THE LANDSCAPE OF THEORETICAL PHYSICS: A GLOBAL VIEW

From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle

MATEJ PAVŠIČ

Department of Theoretical Physics
Jožef Stefan Institute
Ljubljana, Slovenia

Kluwer Academic Publishers
Boston/Dordrecht/London

2.1. INTRODUCTION TO GEOMETRIC CALCULUS BASED ON CLIFFORD ALGEBRA

We have seen that point particles move in some kind of space. In non relativistic physics the space is 3-dimensional and Euclidean, while in the theory of relativity space has 4-dimensions and pseudo-Euclidean signature, and is called spacetime. Moreover, in general relativity spacetime is curved, which provides gravitation. If spacetime has even more dimensions —as in Kaluza–Klein theories— then such a higher-dimensional gravitation contains 4-dimensional gravity and Yang–Mills fields (including the fields associated with electromagnetic, weak, and strong forces). Since physics happens to take place in a certain space which has the role of a stage or arena, it is desirable to understand its geometric properties as deeply as possible.

Let V_n be a continuous space of arbitrary dimension n. To every point of V_n we can ascribe n *parameters* x^μ, $\mu = 1, 2, ..., n$, which are also called *coordinates*. Like house numbers they can be freely chosen, and once being fixed they specify points of the space[2].

When considering points of a space we ask ourselves what are the distances between the points. *The distance* between two infinitesimally separated points is given by

$$\mathrm{d}s^2 = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu. \tag{2.1}$$

Actually, this is the square of the distance, and $g_{\mu\nu}(x)$ is the metric tensor. The quantity $\mathrm{d}s^2$ is *invariant* with respect to general coordinate transformations $x^\mu \to x'^\mu = f^\mu(x)$.

Let us now consider *the square root* of the distance. Obviously it is $\sqrt{g_{\mu\nu}\mathrm{d}x^\mu\,\mathrm{d}x^\nu}$. But the latter expression is not linear in $\mathrm{d}x^\mu$. We would like to define an object which is *linear* in $\mathrm{d}x^\mu$ and whose square is eq. (2.1). Let such object be given by the expression

$$\mathrm{d}x = \mathrm{d}x^\mu\,e_\mu \tag{2.2}$$

It must satisfy

$$\mathrm{d}x^2 = e_\mu e_\nu\,\mathrm{d}x^\mu \mathrm{d}x^\nu = \tfrac{1}{2}(e_\mu e_\nu + e_\nu e_\mu)\,\mathrm{d}x^\mu \mathrm{d}x^\nu = g_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu = \mathrm{d}s^2, \tag{2.3}$$

from which it follows that

$$\tfrac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = g_{\mu\nu} \tag{2.4}$$

[2]See Sec. 6.2, in which the subtleties related to specification of spacetime points are discussed.

The quantities e^μ so introduced are a new kind of number, called *Clifford numbers*. They do not commute, but satisfy eq. (2.4) which is a characteristic of *Clifford algebra*.

In order to understand what is the meaning of the object $\mathrm{d}x$ introduced in (2.2) let us study some of its properties. For the sake of avoiding use of differentials let us write (2.2) in the form

$$\frac{\mathrm{d}x}{\mathrm{d}\tau} = \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\, e_\mu \,, \tag{2.5}$$

where τ is an arbitrary parameter invariant under general coordinate transformations. Denoting $\mathrm{d}x/\mathrm{d}\tau \equiv a$, $\mathrm{d}x^\mu/\mathrm{d}\tau = a^\mu$, eq. (2.5) becomes

$$a = a^\mu e_\mu \,. \tag{2.6}$$

Suppose we have two such objects a and b. Then

$$(a+b)^2 = a^2 + ab + ba + b^2 \tag{2.7}$$

and

$$\tfrac{1}{2}(ab+ba) = \tfrac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) a^\mu b^\nu = g_{\mu\nu} a^\mu b^\nu. \tag{2.8}$$

The last equation algebraically corresponds to *the inner product* of two *vectors* with components a^μ and b^ν. Therefore we denote

$$a \cdot b \equiv \tfrac{1}{2}(ab+ba). \tag{2.9}$$

From (2.7)–(2.8) we have that the sum $a+b$ is an object whose square is also a scalar.

What about the antisymmetric combinations? We have

$$\frac{1}{2}(ab - ba) = \tfrac{1}{2}(a^\mu b^\nu - a^\nu b^\mu) e_\mu e_\nu \tag{2.10}$$

This is nothing but *the outer product* of the vectors. Therefore we denote it as

$$a \wedge b \equiv \tfrac{1}{2}(ab - ba) \tag{2.11}$$

In 3-space this is related to the familiar vector product $a \times b$ which is the dual of $a \wedge b$.

The object $a = a^\mu e_\mu$ is thus nothing but a *vector*: a^μ are its components and e^μ are n linearly independent basic vectors of V_n. Obviously, if one changes parametrization, a or $\mathrm{d}x$ remains the same. Since under a general coordinate transformation the components a^μ and $\mathrm{d}x^\mu$ do change, e_μ should also change in such a way that the vectors a and $\mathrm{d}x$ remain invariant.

