

TOWARDS A NEW PARADIGM: RELATIVITY IN CONFIGURATION SPACE

Matej Pavšič

J. Stefan Institute, Ljubljana, Slovenia

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- Introduction

Why a new paradigm?

Persisting puzzles: quantum gravity, problem of time, unification of interactions, the nature of dark matter, dark energy, etc.

From history we know that such situation calls for 'paradigm shift'

We also know that often a formalism is more powerful than initially envisaged

Examples: Hamilton-Jacobi function (hints of wave mechanics)

Clifford algebra (implies spin)

Line element in Minkowski spacetime

(suggests generalization to curved spacetime)

Common: In all those cases the formalism itself pointed to its own generalization!

This introduced important new physics

Proposal: To do something analogous with the formalism describing configurations of physical systems

- Generalising relativity

Configuration space replaces spacetime

Action for a system of point particles

$$I[\dot{X}_i^\mu] = \sum_i \int d\tau [\dot{X}_i^\mu \dot{X}_i^\nu m_i g_{\mu\nu}(X_i^\mu)]^{1/2}$$

The Schild action for a system of point particles

$$I[\dot{X}_i^\mu] = \int d\tau \sum_i \dot{X}_i^\mu \dot{X}_i^\nu \frac{m_i}{k_i} g_{\mu\nu}(X_i^\mu)$$

$$\begin{aligned} \dot{X}_i^\mu &\equiv \dot{X}^{(i\mu)} \equiv \dot{X}^M, & M &= (i\mu) \\ \frac{m_i}{k_i} g_{\mu\nu} &\equiv \frac{M}{K} g_{(i\mu)(j\nu)} \equiv \frac{M}{K} g_{MN} \end{aligned}$$

$$I[X^M] = \int d\tau \dot{X}^M \dot{X}^N \frac{M}{K} g_{MN}(X^M)$$

The Schild action in configuration space C .

It is equivalent to the reparametrization invariant action in C :

$$I[X^M] = M \int d\tau [\dot{X}^M \dot{X}^N g_{MN}(X^M)]^{1/2}$$

$$M^2 = \sum_i m_i^2$$

Action for a point particle:

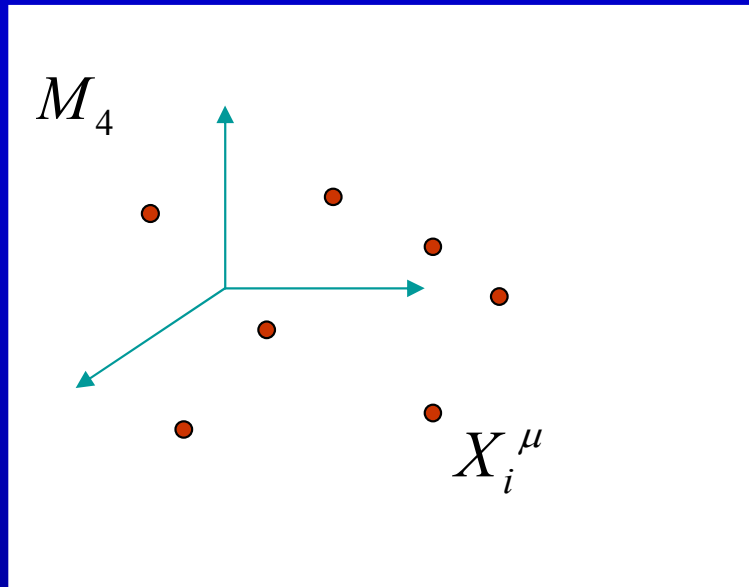
$$I[X^\mu] = \int d\tau m (\dot{X}^\mu \dot{X}^\nu g_{\mu\nu})^{1/2}$$

Gauge fixed action (the Schild action):

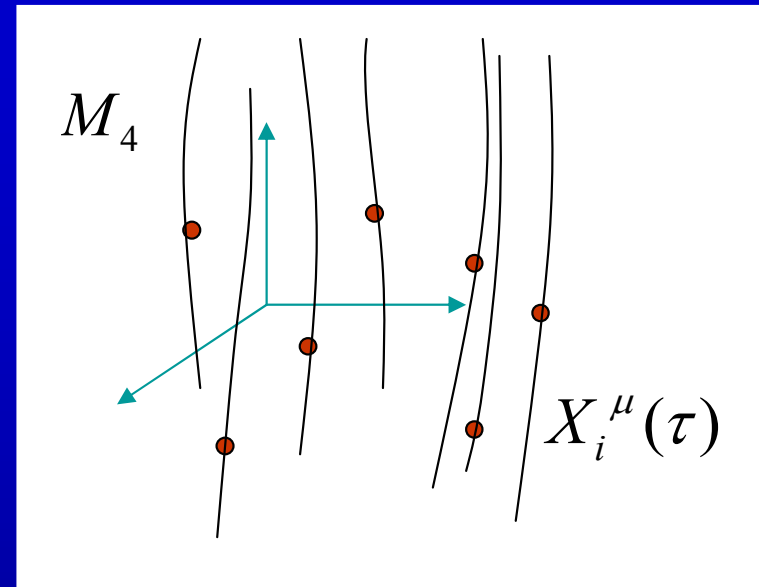
$$I[X^\mu] = \int d\tau \frac{m}{k} \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$$

$$\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} = k^2 \quad \text{Is constant}$$

