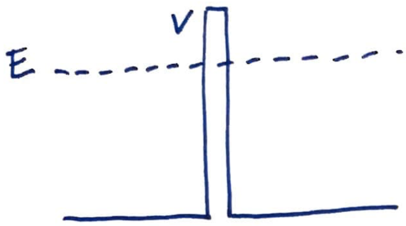
(32) Kolikšna je prepustnost zelo tanke in visoke potencialne plasti?

$$v = 2000\ \frac{m}{s} \qquad \beta = \frac{2\cdot 10^3}{3\cdot 10^8} = 0{,}66\cdot 10^{-5} \Rightarrow E \sim \frac{\overset{m\beta c}{p^2}}{2m} \sim \frac{\beta^2 mc^2}{2}$$

$$V_0 a = 10^{-3}\,\mathrm{eV\,nm} = \tilde{h}c$$



$$k = \sqrt{2mE}/\hbar \quad , \quad K = \sqrt{2m(V-E)}/\hbar$$

- visoka : $V \gg E$
- ozka : $Ka \simeq \sqrt{2mV}\,a/\hbar \ll 1$

$$T = \frac{1}{1 + \frac{(\cancel{E} + V - \cancel{E})^2}{4E(V-E)}\,\mathrm{sh}^2 Ka}$$

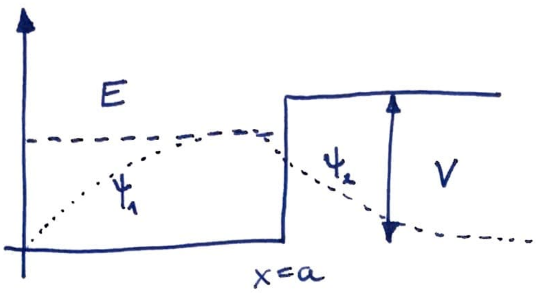
$$\mathrm{sh}^2 Ka \sim (Ka)^2 \sim \frac{2mVa^2}{\hbar^2}$$

$$= \frac{1}{1 + \frac{V}{\cancel{4}2E}\frac{\cancel{2}mVa^2}{\hbar^2}} = \frac{1}{1 + \frac{mc^2(\tilde{h}c)^2}{2E\,(\hbar c)^2}} = \frac{1}{1 + \left(\frac{\tilde{h}c}{\beta\,\hbar c}\right)^2}$$

$$\frac{\tilde{h}c}{\beta\,\hbar c} = \frac{10^{-3}}{\beta\ 200} = \frac{3\cdot\cancel{10^{-3}}}{4\cdot\cancel{10^{-3}}} \Rightarrow T = \frac{1}{1+\frac{9}{16}} = \frac{16\cdot 4}{25\cdot 4}$$

$$= 0{,}64$$

(39) Pobeg iz neskončne potencialne jame



$E = \frac{3}{4} V$

$ka = \frac{2\pi}{3} = 120°$

$\psi_1 = A \sin kx \;, \; k = \sqrt{2mE}/\hbar$

$\psi_2 = A \sin ka \; e^{-K(x-a)} \;, \; K = \sqrt{2m(V-E)}/\hbar$

$\psi_1(a) = \psi_2(a)$ ✓

$\psi_1' = \psi_2' \big|_{x=a} \Rightarrow \cancel{A} k \cos ka = \cancel{A} \sin ka \, (-K)$

$\Rightarrow \tan ka = -\frac{k}{K} = \tan\frac{2\pi}{3} = -\sqrt{3}$

$\Rightarrow Ka = -\frac{ka}{\tan ka} = -\frac{2\pi}{3(-\sqrt{3})} = \frac{2\pi}{3\sqrt{3}}$

- Verjetnost, da se delec nahaja zunaj jame

$P(x>a) = \int_a^\infty |\psi_2|^2 dx \;, \; P(x \in (0,\infty)) = \underbrace{\int_0^a |\psi_1|^2 dx}_{I_1} + \underbrace{\int_a^\infty |\psi_2|^2 dx}_{I_2} = 1$

