(50) Harmonski oscilator : $H = \hat{T} + \hat{V}$, $\hat{V} = \frac{1}{2} m\omega^2 \hat{x}^2$

$H \psi_n = E_n \psi_n$, $E_n = \hbar\omega (n + 1/2)$, $n = 0, 1, 2, \ldots$

$$\psi_n = \frac{1}{\sqrt{2^n n! \sqrt{\pi} a}} e^{-y^2/2} H_n(y) \quad , \; y = \frac{x}{a} \; , \; a = \sqrt{\frac{\hbar}{m\omega}}$$

↳ ortonormiran sistem : $\int_{-\infty}^{\infty} dx \, \psi_n^* \psi_m = \delta_{nm}$, dobimo jih

z generacijsko formulo : $H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} \left(e^{-y^2} \right)$

n sodi : $H_n(-y) = H_n(y)$, n lihi : $H_n(-y) = -H_n(y)$

n	0	1	2	3	4	5
$H_n(y)$	1	$2y$	$4y^2 - 2$	$8y^3 - 12y$	$16y^4 - 48y^2 + 12$	$32y^5 - 160y^3 + 120y$

(50) Ob času $t = 0$, se valovna funkcija nahaja v

$$\psi(x,0) = A \left(2y^2 + iy \right) e^{-y^2/2}$$

$$\psi_0 = \frac{1}{\sqrt{\sqrt{\pi} a}} e^{-y^2/2} \; , \; \psi_1 = \frac{1}{\sqrt{2\sqrt{\pi} a}} e^{-y^2/2} \cdot 2y = \sqrt{2} y \psi_0 ,$$

$$\psi_2 = \frac{1}{\sqrt{4 \cdot 2! \sqrt{\pi} a}} e^{-y^2/2} 2(2y^2 - 1) = \frac{2y^2 - 1}{\sqrt{2}} \psi_0 .$$

$$\psi = B \left(2y^2 + iy \right) \psi_0 = c_0 \psi_0 + c_1 \psi_1 + c_2 \psi_2 \; , \; c_i = ?$$

$$\sum c_i^2 = 1$$

vemo : $2y^2 \psi_0 = \sqrt{2} \psi_2 + \psi_0$, $iy \psi_0 = \frac{i}{\sqrt{2}} \psi_1$.

↪ iz prejšnjih zvez smo dobili

$$\psi = B\left(\sqrt{2}\,\psi_2 + \psi_0 + \frac{i}{\sqrt{2}}\psi_1\right), \quad c_0 = 1/B,\ c_1 = \frac{iB}{\sqrt{2}},\ c_2 = B\sqrt{2}$$

$$\Rightarrow B^2\left(1 + \frac{1}{2} + 2\right) = 1 = B^2\left(\frac{7}{2}\right), \quad B = \sqrt{\frac{2}{7}}.$$

$$\langle E\rangle = \sum_n |c_n|^2 E_n = c_0^2 E_0 + c_1^2 E_1 + c_2^2 E_2, \quad E_n = \hbar\omega\left(n + \frac{1}{2}\right)$$

$$= \frac{2}{7}(\hbar\omega)\left(1\cdot\frac{1}{2} + \frac{1}{2}\,\frac{3}{2} + \cancel{2}\cdot\frac{5}{\cancel{2}}\right) = \frac{\cancel{2}}{7}\hbar\omega\frac{1}{\cancel{4}_2}\underbrace{(2+3+20)}_{25}$$

$$\underline{= \frac{25}{14}\hbar\omega.}$$

Časovni razvoj sledi iz: $i\hbar\frac{d\psi}{dt_n} = H\psi_n = E_n\psi_n$

$$\psi(x,t) = \sqrt{\frac{2}{7}}\left(\psi_0 e^{-i\frac{E_0 t}{\hbar}} + \frac{i}{\sqrt{2}}\psi_1 e^{-i\frac{E_1 t}{\hbar}} + \sqrt{2}\,\psi_2 e^{-i\frac{E_2 t}{\hbar}}\right)$$

$$= \sqrt{\frac{2}{7}}\left(\psi_0 e^{-i\frac{\cancel{\hbar}\omega t}{\cancel{\hbar}2}} + \frac{i}{\sqrt{2}}\psi_1(x) e^{-i\frac{3}{2}\omega t} + \sqrt{2}\,\psi_2 e^{-i\frac{5}{2}\omega t}\right).$$