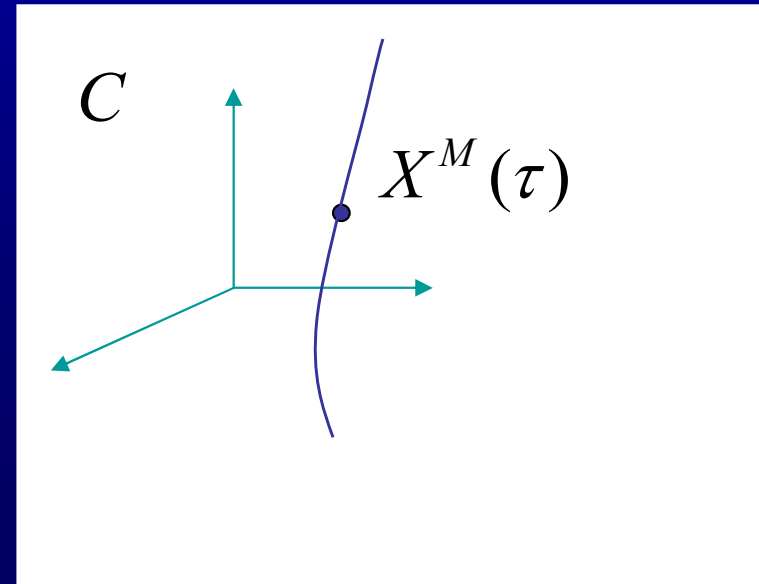
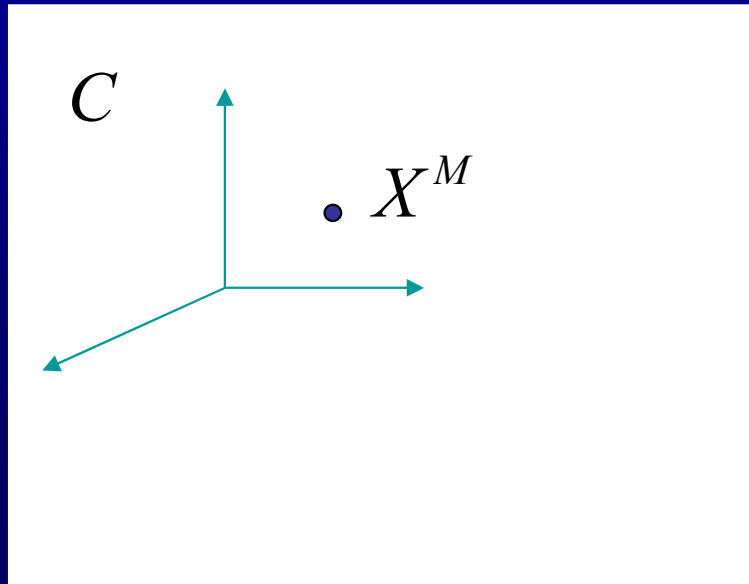
'Instantaneous' configuration in M_4



'Evolution' of configuration in M_4

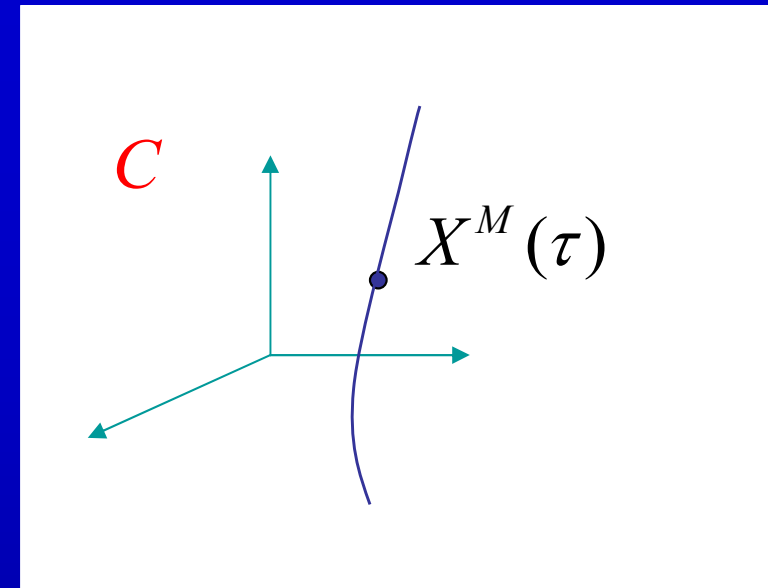


Representation in configuration space C



A given configuration, described by coordinates X^M , traces a world line $X^M(\tau)$ in configuration space C .

We assume that, in general, the metric g_{MN} of C can be arbitrary. The world line is a geodesic of C .



In particular, for the block diagonal metric

$$G_{MN} \equiv G_{(i\mu)(j\nu)} = \begin{pmatrix} g_{\mu\nu}(x_1) & 0 & 0 & \dots \\ 0 & g_{\mu\nu}(x_2) & 0 & \dots \\ 0 & 0 & g_{\mu\nu}(x_3) & \dots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

we obtain the ordinary relativistic theory for a many particle system in a gravitational field.

By allowing for more general metric, we go beyond the ordinary theory.

Now we have **general relativity in configuration space**, given by the action

$$I[X^M, G_{MN}] = I_m + I_g$$

$$I_m = \int d\tau M (G_{MN} \dot{X}^M \dot{X}^N)^{(1/2)} = \int d\tau M (G_{MN} \dot{X}^M \dot{X}^N)^{(1/2)} \delta^D(x - X(\tau)) d^D x$$

$$I_g = \frac{1}{16\pi G_D} \int d^D x \sqrt{|g^{(C)}|} R^{(C)}$$

Formally we arrived at a theory which is analogous to **Kaluza-Klein theory**.

Configuration space C is a higher dimensional space.

A 4-dimensional subspace, associated with a chosen particle, is spacetime M_4

The concept of configuration space can take place:

In macrophysics: The theory predicts deviations from the conventional theory. New effects. Possible theoretical basis for MONDs.