$I_1 = A^2 \int_0^a \sin^2 kx \, dx = \frac{A^2}{2} \int_0^a (1 - \cos 2kx) \, dx = \frac{A^2}{2}\left(a - \frac{\sin 2ka}{k}\right)$

$I_2 = A^2 \sin^2 ka \int_a^\infty e^{\overbrace{-2K(x-a)}^{t}} dx = A^2 \sin^2 ka \frac{1}{2K}$

$P(x>a) = \frac{I_2}{I_1 + I_2} = \frac{1}{1 + \frac{I_1}{I_2}} = \frac{1}{1 + \frac{Ka(1 - \sin 2ka/2ka)}{\sin^2 ka}} = \underline{0{,}39}$

ATOM VODIKA

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(r) \quad , \quad V = -\frac{\alpha \hbar c}{r}$$

- za sferično simetrične potenciale $\psi_{nlm} = \underbrace{R_{nl}(r)}_{\text{radialni del valovne funkcije}} \underbrace{Y_{lm}(\theta,\varphi)}_{\text{sferični harmoniki}}$

$$R_{nl} = \sqrt{\left(\frac{2}{n r_B}\right)^3 \frac{n-l-1}{2n\,(n+l)!}}\; e^{-\rho/2} \rho^l L_{n-l-1}^{2l+1}(\rho) \quad , \quad \rho = \frac{2r}{n r_B}$$

$n = 1, 2, 3, \ldots$

$l = 0, \ldots, n-1$

L_{n-l-1}^{2l+1}: generalizirani Laguerrovi polinomi

r_B: Bohrov radij $\quad r_B = \frac{\hbar c}{\alpha m_e c^2}$

$$Y_{lm}(\theta, \varphi) = (-1)^m \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}}\; P_{lm}(\cos\theta) e^{im\varphi}$$

$$\hat{H} \psi_{nlm} = E_n \psi_{nlm} \quad , \quad E_n = -\frac{mc^2 \alpha^2}{2n^2} \quad , \quad -E_1 = E_{Ry} = \underline{13.6\,eV}$$

- Lastne valovne funkcije so ortonormirane

$$\int \psi^*_{n'l'm'} \psi_{nlm}\, dV = \delta_{nn'} \delta_{ll'} \delta_{mm'}$$

$$\int Y^*_{l'm'} Y_{lm}\, d\Omega = \int_{-1}^{1} \int_0^{2\pi} Y^*_{l'm'} Y_{lm}\, d(\cos\theta)\, d\varphi = \delta_{ll'} \delta_{mm'}$$

$$\int_0^\infty R_{n'l'} R_{nl}\, r^2 dr = \delta_{nn'}$$

- Sferični harmoniki

$l = 0$, $Y_{00} = \frac{1}{\sqrt{4\pi}}$

$l = 1$, $m = \pm 1, 0$, $\quad Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta$, $\quad Y_{1\pm1} = \mp \sqrt{\frac{3}{8\pi}} \sin\theta\, e^{\pm i\varphi}$

$l=2$ $\qquad Y_{20} = \sqrt{\frac{5}{8\pi}}\left(3\cos^2\theta - 1\right)$

$m = 0, \pm 1, \pm 2$ $\qquad Y_{2\pm1} = \mp\sqrt{\frac{15}{8\pi}}\sin\theta\cos\theta\, e^{\pm i\varphi}$

$\downarrow$

v splošnem je

degeneracija $2l+1$ $\qquad Y_{2\pm2} = \sqrt{\frac{15}{32\pi}}\sin^2\theta\, e^{\pm 2i\varphi}$

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(54) Atom vodika : $\psi = \frac{1}{2\sqrt{\pi}} f(r,\theta)\left(\cos\varphi + i\sqrt{3}\sin\varphi\right)$

