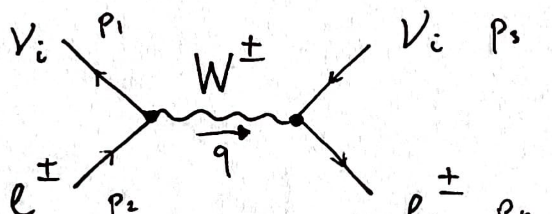
Neutrino Decoupling

- With the simplified Boltzmann and our rough understanding of $\langle \sigma v \rangle$, we can estimate when neutrinos decouple from the plasma of SM particles.
- Suppose we are at $T \sim \mathrm{GeV}$, so e^+, e^-, γ and all species of ν are present. Weak interactions proceed via:



$$; \quad \mathcal{A} \simeq \left(\frac{g}{\sqrt{2}}\right)^2 \left(\bar{u}_{1L}\gamma^\mu u_{2L}\right)\left(\bar{u}_{3L}\gamma^\nu u_{4L}\right) \left(\frac{-g_{\mu\nu} + \left(P_\mu P_\nu / M_W^2\right)^{\propto m}}{q^2 - M_W^2}\right)$$

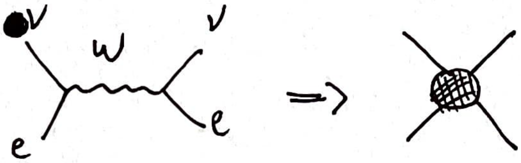
- At high momenta $p_i \gg M_W$, the propagator is roughly like for photons. At low momenta, we can ignore q^2 wrt. M_W^2. Dimensionally $[u] = [\bar{u}] = 1/2$, remember: $\sum_{\text{spins}} u\bar{u} = \not{p}_i + m_i$, then:

$$\overline{|\mathcal{A}|^2} \simeq \frac{g^4}{M_W^4} \text{ or } \frac{p^4}{M_W^4} \quad \text{but } \sigma \simeq \frac{1}{q^2}\overline{|\mathcal{A}|^2}$$

$$\Rightarrow \langle \sigma v \rangle \xrightarrow{p \to T} \frac{q^2}{M_W^4} \to \frac{T^2}{M_W^4} \text{ or: } \boxed{\langle \sigma v \rangle \sim G_F^2 T^2}$$

because $p_i \gg m_i$, for m_e & m_ν

- We could have guessed this even more directly



SM FERMI

$\mathcal{A} \propto G_F$ (or $\frac{g^2}{M_W^2}$, modulo factors of $\sim$ few)

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2}$$

So: $\overline{|\mathcal{A}|^2} \propto G_F^2$ but to get units of cm^2 for σ, or σv because $[v]=0$, we need q^2, or T^2, thus

$$\langle \sigma v \rangle \sim G_F^2 T^2 .$$

We now get the interaction rate $\Gamma = n_2^{(0)} \langle \sigma v \rangle$,

where $n_2^{(0)} = n_e^{(0)} \simeq \# \, T^3$ because $m_e \simeq 0$, thus

$$\Gamma \sim T^3 G_F^2 T^2 = G_F^2 T^5 \quad , \; [\Gamma] = 1$$

- This takes care of the RHS and we got a good heuristic understanding of σ, $\langle \sigma v \rangle$. Now we focus on the LHS. At $T \sim GeV$, we are in radiation domination ($z_{eq} \simeq 3000$, $T_{eq} \simeq z T_0 \sim eV$)

$$H = \sqrt{\frac{8\pi G}{3} \rho_\gamma} = \sqrt{\frac{8\pi}{3M_{Pe}^2} \frac{\pi^2}{30} g_*(T) \, T^4} \simeq \frac{T^2}{M_{Pe}} \times \mathcal{O}(1)$$

$$g_* = \frac{21}{2}: \quad 2 + 4\,\frac{7}{8} = \frac{11}{2} \qquad \text{or } \sqrt{G_N}\, T^2$$

Radiation domination: $H \sim \sqrt{G_N}\, T^2$,

Thermal averaging for: $\Gamma \sim G_F^2 T^5$,

$m_e \ll T \ll M_W$

Decoupling or FREEZE-OUT : $\Gamma \sim H$,

$$G_F^2 T^{\cancel{5}\,3} \simeq \sqrt{G_N}\, \cancel{T^2}$$

$$T_{dec} \simeq \left(\frac{G_N}{G_F^4} \right)^{1/3 \cdot 1/2}$$

$$T_{dec} \simeq \left(\frac{M_W^8}{M_{Pl}^2} \right)^{1/6} = \left(\frac{M_W^4}{M_{Pl}} \right)^{1/3} = \left(\frac{\overbrace{(80\ \mathrm{GeV})^4}^{\sim 10^8\ \mathrm{GeV}}}{10^{18}\ \mathrm{GeV}} \right)^{1/3}$$

$$= 10^{-\frac{10}{3}}\ \mathrm{GeV} = \text{a few MeV}.$$

- A precise numerical treatment with 

and a more general Boltzmann equation gives $T_{\nu dec} = 1.5$ MeV, in excellent agreement with our estimate.