In microphysics: We can consider a configuration space associated with extended objects, e.g., strings and branes. This gives fundamental interactions (gravity + YM)

Configuration space can be:

Finite dimensional: Associated with a system of point particles, dust,

Infinite dimensional: Associated with a fluid, string, brane

Equations of motion for a configuration of point particles

Quadratic form in C

$$\dot{X}^M \dot{X}^N G_{MN} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \text{extra terms}$$

$$X^M = (X^\mu, X^{\bar{M}}), \quad X^\mu \equiv X^{1\mu}$$

Ansatz for the metric

$$G_{MN} = \begin{pmatrix} g_{\mu\nu} + A_\mu^{\bar{M}} A_\nu^{\bar{N}} \phi_{\bar{M}\bar{N}}, & A_\mu^{\bar{N}} \phi_{\bar{M}\bar{N}} \\ A_\nu^{\bar{N}} \phi_{\bar{M}\bar{N}}, & \phi_{\bar{M}\bar{N}} \end{pmatrix}$$

$$\dot{X}^M \dot{X}^N G_{MN} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \dot{X}_{\bar{M}} \dot{X}_{\bar{N}} \phi^{\bar{M}\bar{N}}$$

$$\dot{X}_{\bar{M}} = G_{\bar{M}N} \dot{X}^N = A_{\bar{M}\mu} \dot{X}^\mu + \phi_{\bar{M}\bar{N}} \dot{X}^{\bar{N}}$$

Split action

$$I = M \int d\tau \left[\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \phi^{\bar{M}\bar{N}} (A_{\bar{M}\mu} \dot{X}^\mu + \phi_{\bar{M}\bar{J}} \dot{X}^{\bar{J}}) (A_{\bar{N}\nu} \dot{X}^\nu + \phi_{\bar{N}\bar{K}} \dot{X}^{\bar{K}}) \right]^{1/2}$$

Variation with respect to X^μ

$$\frac{1}{(\dot{X}^2)^{1/2}} \frac{d}{d\tau} \left(\frac{\dot{X}^\mu}{(\dot{X}^2)^{1/2}} \right) + \frac{1}{\dot{X}^2} \Gamma^\mu_{\rho\sigma} \dot{X}^\rho \dot{X}^\sigma + \text{extra terms} = 0$$

$$\dot{X}^2 \equiv g_{\rho\sigma} \dot{X}^\rho \dot{X}^\sigma$$

Phase space action

$$I[X^M, P_M, \Lambda] = \int d\tau (P_M \dot{X}^M - H)$$

$$H = \frac{\Lambda}{2M} (P_M P_N G^{MN} - M^2)$$

Splitting $X^M = (X^\mu, X^{\bar{M}})$

$$I[X^\mu, X^{\bar{M}}, p_\mu, P_{\bar{M}}, \Lambda] = \int d\tau [p_\mu \dot{X}^\mu + P_{\bar{M}} \dot{X}^{\bar{M}} - H]$$

$$H = \frac{\Lambda}{2M} \left[g^{\mu\nu} (p_\mu - A_\mu^{\bar{J}} P_{\bar{J}}) (p_\nu - A_\nu^{\bar{K}} P_{\bar{K}}) + \phi^{\bar{M}\bar{N}} P_{\bar{M}} P_{\bar{N}} - M^2 \right]$$

Hamiltonian

We assume that the extra (or 'internal') space admits isometries given by Killing vector fields $k_\alpha^{\bar{J}}$

Projection of momentum onto Killing vector

$$k_\alpha^{\bar{J}} P_{\bar{J}} \equiv p_\alpha$$

Charge

$$A_\mu^{\bar{J}} = k_\alpha^{\bar{J}} A_\mu^\alpha$$

$$\phi^{\bar{M}\bar{N}} = \varphi^{\alpha\beta} k_\alpha^{\bar{M}} k_\beta^{\bar{N}}$$

$$H = \frac{\Lambda}{2M} \left[g^{\mu\nu} (p_\mu - A_\mu^\alpha p_\alpha) (p_\nu - A_\nu^\beta p_\beta) + \varphi^{\alpha\beta} p_\alpha p_\beta - M^2 \right]$$

$$H = \frac{\Lambda}{2M} \left[g^{\mu\nu} (p_\mu - A_\mu^\alpha p_\alpha) (p_\nu - A_\nu^\beta p_\beta) + \varphi^{\alpha\beta} p_\alpha p_\beta - M^2 \right]$$

$$k_\alpha^{\bar{J}} P_{\bar{J}} \equiv p_\alpha$$

$$\dot{p}_\alpha = \{p_\alpha, H\}$$

$$\{p_\alpha, p_\beta\} = \frac{\partial p_\alpha}{\partial X^J} \frac{\partial p_\beta}{\partial X_J} - \frac{\partial p_\beta}{\partial X^J} \frac{\partial p_\alpha}{\partial X_J} = (k_{\alpha,J}^M k_\beta^J - k_{\beta,J}^M k_\alpha^J) p_M = -C_{\alpha\beta}^\gamma p_\gamma$$

$$(k_{\alpha,J}^M k_\beta^J - k_{\beta,J}^M k_\alpha^J) = -C_{\alpha\beta}^\gamma k_\gamma^M$$

$$p_\mu - A_\mu^{\bar{J}} P_{\bar{J}} \equiv \pi_\mu, \quad g^{\mu\nu} \pi_\nu = \frac{M}{\Lambda} \dot{X}^\mu$$

$$\dot{p}_\alpha = C_{\alpha\beta}^\gamma p_\gamma A_\mu^\beta \dot{X}^\mu - \frac{\Lambda}{2M} \varphi^{\alpha'\beta'} {}_{,\bar{J}} P_{\alpha'} p_{\beta'} k_\alpha^{\bar{J}}$$

Wong equation

One can choose a frame in which

$$k_\alpha^M = (k_\alpha^\mu, k_\alpha^{\bar{M}}), \quad k_\alpha^\mu = 0, \quad k_\alpha^{\bar{M}} \neq 0$$

$$\dot{p}_\mu = \{p_\mu, H\} = -\frac{\partial H}{\partial X^\mu}$$

$$F_{\mu\nu}{}^\alpha = \partial_\mu A_\nu{}^\alpha - \partial_\nu A_\mu{}^\alpha + C_{\alpha'\beta'}{}^\alpha A_\mu{}^{\alpha'} A_\nu{}^{\beta'}$$