$\hat{L}_z = -i\hbar\frac{d}{d\varphi}$, $\langle L_z\rangle = ?$, $\int_0^\infty\int_{-1}^{1} f^2 r^2 dr\, d(\cos\theta) = 1$

$$\langle L_z\rangle = \int \psi^* \hat{L}_z \psi\, dV$$

$$= \frac{-i\hbar}{4\pi}\underbrace{\int f^2 r^2 dr\, d\cos\theta}_{1}\int_0^{2\pi}\left(\cos\varphi - i\sqrt{3}\sin\varphi\right)\left(-\sin\varphi + i\sqrt{3}\cos\varphi\right)d\varphi$$

$$= -\frac{i\hbar}{4\pi}\left(\int_0^{2\pi}(\underbrace{\cos^2\varphi + \sin^2\varphi}_{1})\, i\sqrt{3}\, d\varphi + \underbrace{\int_0^{2\pi} 2\sin\varphi\cos\varphi\, d\varphi}_{0}\right)$$

$$= \frac{\hbar\, \cancel{2\pi}\sqrt{3}}{\cancel{4\pi}\, 2} = \frac{\hbar\sqrt{3}}{2}.$$

ROTATOR: VRTILNA KOLIČINA, OPERATORJI, VALOVNE FUNKCIJE in PRIČAKOVANE VREDNOSTI

$$\vec{L} = \vec{r} \times \vec{p} \longrightarrow \hat{L}_i = -i\hbar\, \varepsilon_{ijk}\, x_j \frac{d}{dx_k}$$

- iz kartezičnih v sferične koordinate

$$x = R \sin\theta \cos\varphi \qquad x_i = (x, y, z), \quad y_i = (R, \theta, \varphi)$$

$$y = R \sin\theta \sin\varphi$$

$$z = R\cos\theta \qquad \text{Jacobi matrika } J_{ij} = \frac{dx_i}{dy_j}$$

$$\frac{d}{dx_i} = \frac{d}{dy_j}\frac{dy_j}{dx_i} = (J^{-1})_{ji} \frac{d}{dy_j}$$

$$\Rightarrow \hat{L}_x = i\hbar\left(\sin\varphi \frac{d}{d\theta} + \frac{\cos\varphi}{\tan\theta}\frac{d}{d\varphi}\right)$$

$$\hat{L}_y = i\hbar\left(-\cos\varphi \frac{d}{d\theta} + \frac{\sin\varphi}{\tan\theta}\frac{d}{d\varphi}\right)$$

$$\hat{L}_z = -i\hbar\left(\frac{d}{d\varphi}\right)$$

$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 = -\hbar^2\left(\frac{d^2}{d\theta^2} + \cot\theta \frac{d}{d\theta} + \frac{1}{\sin^2\theta}\frac{d^2}{d\varphi^2}\right)$$

Lastne funkcije $\hat{L}^2$ in $\hat{L}_z$ so sferični harmoniki Y_{lm}

$$Y_{lm} = (-1)^m \sqrt{\frac{2l+1}{4\pi}\frac{(l-m)!}{(l+m)!}}\, P_l^m(\cos\theta)\, e^{im\varphi}, \qquad \int_\Omega Y_{l'm'}^* Y_{lm} = \delta_{l'l}\,\delta_{m'm}$$

Legendrovi polinomi

$$P_1^1 = \sqrt{1-x^2}$$

$$P_0^0 = 1, \quad P_1^0 = x$$

$$P_1^{-1} = -\frac{1}{2}P_1^1$$

LASTNE VREDNOSTI:

$$\hat{L}^2 Y_{lm} = \hbar^2 l(l+1) Y_{lm}$$

$$\hat{L}_z Y_{lm} = \hbar m Y_{lm}$$

PRIMER / ROTATOR : $\psi = \frac{1}{\sqrt{2}}\left(Y_1^0 + Y_1^1\right)$

DIREKTNI RAČUN $= \sqrt{\frac{3}{8\pi}}\left(c_\theta - \frac{1}{\sqrt{2}} s_\theta e^{i\varphi}\right)$