- This implies neutrinos froze-out while being very relativistic with $E_\nu \sim p_\nu \sim$ MeV $\gg m_\nu \lesssim 0.1$ eV by at least 7 orders of magnitude.

- Neutrinos after freeze-out

→ No interactions, they keep $f_\nu^{FD}(T) = \frac{1}{e^{p/T}+1}$

and their $n_\nu \propto a^{-3}$

- When e^+e^- annihilate, they heat up the photons but not ν's, so $T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$ as follows from entropy conservation.

N_{eff}

The N_{eff} is a relevant cosmoligical parameter that "counts" the number of additional relativistic degrees of freedom. This is lumped in $\rho_{rad} = \rho_\gamma + \rho_{extra}$,

radiation photons all other species

such that: $$N_{eff} = \frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\left(\frac{\rho_{rad}-\rho_\gamma}{\rho_\gamma}\right) = \frac{8}{7}\left(\frac{11}{4}\right)^{4/3}\frac{\rho_{extra}}{\rho_\gamma}$$

It is normalized to fermions, such that neutrinos, after being cooled down, give: $\rho_{extra} = \rho_\nu$

and $$N_{eff}^{SM} = 3\left(\frac{11}{4}\right)^{4/3}\left(\frac{T_\nu}{T_\gamma}\right)^4 = 3$$

- A more precise numerical calculation of N_{eff} in the SM includes all EW interactions, T-corrections and neutrino oscillations, such that $N_{eff}^{SM} = 3.044$.

- Finally, N_{eff} can be used as a constraint on certain models. If there is a species (fermion or a boson) that is long-lived enough and light enough to contribute to ρ_{extra}, it can affect $N_{eff}(T_{BBN})$, $N_{eff}(T_{CMB})$, @ γ decoupling

$\rho_{extra} = \rho_\nu + \Delta\rho$

$N_{eff} \sim \left(\frac{8}{7}\right)\left(\frac{11}{4}\right)^{4/3} \frac{\rho_{extra}}{\rho_\gamma} \simeq 3 + \Delta N_{eff}$

$$\Delta N_{eff}(T_{BBN}) \lesssim 0.3,$$

$$\Delta N_{eff}(T_{CMB}) \simeq 1.$$

Neutrino number and energy density

- For photons $$n_\gamma = 2\int_p \frac{1}{e^{p/T}-1} = \frac{2\zeta(3)}{\pi^2} T_\gamma^3$$

- For neutrinos $f = f_\nu^{FD}(p,T)$

for one generation: $$n_{\nu_i} = 2\int_p \frac{1}{e^{p/T}+1} = \frac{2\zeta(3)}{\pi^2}\underbrace{\left(1-\frac{1}{4}\right)}_{=3/4} \underbrace{T_\nu^3}_{=\frac{4}{11}T_\gamma^3}$$

for all three : $$n_\nu = 3\cdot\frac{3}{4} n_\gamma \left(\frac{T_\nu}{T_\gamma}\right)^3 = \frac{9}{4}\cdot\frac{4}{11} n_\gamma = \frac{9}{11}n_\gamma$$

$n_\gamma = 410\ \mathrm{cm}^{-3}$, $\frac{9}{11}n_\gamma = 340\ \mathrm{cm}^{-3}$

- Neutrino energy density depends on m_ν $\qquad \rho_\gamma = \frac{\pi^2}{15} T_\gamma^4$

$$\rho_{\nu_i} = 2 \int_p \sqrt{p^2 + m_{\nu_i}^2}\, f_\nu^{FD}(p,T)$$

$$= \begin{cases} \frac{7}{8} \rho_\gamma(T_\nu) = \frac{7\pi^2}{120} T_\nu^4 & , \text{ for } m_{\nu_i} \ll T_{CMB} \\ m_{\nu_i} n_{\nu_i} & , \text{ for } m_{\nu_i} > T_\gamma \end{cases}$$