Yang-Mills
field strength

$$g^{\mu\nu} \pi_\nu = \frac{M}{\Lambda} \dot{X}^\mu, \quad \pi_\mu = \frac{M}{\Lambda} g_{\mu\nu} \dot{X}^\nu$$

$$\dot{\pi}_\mu - \frac{\Lambda}{2M} g_{\rho\sigma,\mu} \pi^\rho \pi^\sigma + F_{\mu\nu}{}^\alpha p_\alpha \dot{X}^\nu + \frac{\Lambda}{2M} \left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'} \right) p_\alpha p_\beta = 0$$

Wong equation
(Equation of geodesic
+ Yang-Mills)

Extra contribution
due to 'scalar' fields

Relation between higher dimensional and 4-dimensional mass

$$\dot{X}^M \dot{X}^N G_{MN} = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu} + \dot{X}_{\bar{M}} \dot{X}_{\bar{N}} \phi^{\bar{M}\bar{N}}$$

$$g_{\mu\nu} = G_{\mu\nu} - \phi^{\bar{M}\bar{N}} k_{\bar{M}}^\alpha k_{\bar{N}}^\beta A_\mu^\alpha A_\nu^\beta$$

$$\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} = \dot{X}^M \dot{X}^N G_{MN} - \dot{X}_{\bar{M}} \dot{X}_{\bar{N}} \phi^{\bar{M}\bar{N}}$$

$$\frac{\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}}{\dot{X}^M \dot{X}^N G_{MN}} = 1 - \frac{\dot{X}_{\bar{M}} \dot{X}_{\bar{N}} \phi^{\bar{M}\bar{N}}}{\dot{X}^M \dot{X}^N G_{MN}}$$

Multiplying by M^2

$$M^2 \frac{\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}}{\dot{X}^M \dot{X}^N G_{MN}} = M^2 - \phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}} = g^{\mu\nu} p_\mu p_\nu = m^2$$

$$\frac{m}{M} = \left(\frac{\dot{X}^\mu \dot{X}^\nu g_{\mu\nu}}{\dot{X}^M \dot{X}^N G_{MN}} \right)^{1/2}$$

$$m^2 = g^{\mu\nu} p_\mu p_\nu = M^2 - \phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}}$$

Four dimensional mass m is given by the higher dimensional mass M and the contribution due to the extra components of momentum $p_{\bar{M}}$

From the perspective of 4-dimensional spacetime, m has the role of inertial mass. This can be seen if we rewrite the equation of motion

$$\dot{\pi}_\mu - \frac{\Lambda}{2M} g_{\rho\sigma,\mu} \pi^\rho \pi^\sigma + F_{\mu\nu}{}^\alpha p_\alpha \dot{X}^\nu + \frac{\Lambda}{2M} \left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'} \right) p_\alpha p_\beta = 0$$

$$g^{\mu\nu} \pi_\nu = \frac{M}{\Lambda} \dot{X}^\mu, \quad \pi_\mu = \frac{M}{\Lambda} g_{\mu\nu} \dot{X}^\nu$$

$$\Lambda^2 = \dot{X}^M \dot{X}^N G_{MN}, \quad \lambda^2 = \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$$

$$\frac{m}{M} = \frac{\lambda}{\Lambda}$$

$$\frac{1}{\lambda} \frac{d}{d\tau} \left(\frac{\dot{X}^\mu}{\lambda} \right) + {}^{(4)}\Gamma^\mu{}_{\rho\sigma} \frac{\dot{X}^\rho \dot{X}^\sigma}{\lambda^2} + \frac{p_\alpha}{m} F_{\mu\nu}{}^\alpha \frac{\dot{X}^\nu}{\lambda} + \frac{1}{2m^2} \left(\varphi^{\alpha\beta}{}_{,\mu} - \varphi^{\alpha\beta}{}_{,\bar{J}} k_{\alpha'}{}^{\bar{J}} A_\mu{}^{\alpha'} \right) p_\alpha p_\beta + \frac{1}{\lambda m} \frac{dm}{d\tau} = 0$$

4-dimensional mass

$$m^2 = g^{\mu\nu} p_\mu p_\nu = M^2 - \phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}} = M^2 - \phi^{\alpha\beta} p_\alpha p_\beta + \text{extra terms}$$

$$p_\alpha \equiv k_\alpha^{\bar{M}} P_{\bar{M}}$$

In general m is not constant. In particular, when the extra terms vanish, m may be constant.

A configuration under consideration can be the universe. Then, according to this theory, the motion of a subsystem, approximated as a point particle, obeys the law of motion given in previous slide.

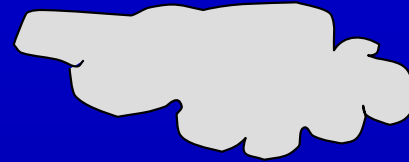
Besides the usual 4-dimensional gravity, there are extra forces. They come from the generalized metric, i.e., the metric of configuration space.

The inertial mass depends on momenta of all other particles within the configuration. This is reminiscent of the Mach principle.

Such approach opens a Pandora's box of possibilities to revise our current views on the universe, dark matter, dark energy, MOND, the Pioneer effect, etc.

- Strings, branes

Theories of strings and higher dimensional extended objects, branes
- very promising in explaining the origin and interrelationship of the fundamental interactions,
including gravity



But there is a cloud:

- what is a geometric principle behind string and brane theories and how to formulate them in a background independent way

The diagram consists of a white rectangular area divided by a vertical green line. On the left side of the line, there is a drawing of a wavy vertical line representing a string and a drawing of a cylinder representing a brane. On the right side of the line, the Einstein-Hilbert action is written as $I[g_{\mu\nu}] = \int \sqrt{-g} R d^4x$. Below the action, a red question mark is placed, indicating a question about the applicability of this action to string and brane theories.

