In the (x, y) representation the vacuum state $\langle x, y|0\rangle \equiv \psi_0(x, y)$ satisfies

$$\frac{1}{\sqrt{2}}\left(\sqrt{\omega}\,x + \frac{1}{\sqrt{\omega}}\frac{\partial}{\partial x}\psi_0(x,y)\right) = 0\ , \quad \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\,y + \frac{1}{\sqrt{\omega}}\frac{\partial}{\partial y}\psi_0(x,y)\right) = 0\ , \tag{3.15}$$

which comes straightforwardly from (3.13). A solution which is in agreement with the probability interpretation,

$$\psi_0 = \frac{2\pi}{\omega}\exp[-\tfrac{1}{2}\omega(x^2 + y^2)] \tag{3.16}$$

is normalized according to $\int \psi_0^2\,\mathrm{d}x\,\mathrm{d}y = 1$.

We see that our particle is localized around the origin. The excited states obtained by applying $c_x^\dagger$, $c_y^\dagger$ to the vacuum state are also localized. This is in agreement with the property that also according to the classical equations of motion (3.2), the particle is localized in the vicinity of the origin. All states $|\psi\rangle$ have positive norm. For instance,

$$\langle 0|cc^\dagger|0\rangle = \langle 0|[c, c^\dagger]|0\rangle = \langle 0|0\rangle = \int \psi_0^2 \mathrm{d}x\,\mathrm{d}y = 1.$$

3.2. HARMONIC OSCILLATOR IN *d*-DIMENSIONAL PSEUDO-EUCLIDEAN SPACE

Extending (3.1) to arbitrary dimension it is convenient to use the compact (covariant) index notation

$$L = \frac{1}{2}\dot{x}^\mu \dot{x}_\mu - \tfrac{1}{2}\omega^2 x^\mu x_\mu\ , \tag{3.17}$$

where for arbitrary vector A^μ the quadratic form is $A^\mu A_\mu \equiv \eta_{\mu\nu} A^\mu A^\nu$. The metric tensor $\eta_{\mu\nu}$ has signature $(+++...---...)$. The Hamiltonian is

$$H = \tfrac{1}{2}p^\mu p_\mu + \tfrac{1}{2}\omega^2 x^\mu x_\mu \tag{3.18}$$

Conventionally one introduces

$$a^\mu = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}\,x^\mu + \frac{i}{\sqrt{\omega}}\,p^\mu\right)\ , \qquad a^{\mu\dagger} = \frac{1}{\sqrt{2}}\left(\sqrt{\omega}x^\mu - \frac{i}{\sqrt{\omega}}p^\mu\right). \tag{3.19}$$

In terms of a^μ, $a^{\mu\dagger}$ the Hamiltonian reads

$$H = \frac{\omega}{2}(a^{\mu\dagger}a_\mu + a_\mu a^{\mu\dagger}). \tag{3.20}$$

Upon quantization we have

$$[x^\mu, p_\nu] = i\delta^\mu{}_\nu \qquad \text{or} \qquad [x^\mu, p^\nu] = i\eta^{\mu\nu} \tag{3.21}$$

and

$$[a^\mu, a^\dagger_\nu] = \delta^\mu{}_\nu \qquad \text{or} \qquad [a^\mu, a^{\nu\dagger}] = \eta^{\mu\nu}. \tag{3.22}$$

We shall now discuss two possible definitions of the vacuum state. The first possibility is the one usually assumed, whilst the second possibility [44, 45] is the one I am going to adopt.

