Gibanje nabitega delca v elektro-magnetnem polju

Klasično: $\vec{F} = m\vec{a}$ ← pospešek v lab. času $\frac{d^2\vec{x}}{dt^2}$

E.M. sila $= e(\vec{E} + \vec{v} \times \vec{B})$

Relativistično: $F^\mu = \frac{dp^\mu}{d\tau}$ ← rel. gibalna količina; $d\tau$ ← lastni čas

$cp^\mu = \gamma mc^2(1, \beta)$ (samo v x-smeri)

$$c^2 d\tau^2 = dx^\mu dx_\mu = c^2dt^2 \underbrace{- dx^2 - dy^2 - dz^2}_{-d\vec{x}\cdot d\vec{x}}$$

• v x-smeri (in v splošnem) $= c^2dt^2\left(1 - \left(\frac{dx}{c\,dt}\right)^2\right) = c(1-\beta^2)dt^2$

$= c^2\frac{dt^2}{\gamma}$

$d\tau = \frac{dt}{\gamma}$ ← laboratorijski čas

lastni čas (čas ki teče za opazovalca, ki je na miru, sicer je to invarianta za vse opazovalce v inercialnih sistemih)

$p^\mu = m\frac{dx^\mu}{d\tau}$ in $F^\mu = \frac{dp^\mu}{d\tau} = \left(\frac{e}{c}\gamma\vec{E}\cdot\vec{v}, e\gamma(\vec{E} + \vec{v}\times\vec{B})\right)$

relativistična EM sila

$$\frac{dp^\mu}{dt} = e\gamma\left(\vec{E}\cdot\frac{\vec{v}}{c}, \vec{E} + \vec{v}\times\vec{B}\right), \quad d\tau = \frac{dt}{\gamma}.$$

(23) Gibanje v električnem polju $\vec{E}$, $\vec{B}=0$

Reševali bomo na <u>dva</u> različna načina:

a) z lastnim časom τ,

b) z laboratorijskim časom t,

c) z ohranitvijo energije.

$\vec{E} = (E, 0, 0)$, $E = 1{,}75\ \frac{kV}{m}$.

a) Zapišimo enačbe gibanja in začetne pogoje za posamezne komponente p^μ: $p^0 = \gamma mc$, $p^1 = \gamma m v$

$$\frac{dp^0}{d\tau} = \dot{p}^0 = e\gamma E \frac{v}{c} = \underbrace{\frac{eE}{mc}}_{\alpha} \underbrace{\gamma m v}_{p^1} = \alpha p^1$$

$$\frac{dp^1}{d\tau} = \dot{p}^1 = e\gamma E = \frac{eE}{mc}\gamma mc = \alpha p^0$$

Dobili smo sistem:

$$\dot{p}^0 = \alpha p^1, \quad p^0(0) = mc,$$

$$\dot{p}^1 = \alpha p^0, \quad p^1(0) = 0.$$

Rešimo ga tako, da ostanimo

$$\ddot{p}^0 = \alpha \dot{p}^1 = \alpha^2 p^0, \quad p^0 = A \operatorname{ch} \alpha\tau + B \operatorname{sh} \alpha\tau$$

$$\ddot{p}^1 = \alpha \dot{p}^0 = \alpha^2 p^1, \quad p^0(0) = A = mc$$

Dobili smo: $p^0(\tau) = mc\,\mathrm{ch}(\alpha\tau)$ in od tod sledi

$$p^1(\tau) = \frac{1}{\alpha}\dot{p}^0 = \frac{1}{\alpha} mc\,\alpha\,\mathrm{sh}(\alpha\tau) + B\,\mathrm{ch}\,\alpha\tau, \text{ ki}$$

bo ustrezal robnemu pogoju

$$p^1(0) = 0 = B, \text{ če } B = 0.$$

Rešitev z lastnim časom: $p^0(\tau) = mc\,\mathrm{ch}(\alpha\tau)$,

$p^1(\tau) = mc\,\mathrm{sh}(\alpha\tau)$.

S temi rešitvami dobimo $t(\tau)$ in $x(\tau)$, torej laboratorijski čas in prepotovano pot, ker $p^1 = m\frac{dx^1}{dt}$

$$p^0 = mc\frac{dt}{d\tau} = mc\,\mathrm{ch}(\alpha\tau) \Rightarrow \int_0^t dt' = \int_0^\tau \mathrm{ch}(\alpha\tau') \Rightarrow t = \frac{1}{\alpha}\,\mathrm{sh}\,\alpha\tau'\Big|_0^\tau$$

$$\Rightarrow \alpha t = \mathrm{sh}(\alpha\tau).$$

Iz podatkov vemo: $\alpha t = \frac{eEtc}{mc^2} = \frac{1.75\,\mathrm{keV/m}\; 10^{-6}\,\mathrm{s}\; 3\cdot 10^{8}\,\mathrm{m/s}}{511\,\mathrm{keV}} = \underline{1{,}03}$

Prepotovano pot dobimo z integracijo p^1 po τ:

$$p^1 = m\frac{dx}{d\tau} = mc\,\mathrm{sh}(\alpha\tau)$$

$$\int_0^L dx = c\int_0^\tau \mathrm{sh}(\alpha\tau')\,d\tau' \Rightarrow L = \frac{c}{\alpha}\,\mathrm{ch}(\alpha\tau')\Big|_0^\tau = \frac{c}{\alpha}\left(\mathrm{sh}(\alpha\tau) - 1\right)$$