Configuration space for infinite dimensional objects - branes

A brane can be considered as a point in infinite dimensional space with coordinates

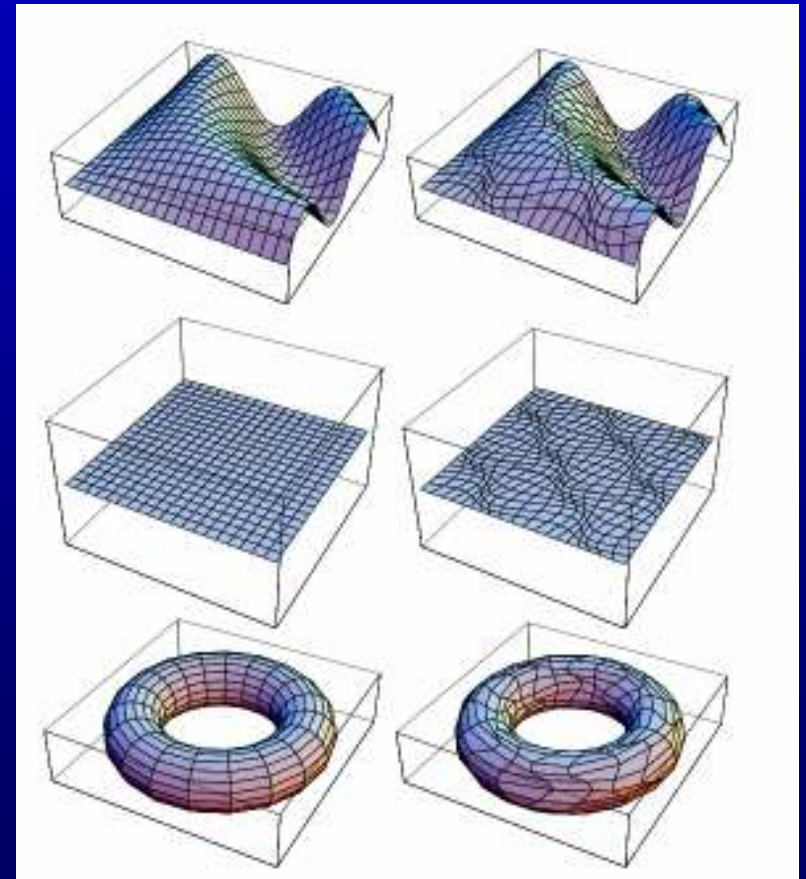
$$X^\mu(\xi^a) \equiv X^{\mu(\xi)} \equiv X^M$$

This includes classes of tangentially deformed branes which we can interpret as physically different objects, not just reparametrizations.

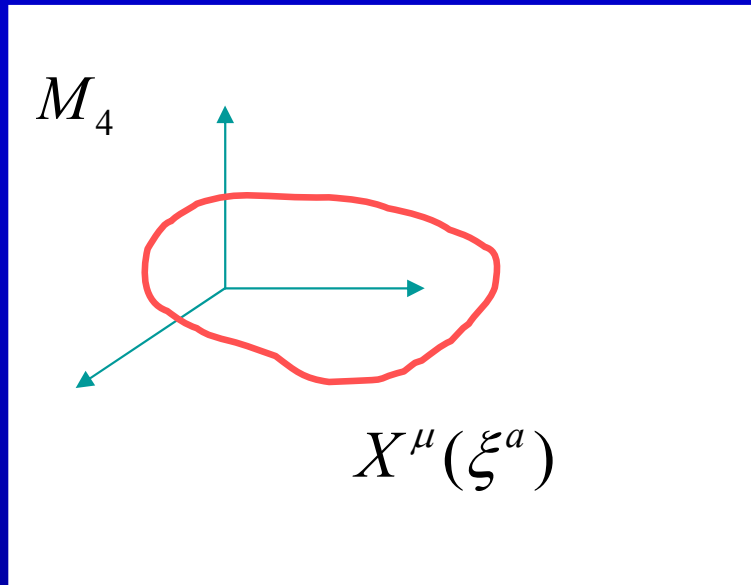
Mathematically the surfaces on the left and the right are the same.
Physically they are different.

They are represented by two different points in \mathcal{C}

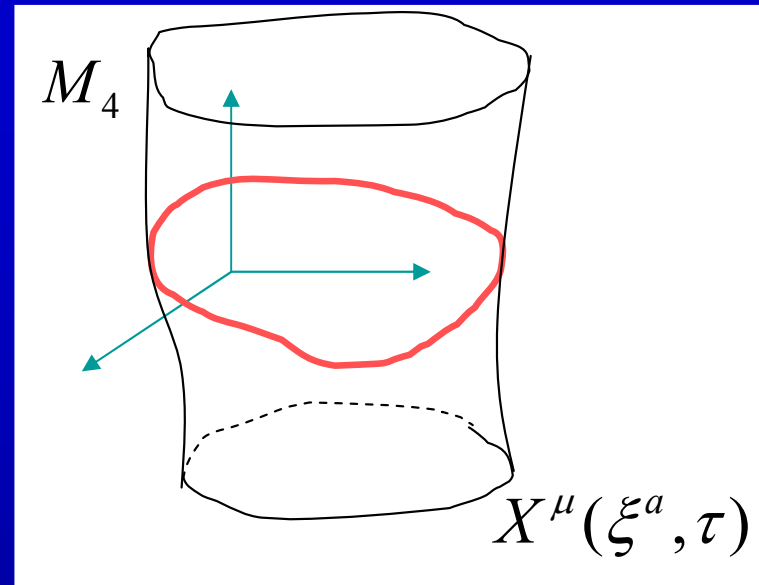
For the configuration space associated with a brane we will also use the name brane space \mathcal{M}