(61) $m = 10^{-30}\,kg$

$k = 50\ \frac{eV}{nm^2}$

$a = 0{,}3\ nm$

$\alpha = \sqrt[4]{\frac{mk}{\hbar}}$

$$\psi(x) = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\, e^{-\frac{\alpha^2}{2}(x-a)^2},$$

premaknjen harmonski oscilator

Lastne funkcije:

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}k\hat{x}^2, \quad \alpha = \sqrt[4]{\frac{mk}{\hbar}},$$

$$\psi_n = \sqrt{\frac{\alpha}{2^n n! \sqrt{\pi}}}\, H_n(y)\, e^{-y^2/2}, \quad y = \alpha x.$$

Uporabimo formulo za koeficiente c_n v $\psi = \sum_n c_n \psi_n$

$$c_n = \int_{-\infty}^{\infty} \psi_n^* \psi\, dx = \frac{\cancel{\alpha}}{\sqrt{2^n n!\, \pi}} \underbrace{\int_{-\infty}^{\infty} \underbrace{dx}_{=\frac{1}{\cancel{\alpha}}dy}\, H_n(y)\, \underbrace{e^{-y^2/2 - y^2/2 + yy_0 - \frac{y_0^2}{2}}}_{e^{-y^2 + yy_0 - y_0^2/2}}}_{\sqrt{\pi}\, y_0^n\, e^{-y_0^2/4}}$$

$$= \frac{1}{\sqrt{2^n n!}}\, y_0^n\, e^{-y_0^2/4}.$$

Preverimo: $\sum_{n=0}^{\infty} c_n^2 = \sum_{n=0}^{\infty} \frac{1}{2^n n!}(y_0^2)^n e^{-y_0^2/2} = \cancel{e^{-y_0^2/2}} \cdot \cancel{e^{y_0^2/2}} = 1.$

Sedaj lahko izračunamo povprečno energijo $\langle E \rangle$

$$\langle E \rangle = \sum_{n=0}^{\infty} c_n^2 E_n, \quad E_n = \hbar\omega(n + 1/2), \quad \omega = \sqrt{\frac{k}{m}}$$

$$= \sum_{n=0}^{\infty} \hbar\omega\, c_n^2 (n + 1/2) = \hbar\omega\Big(e^{-y_0^2/2} \underbrace{\sum_{n=0}^{\infty} \frac{n}{n!}\Big(\frac{y_0^2}{2}\Big)^n}_{\underbrace{\sum_{n=1}^{\infty} \frac{1}{(n-1)!}\left(\frac{y_0^2}{2}\right)^{n-1} \frac{y_0^2}{2}}_{e^{y_0^2/2} \cdot \frac{y_0^2}{2}}} + \frac{1}{2}\Big)$$

$$= \hbar\omega\left(\frac{y_0^2}{2} + \frac{1}{2}\right) = \frac{\hbar\omega}{2}(y_0^2 + 1).$$

Izračunajmo brezdimenzijski parameter y_0 $= 6\cdot 10^{18}\,\cancel{eV}$

$$y_0 = a\alpha = a\sqrt[4]{\frac{mc^2k}{(\hbar c)^2}} = a\sqrt[4]{\frac{10^{-30}\;9\cdot 10^{16}\,\frac{kg\,m^2}{s^2}\;50\,\frac{\cancel{eV}}{nm^2}}{200^2\,\cancel{eV^2}\,nm^2}}$$

$$= a\sqrt[4]{\frac{5\cdot 10^{-22}\,\cancel{6}\,3\cdot 10^{18}}{\cancel{4}\,10^4\,nm^4}} = 5\,nm^{-1}\,a = 1{,}55$$

$$\boxed{y_0 = 1{,}55}$$

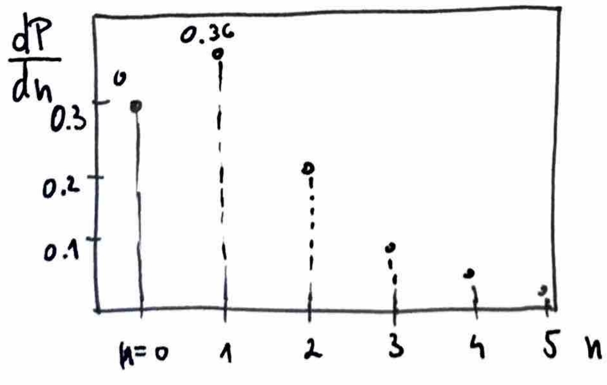
$$\Rightarrow \quad \langle E\rangle = \frac{\hbar\omega}{2}\left(1+y_0^2\right) = 3{,}2\,eV. \quad , \quad \hbar\omega = \sqrt{\frac{(\hbar c)^2 k}{mc^2}}$$