An important lesson we have learnt so far is that

- the "square root" of the distance is a vector;
- vectors are Clifford numbers;
- vectors are objects which, like distance, are *invariant* under general coordinate transformations.

Box 2.1: Can we add apples and oranges?

When I asked my daughter, then ten years old, how much is 3 apples and 2 oranges plus 1 apple and 1 orange, she immediately replied "4 apples and 3 oranges". If a child has no problems with adding apples and oranges, it might indicate that contrary to the common wisdom, often taught at school, such an addition has mathematical sense after all. The best example that this is indeed the case is complex numbers. Here instead of 'apples' we have real and, instead of 'oranges', imaginary numbers. The sum of a real and imaginary number is a complex number, and summation of complex numbers is a mathematically well defined operation. Analogously, in Clifford algebra we can sum Clifford numbers of different degrees. In other words, summation of scalar, vectors, bivectors, etc., is a well defined operation.

The basic operation in Clifford algebra is *the Clifford product* ab. It can be decomposed into the symmetric part $a \cdot b$ (defined in (2.9) and the antisymmetric part $a \wedge b$ (defined in (2.11)):

$$ab = a \cdot b + a \wedge b \tag{2.12}$$

We have seen that $a \cdot b$ is a scalar. On the contrary, eq. (2.10) shows that $a \wedge b$ is not a scalar. Decomposing the product $e_\mu e_\nu$ according to (2.12),

$$e_\mu e_\nu = e_\mu \cdot e_\nu + e_\mu \wedge e_\nu = g_{\mu\nu} + e_\mu \wedge e_\nu \,,$$

we can rewrite (2.10) as

$$a \wedge b = \frac{1}{2}(a^\mu b^\nu - a^\nu b^\mu)\, e_\mu \wedge e_\nu, \tag{2.13}$$

which shows that $a \wedge b$ is a new type of geometric object, called *bivector*, which is neither a scalar nor a vector.

The geometric product (2.12) is thus the sum of a scalar and a bivector. The reader who has problems with such a sum is advised to read Box 2.1.

A *vector* is an algebraic representation of *direction* in a space V_n; it is associated with an oriented line.

A *bivector* is an algebraic representation of an oriented plane.

This suggests a generalization to trivectors, quadrivectors, etc. It is convenient to introduce the name *r-vector* and call r its *degree*:

0-vector	s	scalar
1-vector	a	vector
2-vector	$a \wedge b$	bivector
3-vector	$a \wedge b \wedge c$	trivector
.	.	.
.	.	.
.	.	.
r-vector	$A_r = a_1 \wedge a_2 \wedge ... \wedge a_r$	multivector

In a space of finite dimension this cannot continue indefinitely: an n-vector is the highest r-vector in V_n and an $(n+1)$-*vector* is identically zero. An r-vector A_r represents an oriented r-volume (or r-direction) in V_n.

Multivectors A_r are elements of the *Clifford algebra* $\mathcal{C}_n$ of V_n. An element of $\mathcal{C}_n$ will be called a *Clifford number*. Clifford numbers can be multiplied amongst themselves and the results are Clifford numbers of mixed degrees, as indicated in the basic equation (2.12). The theory of multivectors, based on Clifford algebra, was developed by Hestenes [22]. In Box 2.2 some useful formulas are displayed without proofs.

Let $e_1, e_2, ..., e_n$ be linearly independent vectors, and $\alpha, \alpha^i, \alpha^{i_1 i_2}, ...$ scalar coefficients. A generic Clifford number can then be written as

$$A = \alpha + \alpha^i e_i + \frac{1}{2!}\, \alpha^{i_1 i_2}\, e_{i_1} \wedge e_{i_2} + ... \frac{1}{n!}\, \alpha^{i_1 ... i_n} e_{i_1} \wedge ... \wedge e_{i_n} \, . \qquad (2.14)$$

Since it is a superposition of multivectors of all possible grades it will be called *polyvector*.[3] Another name, also often used in the literature, is *Clifford aggregate*.These mathematical objects have far reaching geometrical and physical implications which will be discussed and explored to some extent in the rest of the book.

[3]Following a suggestion by Pezzaglia [23] I call a generic Clifford number *polyvector* and reserve the name *multivector* for an r-vector, since the latter name is already widely used for the corresponding object in the calculus of differential forms.