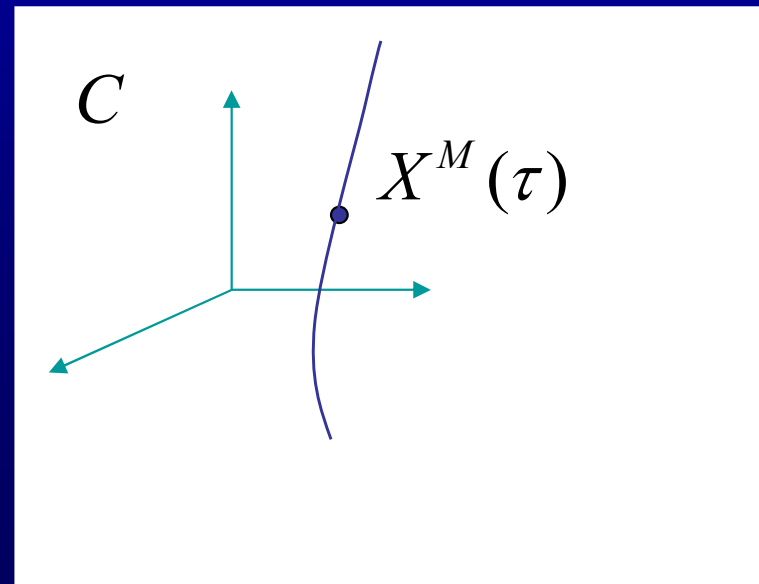
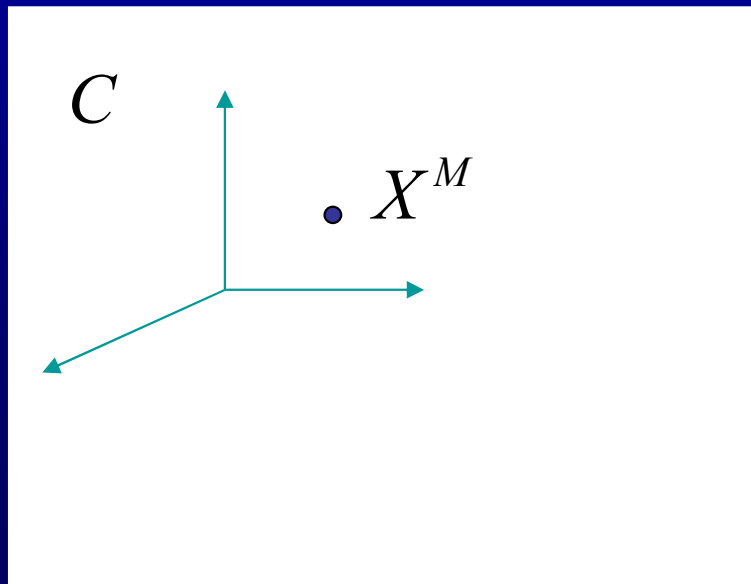
'Instantaneous' brane configuration in M_4



'Evolution' of a brane configuration in M_4



Representation in configuration space C



Action in the brane space \mathcal{M}

$$I[X^M] = \int d\tau (\rho_{MN} \dot{X}^M \dot{X}^N)^{(1/2)}$$

Short hand notation

$$M \equiv \mu(\xi), \quad X^M \equiv X^{\mu(\xi)} \equiv X^\mu(\xi)$$

$$I[X^{\alpha(\xi)}] = \int d\tau (\rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')})^{1/2}$$

More explicit notation

If metric is given by

$$\rho_{\alpha(\xi')\beta(\xi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}} \delta(\xi' - \xi'') \eta_{\alpha\beta}$$

$$f \equiv \det f_{ab}, \quad f_{ab} \equiv \partial_a X^\mu \partial_b X^\nu g_{\mu\nu}$$

$$\dot{X}^2 \equiv \dot{X}^\mu \dot{X}^\nu g_{\mu\nu}$$

then the corresponding equations of motion are precisely those of a **Dirac-Nambu-Goto brane!**

In this theory we assume that the metric above is just one particular chose amongst many other possible metrics that are solution to the Einstein equations in the configuration space.

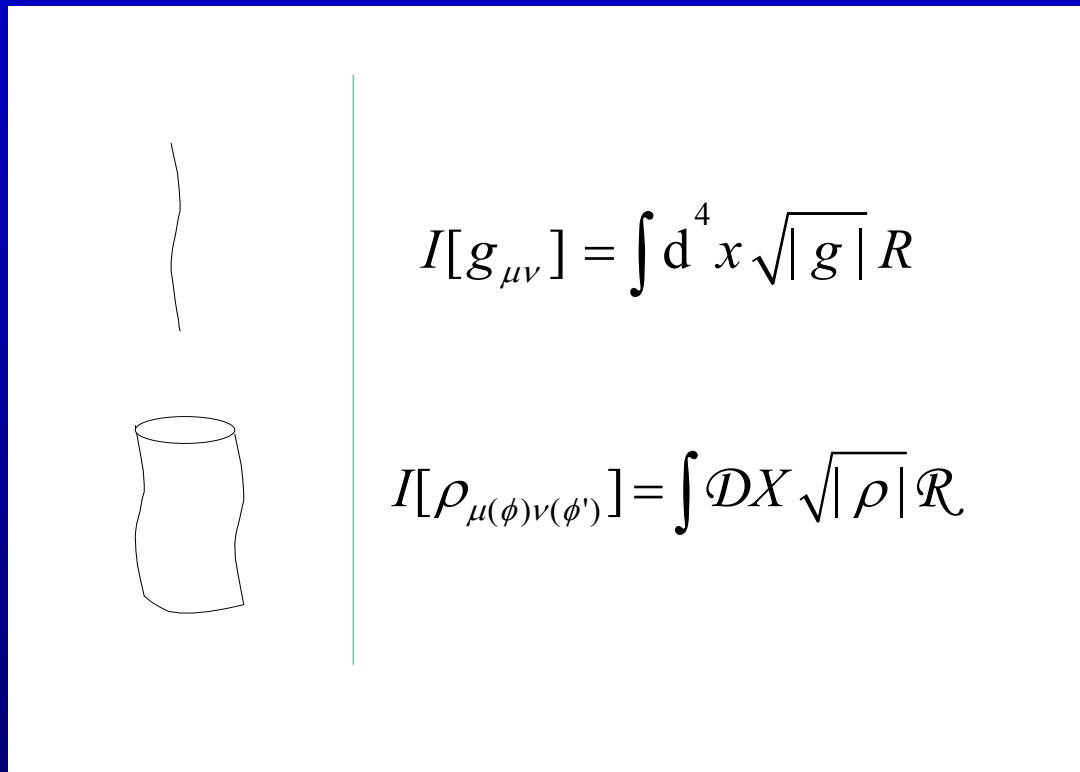
For more details see:

M. Pavšič: The Landscape of theoretical Physics (Kluwer, 2001), gr-qc/0610061 ;
hep-th/0311060

We have taken the brane space \mathcal{M} seriously as an arena for physics.