Possibility I. The vacuum state can be defined according to

$$a^\mu|0\rangle = 0 \tag{3.23}$$

and the Hamiltonian, normal ordered with respect to the vacuum definition (3.23), becomes, after using (3.22),

$$H = \omega\left(a^{\mu\dagger}a_\mu + \frac{d}{2}\right), \qquad d = \eta^{\mu\nu}\eta_{\mu\nu}. \tag{3.24}$$

Its eigenvalues are all positive and there is the non-vanishing zero point energy $\omega d/2$. In the x representation the vacuum state is

$$\psi_0 = \left(\frac{2\pi}{\omega}\right)^{d/2} \exp[-\tfrac{1}{2}\omega x^\mu x_\mu] \tag{3.25}$$

It is a solution of the Schrödinger equation $-\frac{1}{2}\partial^\mu\partial_\mu\psi_0 + (\omega^2/2)x^\mu x_\mu\psi_0 = E_0\psi_0$ with positive $E_0 = \omega(\frac{1}{2} + \frac{1}{2} + ...)$. The state ψ_0 as well as excited states can not be normalized to 1. Actually, there exist negative norm states. For instance, if $\eta^{33} = -1$, then

$$\langle 0|a^3 a^{3\dagger}|0\rangle = \langle 0|[a^3, a^{3\dagger}]|0\rangle = -\langle 0|0\rangle.$$

Possibility II. Let us split $a^\mu = (a^\alpha, a^{\bar\alpha})$, where the indices α, $\bar\alpha$ refer to the components with positive and negative signature, respectively, and define the vacuum according to[1]

$$a^\alpha|0\rangle = 0\,, \qquad a^{\bar\alpha\dagger}|0\rangle = 0. \tag{3.26}$$

[1]Equivalently, one can define annihilation and creation operators in terms of x^μ and the canonically conjugate momentum $p_\mu = \eta_{\mu\nu}p^\nu$ according to $c^\mu = (1/\sqrt{2})(\sqrt{\omega}x^\mu + (i/\sqrt{\omega})p_\mu)$ and $c^{\mu\dagger} = (1/\sqrt{2})(\sqrt{\omega}x^\mu - (i/\sqrt{\omega})p_\mu)$, satisfying $[c^\mu, c^{\nu\dagger}] = \delta^{\mu\nu}$. The vacuum is then defined as $c^\mu|0\rangle = 0$. This is just the higher-dimensional generalization of c_x, c_y (eq. (3.8),(3.9) and the vacuum definition (3.13).

Using (3.22) we obtain the normal ordered Hamiltonian with respect to the vacuum definition (3.26)

$$H = \omega \left(a^{\alpha\dagger} a_\alpha + \frac{r}{2} + a_{\bar{\alpha}} a^{\bar{\alpha}\dagger} - \frac{s}{2} \right) , \tag{3.27}$$

where ${\delta_\alpha}^\alpha = r$ and ${\delta_{\bar{\alpha}}}^{\bar{\alpha}} = s$. If the number of positive and negative signature components is the same, i.e., $r = s$, then the Hamiltonian (3.27) has vanishing zero point energy:

$$H = \omega(a^{\alpha\dagger} a_\alpha + a_{\bar{\alpha}} a^{\bar{\alpha}\dagger}). \tag{3.28}$$

Its eigenvalues are positive or negative, depending on which components (positive or negative signature) are excited. In the x-representation the vacuum state (3.26) is

$$\psi_0 = \left(\frac{2\pi}{\omega} \right)^{d/2} \exp[-\tfrac{1}{2}\omega \delta_{\mu\nu} x^\mu x^\nu], \tag{3.29}$$

where the Kronecker symbol $\delta_{\mu\nu}$ has the values +1 or 0. It is a solution of the Schrödinger equation $-\frac{1}{2}\partial^\mu \partial_\mu \psi_0 + (\omega^2/2) x^\mu x_\mu \psi_0 = E_0 \psi_0$ with $E_0 = \omega(\frac{1}{2} + \frac{1}{2} + ... - \frac{1}{2} - \frac{1}{2} - ...)$. One can also easily verify that there are no negative norm states.

Comparing *Possibility I* with *Possibility II* we observe that the former has positive energy vacuum invariant under pseudo-Euclidean rotations, whilst the latter has the vacuum invariant under Euclidean rotations and having vanishing energy (when $r = s$). In other words, we have: either (i) non-vanishing energy and pseudo-Euclidean invariance or (ii) vanishing energy and Euclidean invariance of the vacuum state. In the case (ii) the vacuum state ψ_0 changes under the pseudo-Euclidean rotations, but its energy remains zero.