- $L_x = i\hbar\left(s_\varphi \partial_\theta + \frac{c_\varphi}{t_\theta}\partial_\varphi\right)$

$$\langle L_x\rangle = \frac{3i\hbar}{8\pi}\int_0^{2\pi}\int_{-1}^{1}\left(c_\theta - \frac{1}{\sqrt{2}} s_\theta e^{-i\varphi}\right)\left(\frac{e^{i\varphi}-e^{-i\varphi}}{2i}\left(-s_\theta - \frac{1}{\sqrt{2}} c_\theta e^{i\varphi}\right)\right.$$
$$\left. + \frac{e^{2i\varphi}+\cancel{1}\cancel{e^{-i\varphi}}}{2}\,\frac{c_\theta}{\cancel{s_\theta}}\left(-\frac{i}{\sqrt{2}}\cancel{s_\theta}\,\cancel{e^{i\varphi}}\right)\right) d\varphi\, dc_\theta$$

- integriramo po $d\varphi$ in upoštevamo: $\int_0^{2\pi} e^{\pm i\varphi} d\varphi = 0$, $\int_0^{2\pi} d\varphi = 2\pi$

$$= \frac{3\cancel{i}\hbar}{\cancel{8\pi}4}\int_{-1}^{1}\left(\frac{-\cancel{i}\,c_\theta^2}{2\sqrt{2}} + \frac{s_\theta^2}{2\sqrt{2}\,\cancel{i}} + \frac{c_\theta^2}{2\sqrt{2}\,\cancel{i}}\right)\cancel{2\pi}\, d\cos\theta$$

$$= \frac{3\hbar}{4\cdot 2\sqrt{2}}\underbrace{\int_{-1}^{1}\left(1 + c_\theta^2\right) dc_\theta}_{2+\frac{2}{3}=\frac{8}{3}} = \frac{\hbar}{\sqrt{2}}.$$

- $L_y = i\hbar\left(-c_\varphi \frac{d}{d\theta} + s_\varphi \frac{c_\theta}{s_\theta}\frac{d}{d\varphi}\right)$

$$\langle L_y\rangle = \frac{3i\hbar}{8\pi}\int_0^{2\pi}\int_{-1}^{1}\left(c_\theta - \frac{1}{\sqrt{2}} s_\theta e^{-i\varphi}\right)\left(-\frac{e^{i\varphi}+e^{-i\varphi}}{2}\left(-s_\theta - \frac{1}{\sqrt{2}} c_\theta e^{i\varphi}\right)\right.$$
$$\left. + \frac{e^{2i\varphi}-\cancel{1}\cancel{e^{-i\varphi}}}{2\cancel{i}}\,\frac{c_\theta}{\cancel{s_\theta}}\left(-\frac{\cancel{i}}{\sqrt{2}}\cancel{s_\theta}\,\cancel{e^{i\varphi}}\right)\right)$$

$$= \frac{3i\hbar}{8\sqrt{2}}\underbrace{\int_{-1}^{1}\left(c_\theta^2 - s_\theta^2 + c_\theta^2\right) dc_\theta}_{\int 3c_\theta^2 - 1\, dc_\theta = 3\frac{2}{3} - 2 = 0} = \underline{\underline{0.}}$$

• $L_z = -i\hbar \frac{d}{d\varphi}$

$$\langle L_z \rangle = \frac{-i\hbar\, 3}{8\pi} \int_0^{2\pi}\int_{-1}^{1} \left(\overset{0}{c_\theta} - \frac{1}{\sqrt{2}} s_\theta e^{-i\varphi} \right) \frac{-i}{\sqrt{2}} s_\theta e^{i\varphi} \, d\varphi \, dc_\theta$$

$$= \frac{3\hbar \cdot \cancel{2\pi}}{8\cancel{\pi} \cdot \cancel{2}} \cdot \int_{-1}^{1} (1 - c_\theta^2)\, dc_\theta = \frac{3\hbar}{8} \left(\underbrace{2 - \frac{2}{3}}_{4/3} \right) = \frac{\hbar}{2} .$$