The arena itself is also a part of the dynamical system, it is not prescribed in advance.

The theory is thus background independent. It is based on the geometric principle which has its roots in the brane space \mathcal{M}


$$I[g_{\mu\nu}] = \int d^4x \sqrt{|g|} R$$
$$I[\rho_{\mu(\phi)\nu(\phi')}] = \int \mathcal{D}X \sqrt{|\rho|} \mathcal{R}$$

$$\phi \equiv \phi^A = (\tau, \xi^A)$$

There is no pre-existing space and metric: they appear dynamically as solutions to the equations of motion.

In summary,

the infinite dimensional brane space has in principle any metric that is a solution to the (generalized) Einstein equations.

For the particular diagonal metric we obtain the ordinary branes, including strings.

The proposed theory goes beyond the usual strings and branes.

It presumably resolves the problem of background independence and the geometric principle behind the string theory.

Geometrical principle behind the string theory is based on the concept of **brane space** \mathcal{M} , i.e., the configuration space for branes.

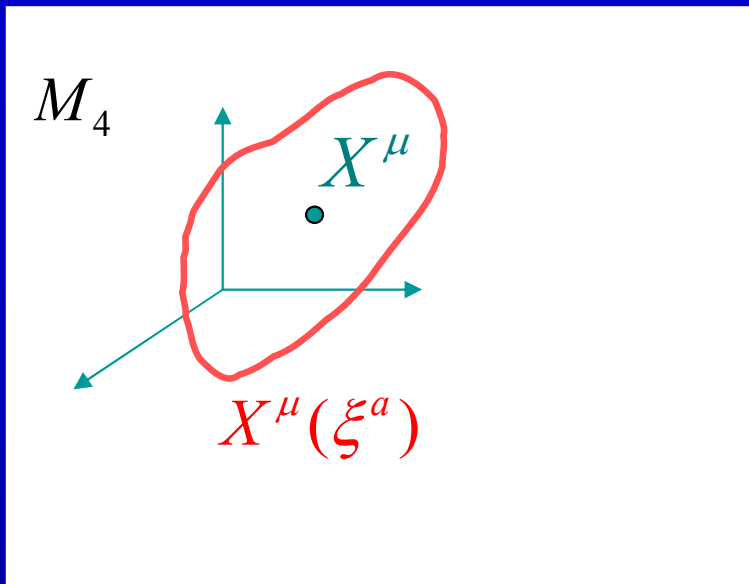
Finite dimensional description of extended objects



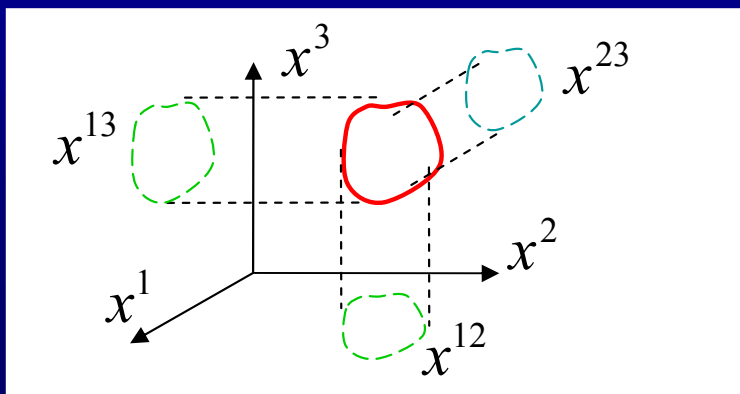
The Earth has a huge (practically infinite) number of degrees of freedom. And yet, when describing the motion of the Earth around the Sun, we neglect them all, except for the coordinates of **the centre of mass**.

Instead of infinitely many degrees of freedom associated with an extended object, we may consider **a finite number of degrees of freedom**.

Strings and branes have infinitely many degrees of freedom.
But at first approximation we can consider just **the centre of mass**.