S kolikšno verjetnostjo izmerijo: $E = \frac{3}{2}\hbar\sqrt{\frac{k}{m}}$

$\| \quad n+\frac{1}{2} = \frac{3}{2} \Rightarrow \underline{n=1.}$

$$P(E=E_1) = |c_1|^2 = \frac{1}{2^1 1!}\,y_0^2\,e^{-y_0^2/2}$$

$$= \frac{y_0^2}{2}\,e^{-y_0^2/2} = 0{,}36.$$



Pri posamezni meritvi izmerimo E_n z verjetnostjo c_n^2.

Povprečje je $\langle E\rangle = \sum_n c_n^2 E_n$

<u>DODATEK</u>: Povprečno energijo $\langle E\rangle$ lahko dobimo tudi z direktnim računom: $\langle E\rangle = \int_{-\infty}^{\infty} dx\, \psi^*\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi$

To lahko naredimo direktno z: $\psi = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\, e^{-\frac{\alpha^2(x-a)^2}{2}}$

$$\psi' = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\left(-\alpha^2(x-a)\right) e^{-\frac{\alpha^2(x-a)^2}{2}}, \quad \psi'' = \sqrt{\frac{\alpha}{\sqrt{\pi}}}\left(-\alpha^2 - \alpha^2(x-a)\cdot(-\alpha^2(x-a))\right) e^{-\frac{\alpha^2(x-a)^2}{2}}$$

$$\langle E\rangle = \cancel{\left[-\frac{\hbar^2}{2m}\right]}\int_{-\infty}^{\infty} dx\, \frac{\alpha}{\sqrt{\pi}}\left(\frac{\hbar^2\alpha^2}{2m}\left(1 - \alpha^2(x-a)^2\right) + \frac{km}{2m}x^2\right) e^{-\alpha^2(x-a)^2}$$

$$= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-t^2}\, \frac{(\hbar\alpha)^2}{2m}\left(1 - t^2 + \alpha^2\left(\frac{t}{\alpha}+a\right)^2\right) \qquad \alpha(x-a) = t, \quad \alpha\, dx = dt.$$

- Kvadratični člen se pokrajša, linearni $\int_{-\infty}^{\infty} dt \to 0$.

$$\langle E\rangle = \frac{(\hbar\alpha)^2}{2m}\,\frac{1}{\cancel{\sqrt{\pi}}}\underbrace{\int_{-\infty}^{\infty} e^{-t^2} dt}_{\cancel{\sqrt{\pi}}}\left(1 + (\alpha a)^2\right) = \frac{(\hbar\alpha)^2}{2m}\left(1 + y_0^2\right).$$

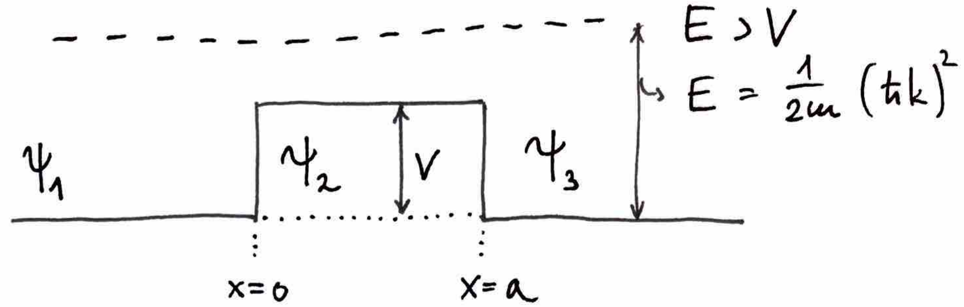
Note: $I = \int_{-\infty}^{\infty} e^{-t^2} dt$ iz: $\int e^{-x^2} dx \int e^{-y^2} dy = I^2$

$$I^2 = \int dx\, dy\, e^{-(x^2+y^2)} = \int_0^{2\pi} d\varphi \int_0^{\infty} dr\, r\, e^{-r^2}, \quad r^2 = u, \quad 2r\,dr = du$$

$$= \frac{\cancel{2}\pi}{\cancel{2}}\cdot\int_0^{\infty} e^{-u} du = \pi \quad \Rightarrow \boxed{I = \sqrt{\pi}.}$$

27 $a = 0{,}4\ \mathrm{nm}$

$E = 0{,}7\ \mathrm{eV}$



$$\psi_1 = Ae^{ikx} + Be^{-ikx}, \quad \psi_2 = Ce^{ik'x} + De^{-ik'x}, \quad \psi_3 = Ee^{ikx},$$

kjer sta: $k = \sqrt{\frac{2mE}{\hbar^2}}$, $k' = \sqrt{\frac{2m(E-V)}{\hbar^2}}$.