• Z uporabo $ch^2 x - sh^2 x = 1$, dobimo:

$$L = \frac{c}{\alpha}\left(\sqrt{sh^2(\alpha\tau)+1} - 1\right) = \frac{c}{\alpha}\left(\sqrt{(\alpha t)^2+1} - 1\right) = \underline{126\ m}.$$

b) z laboratorijskim časom t

$$\dot{p}^1 = \alpha p^0 \quad , \quad p^0 = m\frac{dx^0}{d\tau} = mc\frac{dt}{d\tau}$$

$$p^1 = \gamma m v$$

Poberemo skupaj in dobimo

$$\frac{d}{d\tau}(\gamma m v) = \alpha \frac{d}{dt}(mc\, dt)$$

$$\gamma v(t) = \alpha c t \quad \text{oz.:} \quad \gamma\beta = \alpha t,$$

nas pa zanima $\beta(t)$: $\frac{\beta^2}{1-\beta^2} = (\alpha t)^2$,

$$\Rightarrow \beta = \frac{\alpha t}{\sqrt{1+(\alpha t)^2}}$$ ... hitrost izražena z laboratorijskim časom t

$$\beta = \frac{1}{c}\frac{dx}{dt} \quad \int_0^L dx = c\int_0^t \beta(t')\,dt' = \frac{c}{\alpha}\int_0^t \frac{\alpha t'\, d(\alpha t')}{\sqrt{\underbrace{1+(\alpha t')^2}_{u^2}}}$$

$$2u\,du = 2\alpha^2 t'\,dt' \qquad = \frac{c}{\alpha}\int_1^{\sqrt{1+(\alpha t)^2}} \frac{u\,du}{\sqrt{u^2}} = \frac{c}{\alpha}\left(\sqrt{(\alpha t)^2+1} - 1\right)$$

$$\text{oz.:} \quad L = \frac{c}{\alpha}\left(\sqrt{(\alpha t)^2+1} - 1\right).$$

c) Vsa energija, ki jo odda električno polje, torej $eU = eEL$, se pretvori v relativistično kinetično energijo $T = (\gamma - 1)mc^2$.

spomnimo: $(\gamma\beta)^2 = \gamma^2 - 1 = (\alpha t)^2$,

$$\Rightarrow T = mc^2\left(\sqrt{(\alpha t)^2 + 1} - 1\right) = eEL,$$

oz.: $L = \frac{mc}{eE} \cdot c\left(\sqrt{(\alpha t)^2 + 1} - 1\right)$, $\alpha = \frac{eE}{mc}$.

MAGNETNO POLJE $\rightarrow$ RELATIVISTIČNO KROŽENJE

(27) $E = 0$, $\vec{B} = B_z = (0, 0, B)$, $U = 1{,}02\ MV$, $B = 0{,}048\ T$

Najprej potrebujemo enačbe gibanja, potem začetne pogoje.

$$\dot{p}^\mu = e\gamma\left(\vec{E} \cdot \frac{\vec{v}}{c}, \vec{E} + \vec{v} \times \vec{B}\right) = \left(0, e\gamma\, \vec{v} \times \vec{B}\right)$$

$$\vec{v} \times \vec{B} = \begin{vmatrix} i & j & k \\ v_x & v_y & v_z \\ 0 & 0 & B \end{vmatrix} = (v_y B, -v_x B, 0)$$

$\mu = 0$: $\dot{p}^0 = 0 \Rightarrow p^0 = \text{konst.} = \gamma mc$

$\Downarrow$

$\gamma = \text{konst.}$ ali $\beta = \text{konst.}$ (KROŽENJE)

- Ostaneta še enačbi v x in y smeri:

$$\dot{p}^1 = e\gamma v_y B = \frac{eB}{m} p^2 \qquad p^1 = p_x = \gamma m v_x$$

$$\dot{p}^2 = -e\gamma v_x B = -\underbrace{\frac{eB}{m}}_{\omega} p^1 \qquad p^2 = p_y = \gamma m v_y$$

- Sistem rešujemo podobno kot prej, $\frac{d}{d\tau}$, $2 \to 1$

$$\ddot{p}^1 = \omega \dot{p}^2 = -\omega^2 p^1 \Rightarrow p^1 = A\cos(\omega\tau) + B\sin(\omega\tau)$$

$$p^2 = \frac{1}{\omega}\dot{p}^1 = -A\sin(\omega\tau) + B\cos(\omega\tau).$$

- Začetni pogoj ob $\tau = t = 0$: $p^2(0) = 0$

- $p^1(\tau=0) = \frac{1}{c}\sqrt{T(T+2mc^2}} = p^1_z,$

$$p^2(\tau=0) = -B = 0 \Rightarrow B = 0,$$

$$p^1(\tau=0) = A = p^1_z.$$

Rešitev:

$$p^1(\tau) = p^1_z \cos(\omega\tau) = m\frac{dx}{d\tau}$$

$$p^2(\tau) = -p^1_z \sin(\omega\tau) = m\frac{dy}{d\tau}$$

$$\left.\begin{aligned} x(\tau) &= x_0 + \frac{p^1_z}{m\omega}\sin(\omega\tau) \\ y(t) &= y_0 - \frac{p^1_z}{m\omega}\cos(\omega\tau) \end{aligned}\right\} \quad r = \frac{p^z_1}{m\omega} = \frac{p^z_1}{eB}$$

$$\hookrightarrow \boxed{p = erB}$$

$$\Rightarrow 2r = \frac{2e p^z_1}{B} = \frac{2e}{Bc}\sqrt{T(T+2mc^2)} = \underline{\underline{0{,}2\,m}}.$$