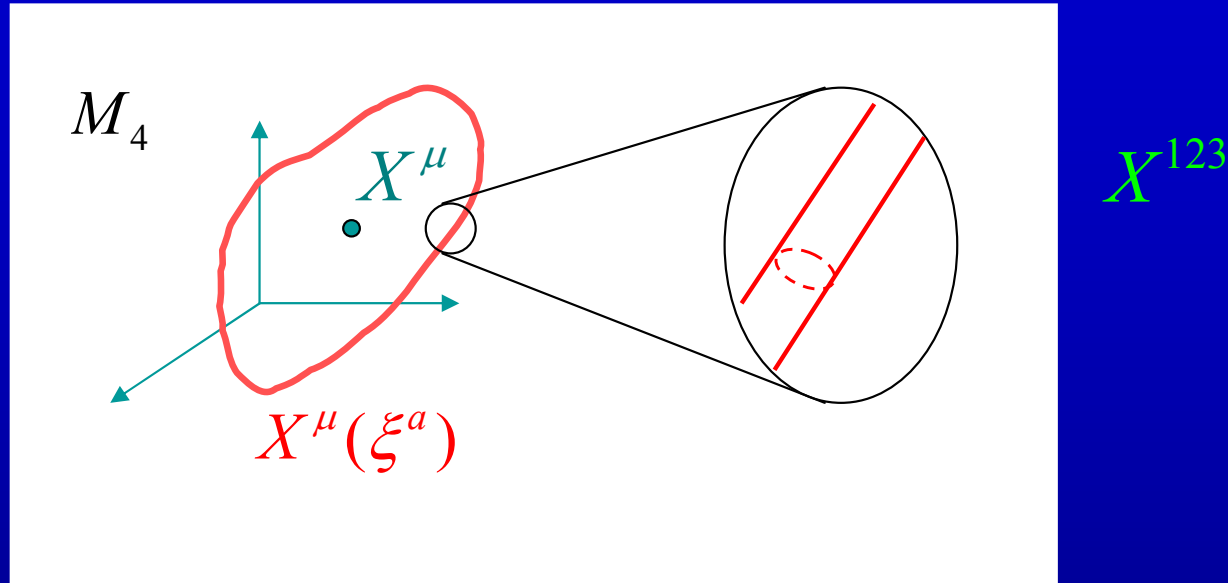


Next approximation is in considering the holographic coordinates of the **oriented area** enclosed by the string.



We may go further and search for eventual thickness of the object.

If the string has finite thickness, i.e., if actually it is not a string, but a 2-brane, then there exist the corresponding **volume degrees of freedom**.



In general, for an extended object in M_4 , we have 16 coordinates

$$x^M \equiv x^{\mu_1 \dots \mu_r}, \quad r = 0, 1, 2, 3, 4$$

They are the projections of r-dimensional volumes (areas) onto the coordinate planes.

Oriented r-volumes can be elegantly described by Clifford algebra.

Instead of the usual relativity formulated in spacetime in which the interval is

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu$$

we are studying the theory in which the interval is extended to the space of r-volumes (called Clifford space):

$$dS^2 = G_{MN} dx^M dx^N \quad dx^M \equiv dx^{\mu_1 \dots \mu_r}, \quad r = 0, 1, 2, 3, 4$$

Coordinates of Clifford space can be used to model extended objects. They are a generalization of the concept of center of mass.

Instead of describing an extended object in "full detail", we can describe them in terms of the center of mass, area and volume coordinates.

In particular, extended object can be a fundamental string or brane.

Dynamics

Action:

$$I = \int d\tau (\eta_{MN} \dot{X}^M \dot{X}^N)^{1/2}$$

Generalization of ordinary relativity

Equations of motion:

$$\ddot{X}^M \equiv \frac{d^2 X^M}{d\tau^2} = 0$$

These equations imply area (volume) motion

Metric:

$$\eta_{MN}$$

Diagonal metric

Signature:

$$+ + + + + + + + - - - - - - - -$$

(8,8)

The above dynamics holds for tensionless branes. For the branes with tension one has to introduce curved Clifford space.

Summary

- We have considered a theory in which spacetime is replaced by a larger space, namely the configuration space associated with a system under consideration.
- The ordinary special and general relativity are recovered for a particular metric of the configuration space.
- Since configuration space has extra dimensions, its metric provides description of additional interactions, beside the 4-dimensional gravity, just as in Kaluza-Klein theories
- In this theory there is no need for extra dimensions of spacetime. The latter space is a subspace of the configuration space.

All dimensions of the configuration space C are physical. Therefore there is no need for a compactification of the extra dimensions of C .

All this was just an introduction. **Much more can be found in a book**

M. Pavšič: *The Landscape of Theoretical Physics: A Global view; From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle* (Kluwer Academic, 2001)

where the description with a metric tensor has been surpassed.

Very promising is the description in terms of the **Clifford algebra equivalent** of the tetrad field which simplifies calculations significantly.

Some other related publications:

Class.Quant.Grav.20:2697-2714,2003, gr-qc/0111092

Kaluza-Klein theory without extra dimensions: Curved Clifford space.
Phys.Lett.B614:85-95,2005, hep-th/0412255

Clifford space as a generalization of spacetime: Prospects for QFT of point particles and strings, Found.Phys.35:1617-1642,2005, hep-th/0501222

Spin gauge theory of gravity in Clifford space: A Realization of Kaluza-Klein theory in 4-dimensional spacetime, Int.J.Mod.Phys.A21:5905-5956,2006, gr-qc/0507053

Example of internal space S_2

Solution to the Killing equation $k^\alpha_{\bar{M};\bar{N}} + k^\alpha_{\bar{N};\bar{M}} = 0$ gives:

$$\begin{aligned} k_1^\vartheta &= \sin \varphi, & k_1^\varphi &= \cot \mathcal{G} \cos \varphi, \\ k_2^\vartheta &= \cos \varphi, & k_2^\varphi &= -\cot \mathcal{G} \sin \varphi, \\ k_3^\vartheta &= 0, & k_3^\varphi &= 1 \end{aligned}$$

$$p_\alpha \equiv k_\alpha^{\bar{M}} P_{\bar{M}}$$

$$P_{\bar{M}} = (p_\vartheta, p_\varphi)$$

$$\begin{aligned} p_1 &= \sin \varphi p_\vartheta + \cot \mathcal{G} \cos \varphi p_\varphi \\ p_2 &= \cos \varphi p_\vartheta - \cot \mathcal{G} \sin \varphi p_\varphi \\ p_3 &= p_\varphi \end{aligned}$$

$$\varphi^{\alpha\beta} = \delta^{\alpha\beta} \frac{1}{r^2}$$

$$\phi^{\bar{M}\bar{N}} = \varphi^{\alpha\beta} k_\alpha^{\bar{M}} k_\beta^{\bar{N}}$$

$$\phi^{\bar{M}\bar{N}} p_{\bar{M}} p_{\bar{N}} = \varphi^{\alpha\beta} p_\alpha p_\beta$$

$$\phi^{\bar{M}\bar{N}} = \begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{1}{r^2 \sin^2 \mathcal{G}} \end{pmatrix}$$