The invariance group of our Hamiltonian (3.18) and the corresponding Schrödinger equation consists of pseudo-rotations. Though a solution of the Schrödinger equation changes under a pseudo-rotation, the theory is covariant under the pseudo-rotations, in the sense that the set of all possible solutions does not change under the pseudo-rotations. Namely, the solution $\psi_0(x')$ of the Schrödinger equation

$$-\tfrac{1}{2}\partial'^\mu {\partial'}_\mu \psi_0(x') + (\omega^2/2) x'^\mu x'_\mu \psi_0(x') = 0 \tag{3.30}$$

in a pseudo-rotated frame S' is

$$\psi_0(x') = \frac{2\pi}{\omega}^{d/2} \exp[-\tfrac{1}{2}\omega \delta_{\mu\nu} x'^\mu x'^\nu]. \tag{3.31}$$

If observed from the frame S the latter solution reads

$$\psi_0'(x) = \frac{2\pi}{\omega}^{d/2} \times \exp[-\tfrac{1}{2}\omega\delta_{\mu\nu}L^\mu{}_\rho x^\rho L^\nu{}_\sigma x^\sigma], \qquad (3.32)$$

where $x'^\mu = L^\mu{}_\rho x^\rho$. One finds that $\psi_0'(x)$ as well as $\psi_0(x)$ (eq. (3.29)) are solutions of the Schrödinger equation in S and they both have the same vanishing energy. In general, in a given reference frame we have thus a degeneracy of solutions with the same energy [44]. This is so also in the case of excited states.

In principle it seem more natural to adopt *Possibility II*, because the classically energy of our harmonic oscillator is nothing but a quadratic form $E = \frac{1}{2}(p^\mu p_\mu + \omega^2 x^\mu x_\mu)$, which in the case of a metric of pseudo-Euclidean signature can be positive, negative, or zero.

3.3. A SYSTEM OF SCALAR FIELDS

Suppose we have a system of two scalar fields described by the action[2]

$$I = \tfrac{1}{2}\int \mathrm{d}^4x\,(\partial_\mu\phi_1\,\partial^\mu\phi_1 - m^2\phi_1^2 - \partial_\mu\phi_2\,\partial^\mu\phi_2 + m^2\phi_2^2). \qquad (3.33)$$

This action differs from the usual action for a charged field in the sign of the ϕ_2 term. It is a field generalization of our action for the point particle harmonic oscillator in 2-dimensional pseudo-Euclidean space.

The canonical momenta are

$$\pi_1 = \dot{\phi}_1\,, \qquad \pi_2 = -\dot{\phi}_2 \qquad (3.34)$$

satisfying

$$[\phi_1(\mathbf{x}), \pi_1(\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}')\,, \qquad [\phi_2(\mathbf{x}), \pi_2(\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}'). \qquad (3.35)$$

The Hamiltonian is

$$H = \tfrac{1}{2}\int \mathrm{d}^3\mathbf{x}\,(\pi_1^2 + m^2\phi_1^2 - \partial_i\phi_1\,\partial^i\phi_1 - \pi_2^2 - m^2\phi_2^2 + \partial_i\phi_2\partial^i\phi_2). \qquad (3.36)$$

We use the spacetime metric with signature $(+\,-\,-\,-)$ so that $-\partial_i\phi_1\,\partial^i\phi_1 = (\nabla\phi)^2$, $i = 1,2,3$. Using the expansion ($\omega_\mathbf{k} = (m^2 + \mathbf{k}^2)^{1/2}$)

$$\phi_1 = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\,\frac{1}{2\omega_\mathbf{k}}\left(c_1(\mathbf{k})e^{-ikx} + c_1^\dagger(\mathbf{k})e^{ikx}\right), \qquad (3.37)$$

[2]Here, for the sake of demonstration, I am using the formalism of the conventional field theory, though in my opinion a better formalism involves an invariant evolution parameter, as discussed in Sec. 1.4