- If all neutrinos were massless ($m_{\nu_i} < T_\gamma = 2.7K = 10^{-4}eV$), which we know is not the case because $\Delta m^2 \sim 10^{-3} eV^2$ from osc.), then we would have:

$$\rho_\nu^{m=0} = 3 \cdot \rho_{\nu_i}^{m=0} = \frac{21 \cancel{\pi^2}}{\cancel{120}8} \left(\frac{T_\nu}{T_\gamma}\right)^4 \cdot \frac{\cancel{15}}{\cancel{\pi^2}} \rho_\gamma$$

$$= 3 \cdot \frac{7}{8} \left(\frac{T_\nu}{T_\gamma}\right)^4 \rho_\gamma = \underbrace{\frac{21}{8}\left(\frac{4}{11}\right)^{4/3}}_{0.7} \rho_\gamma$$

$$\Omega_\nu^{m=0} h^2 = 1.7 \cdot 10^{-5} < \Omega_\gamma h^2$$

- If all neutrinos were massive, we would get

$$\rho_\nu^{m_\nu} = \sum m_{\nu_i} n_{\nu_i} = \sum_{i=1}^{3} m_{\nu_i} \cdot \frac{3}{11} n_\gamma$$

$$= \frac{\sum m_\nu}{93.1\, eV}.$$

RECOMBINATION

- This process happens after BBN, $T \ll MeV$.

  The universe consists of e^- only (e^+ annihilate even before BBN), photons γ and light nuclei ($H = p^+$, D, tritium, ^3He, ^4He, ...)

- When the universe cools down, $e^- + p^+ \to H_{atom} + \gamma$

  @ $E_{Ry} = \frac{1}{2}\alpha^2 m_e c^2 = 13.6\,eV$.

  More precisely, when $\langle p_\gamma \rangle = \langle E_\gamma \rangle \simeq 3T_\gamma$ go below E_{Ry}, the atoms no longer get destroyed by photons.

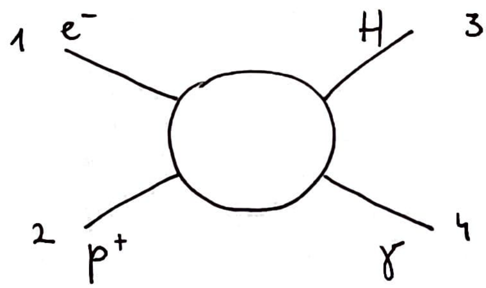
We will use the simplified Boltzmann to calculate how many electrons remain. here $n_e = n_{e^-}$

FIRST : $Q_{tot} = 0$: $Q_e n_e + Q_p n_p = n_p - n_e = 0$.

This is because $U(1)_{em}$ is unbroken and the universe is charge-neutral.

SECOND : The net baryon number is non-zero : $n_b = \eta_b n_\gamma$

and $\eta_b \sim 10^{-10}$.



$$
\begin{array}{lcccc}
 & & e^- \quad p & & H \quad \gamma \\
\Delta B = 0 & : & 0 + 1 & = & 1 + 0 \\
\Delta Q = 0 & : & -1 + 1 & = & 0 + 0
\end{array}
$$

The total baryon number $n_b = B(p)\, n_p + B(H)\, n_H$

$$= n_p + n_H$$

The simplified Boltzmann equation is

$$\mu_\gamma = 0 \qquad n_p = n_e \ (\Delta Q = 0)$$

$$a^{-3} \frac{d}{dt}\left(n_e a^3\right) = n_e^{(0)} n_p^{(0)} \langle \sigma v \rangle \left(\frac{n_H}{n_H^{(0)}} \cancel{\frac{n_\gamma}{n_\gamma^{(0)}}} - \frac{n_e n_p}{n_e^{(0)} n_p^{(0)}} \right)$$

We introduce $X_e = \dfrac{n_e}{n_e + n_H} = \dfrac{n_e}{n_p + n_H} = \dfrac{n_e}{n_b} = \dfrac{n_p}{n_b}$.