Imamo štiri robne pogoje pri $x=0$ (2×) in $x=a$ (2×).

$$\psi_1(0) = \psi_2(0), \quad \psi_1'(0) = \psi_2'(0) \quad \& \quad \psi_2(a) = \psi_3(a), \quad \psi_2'(a) = \psi_3'(a).$$

Sicer so to 4 enačbe za 5 koeficientov, a nas zanima le verjetnost prehoda $T = \left|\frac{E}{A}\right|^2$.

Rešimo sistem:

$$A + B = C + D, \qquad Ce^{ik'a} + De^{-ik'a} = Ee^{ika},$$

$$ik(A-B) = ik'(C-D), \qquad ik'(Ce^{ik'a} - De^{-ik'a}) = ikEe^{ika}.$$

↓ se znebimo B ⇓ dobimo C in D

$$2A = -\frac{k'}{k}(C-D) + C + D, \qquad 2Ce^{ik'a} = E\left(1+\frac{k}{k'}\right)e^{ika},$$

$$2De^{ik'a} = E\left(1-\frac{k}{k'}\right)e^{ika}$$

Torej:

$$A = \frac{1}{2}C\left(1-\frac{k'}{k}\right) + \frac{1}{2}D\left(1+\frac{k'}{k}\right), \qquad C = \frac{1}{2}E\left(1+\frac{k}{k'}\right)e^{i(k-k')a},$$

$$D = \frac{1}{2}E\left(1-\frac{k}{k'}\right)e^{i(k-k')a}.$$

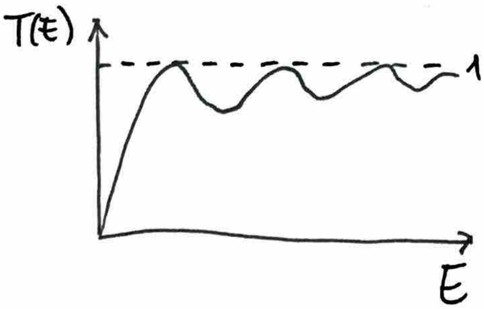
Ko vstavimo C in D v enačbo za A, dobimo:

$$4A = E\left(\underbrace{\left(1+\frac{k'}{k}\right)\left(1+\frac{k}{k'}\right)}_{\frac{(k+k')^2}{kk'}} e^{i(k-k')a} + \underbrace{\left(1-\frac{k'}{k}\right)\left(1-\frac{k}{k'}\right)}_{-\frac{(k-k')^2}{kk'}} e^{i(k+k')a} \right)$$

$$\frac{E}{A} = \frac{4kk'\, e^{-i(k-k')a}}{(k+k')^2 - (k-k')^2 e^{2ik'a}} \Rightarrow T = \left|\frac{E}{A}\right|^2 = \frac{(4kk')^2}{|\quad \dots \quad|^2}$$

Poenostavimo še imenovalec in dobimo:

$$\text{transmisivnost} \rightarrow T = \frac{1}{1+\left(\frac{k'^2-k^2}{2kk'}\right)^2 \sin^2(k'a)} .$$



T ima max pri 1, ko $k'a = n\pi$.

(27) je jama, zato: $k' = \frac{\sqrt{2m(E+V)}}{\hbar}$

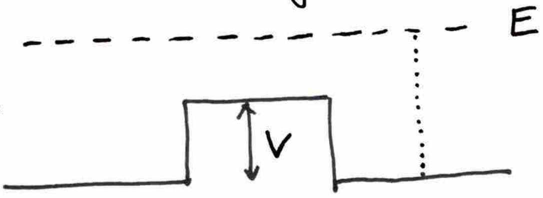
Velja torej: $\frac{2m(E+V)a^2}{\hbar^2} = \pi^2$ za prvi maks. pri $n=1$.

$$V = \frac{\hbar^2\pi^2}{2ma^2} - E = \frac{(\hbar c)^2\pi^2}{2mc^2a^2} - E \cong \frac{10\,(200)^2 (eV\,\cancel{nm})^2}{10^6\,\cancel{eV}\,(0{,}4\,\cancel{nm})^2} - 0{,}7\,eV$$