First, let's see what happens in equilibrium

$$\text{Saha}: \quad \frac{n_H}{n_H^{(0)}} = \frac{n_e^2}{n_e^{(0)} n_p^{(0)}} \Rightarrow \frac{n_e^2}{n_H} = \frac{X_e^2\, n_b^{\cancel{2}}}{(1 - X_e)\cancel{n_b}} = \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}},$$

$$\frac{X_e^2}{1 - X_e} = \underset{\displaystyle \eta n_\gamma \simeq \eta \frac{2\zeta(3)}{\pi^2} T^3}{\underset{\parallel}{n_b}} \left(\frac{m_e T}{2\pi} \right)^{3/2} \left(\frac{\overset{\nearrow 1}{m_p}}{m_H} \right)^{3/2} e^{-\underbrace{(m_e + m_p - m_H)}_{\varepsilon_0 = E_{Ry}}/T}, \quad \eta = 6 \cdot 10^{-10}.$$

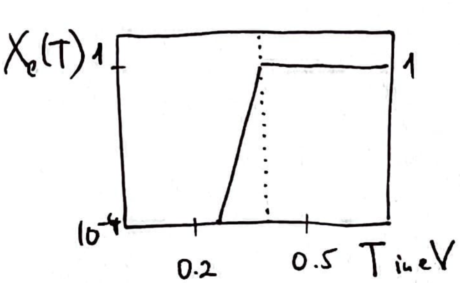
- We can solve the quadratic equation and obtain

$X_e(T)$ from $\dfrac{X_e^2}{1-X_e} \cong 10^9 \left(\dfrac{m_e}{T}\right)^{3/2} e^{-\varepsilon_0/T}$

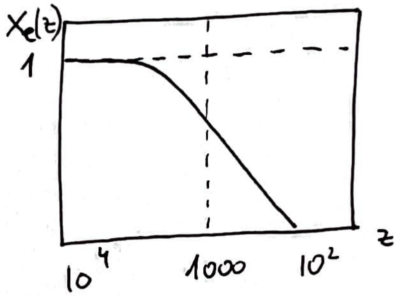
for $T \gtrsim \varepsilon_0$, the exponential is $e^{-0} = 1$ and $T \sim 10\,\mathrm{eV}$

so the RHS is $10^9 \left(\dfrac{10^5 \cancel{eV}}{10 \cancel{eV}}\right)^{3/2} \sim 10^{15} = \dfrac{X_e^2}{1-X_e}$, which

firmly sets $X_e = 1$ $\Rightarrow$ everything is ionized, as expected.



or in z



- Now let's go back to the simplified Boltzmann.

Note $n_b a^3 = \text{const.}$  $\int_V n_b = N_{\text{baryons}} = \text{const.}$ (modulo B-violating interactions = tiny)

$$n_e = X_e n_b \,,\quad a^{-3} \frac{d}{dt}(n_e a^3) = a^{-3} \frac{d}{dt}\left(X_e n_b a^3\right) = n_b \frac{dX_e}{dt}$$

$$\Rightarrow\quad n_b \frac{dX_e}{dt} = \langle \sigma v\rangle \left(n_H \frac{n_e^{(0)} n_p^{(0)}}{n_H^{(0)}} - n_e^2 \right) \qquad n_e^2 = X_e^2 n_b^2$$

$$= \langle \sigma v\rangle (1-X_e)\, n_b \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\varepsilon_0/T} - \langle \sigma v\rangle n_b^2 X_e^2$$

$$X_e = \frac{n_e}{n_b} = \frac{n_p}{n_b} = \frac{n_b - n_H}{n_b} \Rightarrow n_H = 1 - X_e n_b$$

$$\frac{dX_e}{dt} = \underbrace{(1-X_e)\langle\sigma v\rangle \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\varepsilon_0/T}}_{\beta} - X_e^2 \, n_b \underbrace{\langle\sigma v\rangle}_{\alpha^{(2)}}$$

- $\alpha^{(2)} \simeq 10 \left(\frac{\alpha}{m_e}\right)^2 \left(\frac{\varepsilon_0}{T}\right)^{1/2} \ln\left(\frac{\varepsilon_0}{T}\right)$ = recombination rate into an excited state of H (Peebles; see his book "Principles of physical cosmology")

- $\beta = \alpha^{(2)} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\varepsilon_0/T} \simeq 10 \left(\frac{\alpha}{m_e}\right)^2 \sqrt{\frac{\varepsilon_0}{T}} \ln\frac{\varepsilon_0}{T} \left(\frac{m_e T}{2\pi}\right)^{3/2} e^{-\varepsilon_0/T}$

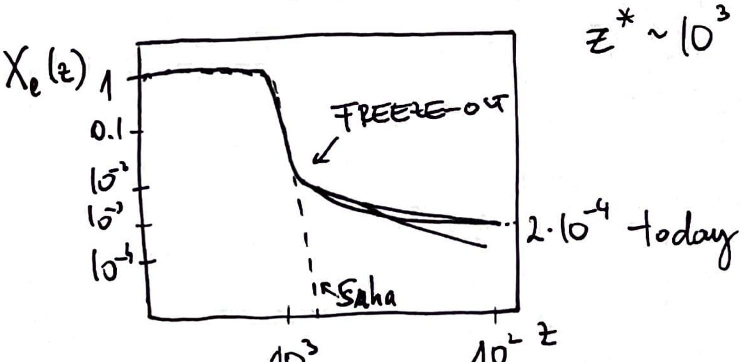
at low T: $T < \varepsilon_0$, the exponential suppresses all the other T^{-n} powers, $\beta \to 0$, no ionization.