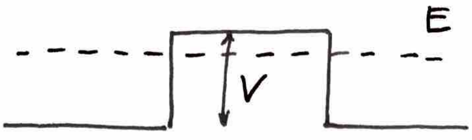
$$= \frac{\cancel{4}\cdot 10^5\, eV}{4\cancel{16}\cdot \cancel{10^4}} - 0{,}7\,eV = (2.23 - 0{,}7)\,eV = \underline{1{,}53\,eV}.$$

(28) Tuneliranje skozi plast

Prej:



Sedaj:



$E > V$ ... sipanje

$E < V$ ... tuneliranje

Veljajo iste formule, le: $k' = \frac{1}{\hbar}\sqrt{2m(E-V)}$

$\Rightarrow \kappa \in \mathbb{R}$, $\kappa = \frac{1}{\hbar}\sqrt{2m(V-E)}$.

$= \frac{i}{\hbar}\sqrt{2m(V-E)} = i\kappa$

$$T = \frac{1}{1 + \left(\frac{k^2 - k'^2}{2kk'}\right)^2 \sin^2(k'a)} = \frac{1}{1 + \cancel{\frac{1}{i^2}} \left(\frac{k^2 + \kappa^2}{2k\kappa}\right)^2 \cancel{i^2}\, \mathrm{sh}^2(\kappa a)}$$

kjer smo uporabili: $\sin(ix) = \frac{e^{i(ix)} - e^{-i(ix)}}{2i} = \frac{i(e^{x} - e^{-x})}{2}$

$= i\,\mathrm{sh}\,x$.

Vemo: $k = \sqrt{\frac{2mc^2E}{(\hbar c)^2}} = 16{,}2\ \mathrm{nm}^{-1}$, $\kappa = \sqrt{\frac{2mc^2(E-V)}{(\hbar c)^2}} = 7{,}3\ \mathrm{nm}^{-1}$.

in $a = \{0.1,\ 1,\ 100\}$ nm

$\Downarrow$

$T = \{0{,}47,\ 1{,}1\cdot 10^{-6},\ 9{,}7\cdot 10^{-631}\}$

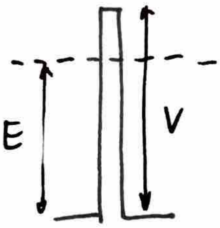
za velike a gre eksponentno: $\mathrm{sh}(\kappa a) \sim \frac{e^{\kappa a}}{2}$

in $T \propto e^{-2\kappa a}$.

32 Prepustnost tanke in visoke potencialne plasti

$$v = 2\cdot 10^{3}\,\frac{m}{s}\,, \quad \beta = \frac{v}{c} = \frac{2}{3}\,\frac{10^{3}}{10^{8}} = \frac{2}{3}\cdot 10^{-5} \ll c \ \ldots \mathrm{NR}$$

$$Va = 10^{-3}\ \mathrm{eV\,nm}\,, \qquad E \simeq \frac{p^2}{2m} \cong \frac{1}{2}mc^2\beta^2$$



- tanka : $\kappa a \ll 1$, $\mathrm{sh}(\kappa a) \sim \kappa a$
- visoka : $V \gg E$.

$$T = \frac{1}{1+\left(\frac{k^2+\kappa^2}{2k\kappa}\right)^2 \mathrm{sh}^2(\kappa a)}\,, \quad k^2 = \frac{1}{\hbar}2mE\,, \quad \kappa^2 = \frac{2m}{\hbar}(V-E)$$

$$= \frac{1}{1+\frac{(\cancel{E}+V-\cancel{E})^2}{2\cancel{4}E\cancel{(V-E)}}\,\frac{\cancel{2}ma^2}{\hbar^2}\cancel{\kappa}} = \frac{1}{1+\frac{mc^2(Va)^2}{2E\hbar^2c^2}}\,, \quad E = \frac{mc^2}{2}\beta^2$$

$$= \frac{1}{1+\left(\frac{Va}{\beta\hbar c}\right)^2}\,, \qquad \frac{aV}{\beta\hbar c} = \frac{\cancel{10^{-3}}\ \cancel{\mathrm{eV\,nm}}}{\frac{2}{3}\cdot\cancel{10^{-5}}\cdot 2\cdot\cancel{10^{2}}\ \cancel{\mathrm{eV\,nm}}} = \frac{3}{4}\,.$$

$$= \frac{1}{1+\frac{9}{16}} = \frac{4\cdot 16}{4\cdot 25} = \frac{64}{100} = \underline{\underline{0{,}64}}\,.$$