- The simplified Boltzmann equation

$$\dot{X}_e = (1-X_e)\beta - X_e^2 \alpha^{(2)}, \quad X_e(x=0) = 1$$

is a non-linear first order differential equation and can easily be integrated numerically.

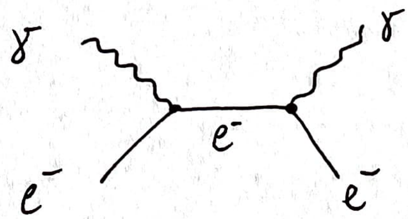
$z^* \sim 10^3 \to T^* \sim 0.27\,eV$



- We will further work out the analytics of freeze-out when we come to DM production.
- There are fast numerical tools CAMB and CLASS [1104.2933] that deal with multiply excited states include the He abundance and produce $X_e(0)$.
- During recombination, universe is becoming neutral and transparent to photons, let us see when the photons decouple and CMB gets created.

Photon Decoupling

Compton scattering



is given by

$$\langle \sigma v \rangle \simeq \sigma_T = \frac{8\pi}{3}\left(\frac{\alpha}{m_e}\right)^2$$

Again, $T \sim$ few eV, $\ll m_e \sim 511\,\text{keV}$, so instead of p in the $d\sigma$, we get m_e, thermal average $\to 1$

$$\Gamma = n_e \sigma_T = \underbrace{X_e}_{\text{drops sharply}} n_b \sigma_T = H$$

- Photon decoupling $T_{\gamma dec}$

$$\Gamma = X_e n_b \sigma_T = H$$

Remember that the number density $\propto a^{-3}$ or

$$\Omega_{matter} = \Omega_m(0) \cdot a^{-3} \;:\; \Omega_b(a) = \Omega_b a^{-3} = \frac{n_b m_p}{\rho_{cr}}$$

OR : $n_b = \Omega_b a^{-3} \frac{\rho_{cr}}{m_p}$, $\Omega_b \simeq 5\%$

Now for the Hubble $H(a)$. We are in a mixed universe because $z_{eq} \sim \mathcal{O}(10^3) \Rightarrow T_{eq} \simeq eV$.

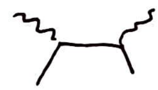
$$\frac{H}{H_0} = \sqrt{\Omega_m}\, a^{-3/2} \sqrt{1 + \frac{a_{eq}}{a}}.$$

Freeze-out happens when $\Gamma = H$, or

$$1 = \frac{\Gamma}{H} = \frac{n_e \sigma_T}{H} = 123\, X_e \left(\frac{\omega_b}{0.02}\right) \sqrt{\frac{0.1}{\omega_m}} \left(\frac{1+z}{1000}\right)^{3/2} \times$$
$$\left(1 + \frac{1+z}{3360} \frac{0.1}{\Omega_m}\right)^{-1/2}$$

The ω_i are constants and z does not vary much. What changes drastically is X_e. To satisfy $\frac{\Gamma}{H} = 1$, we need $X_e \sim 10^{-2}$. At that point (somewhere in the middle between $X_e = 1$ and the relic $X_e \sim 10^{-4}$), the CMB photons get produce, i.e. they decouple from e^-.

DECOUPLING WITHOUT RECOMBINATION

- Similar to weak interactions and neutrinos, the QED interactions (Compton scattering)  will also go out of equilibrium, but much later.

- We can use the same equation as above to see when that would happen. Essentially $X_e = 1$ if no recombination occurs and $z \ll 1000$, so $\frac{1+z}{10^3} \to 0$.

then
$$\left(10^2 \left(\frac{\omega_b}{0.02}\right)\left(\frac{0.1}{\omega_m}\right)^{1/2}\right)^{2/3} \frac{1+z}{1000} = 1$$
$$\Downarrow$$
$$1 + z_\gamma^{dec} \cong 40 \left(\frac{0.02}{\omega_b}\right)^{2/3} \left(\frac{\omega_m}{0.1}\right)^{1/3}$$