Box 2.2: Some useful basic equations

For a vector a and an r-vector A_r the inner and the outer product are defined according to

$$a \cdot A_r \equiv \tfrac{1}{2}\left(aA_r - (-1)^r A_r a\right) = -(-1)^r A_r \cdot a, \tag{2.15}$$

$$a \wedge A_r = \tfrac{1}{2}\left(aA_r + (-1)^r A_r a\right) = (-1)^r A_r \wedge a. \tag{2.16}$$

The inner product has symmetry opposite to that of the outer product, therefore the signs in front of the second terms in the above equations are different.
Combining (2.15) and (2.16) we find

$$aA_r = a \cdot A_r + a \wedge A_r. \tag{2.17}$$

For $A_r = a_1 \wedge a_2 \wedge ... \wedge a_r$ eq. (2.15) can be evaluated to give the useful expansion

$$a \cdot (a_1 \wedge ... \wedge a_r) = \sum_{k=1}^{r} (-1)^{k+1} (a \cdot a_k) a_1 \wedge ... a_{k-1} \wedge a_{k+1} \wedge ... a_r. \tag{2.18}$$

In particular,

$$a \cdot (b \wedge c) = (a \cdot b)c - (a \cdot c)b. \tag{2.19}$$

It is very convenient to introduce, besides the basic vectors e_μ, another set of basic vectors e^ν by the condition

$$e_\mu \cdot e^\nu = \delta_\mu{}^\nu. \tag{2.20}$$

Each e^μ is a linear combination of e_ν:

$$e^\mu = g^{\mu\nu} e_\nu, \tag{2.21}$$

from which we have

$$g^{\mu\alpha} g_{\alpha\nu} = \delta_\mu{}^\nu \tag{2.22}$$

and

$$g^{\mu\nu} = e^\mu \cdot e^\nu = \tfrac{1}{2}(e^\mu e^\nu + e^\nu e^\mu). \tag{2.23}$$

2.2. ALGEBRA OF SPACETIME

In spacetime we have 4 linearly independent vectors e_μ, $\mu = 0, 1, 2, 3$. Let us consider *flat* spacetime. It is then convenient to take orthonormal basis vectors γ_μ

$$\gamma_\mu \cdot \gamma_\nu = \eta_{\mu\nu}, \tag{2.24}$$

where $\eta_{\mu\nu}$ is the diagonal metric tensor with signature (+ - - -).

The Clifford algebra in V_4 is called the *Dirac algebra*. Writing $\gamma_{\mu\nu} \equiv \gamma_\mu \wedge \gamma_\nu$ for a basis bivector, $\gamma_{\mu\nu\rho} \equiv \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho$ for a basis trivector, and $\gamma_{\mu\nu\rho\sigma} \equiv \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho \wedge \gamma_\sigma$ for a basis quadrivector we can express an arbitrary number of the Dirac algebra as

$$D = \sum_r D_r = d + d^\mu \gamma_\mu + \frac{1}{2!}\, d^{\mu\nu} \gamma_{\mu\nu} + \frac{1}{3!}\, d^{\mu\nu\rho} \gamma_{\mu\nu\rho} + \frac{1}{4!}\, d^{\mu\nu\rho\sigma} \gamma_{\mu\nu\rho\sigma}\,, \tag{2.25}$$

where d, d^μ, $d^{\mu\nu}, ...$ are scalar coefficients.

Let us introduce

$$\gamma_5 \equiv \gamma_0 \wedge \gamma_1 \wedge \gamma_2 \wedge \gamma_3 = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \ , \qquad \gamma_5^2 = -1, \tag{2.26}$$

which is the unit element of 4-dimensional volume and is called a *pseudoscalar*. Using the relations

$$\gamma_{\mu\nu\rho\sigma} = \gamma_5 \epsilon_{\mu\nu\rho\sigma}\,, \tag{2.27}$$

$$\gamma_{\mu\nu\rho} = \gamma_{\mu\nu\rho\sigma} \gamma^\rho\,, \tag{2.28}$$

where $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor and introducing the new coefficients

$$S \equiv d\,, \quad V^\mu \equiv d^\mu\,, \quad T^{\mu\nu} \equiv \tfrac{1}{2} d^{\mu\nu}\,,$$
$$C_\sigma \equiv \frac{1}{3!} d^{\mu\nu\rho} \epsilon_{\mu\nu\rho\sigma}\,, \qquad P \equiv \frac{1}{4!} d^{\mu\nu\rho\sigma} \epsilon_{\mu\nu\rho\sigma}\,, \tag{2.29}$$

we can rewrite D of eq. (2.25) as the sum of scalar, vector, bivector, pseudovector and pseudoscalar parts:

$$D = S + V^\mu \gamma_\mu + T^{\mu\nu} \gamma_{\mu\nu} + C^\mu \gamma_5 \gamma_\mu + P \gamma_5. \tag{2.30}$$

POLYVECTOR FIELDS

A polyvector may depend on spacetime points. Let $A = A(x)$ be an r-vector field. Then one can define the *gradient operator* according to

$$\partial = \gamma^\mu \partial_\mu \,, \tag{2.31}$$

where ∂_μ is the usual partial derivative. The gradient operator ∂ can act on any r-vector field. Using (2.17) we have

$$\partial A = \partial \cdot A + \partial \wedge A. \tag{2.32}$$

Example. Let $A = a = a_\nu \gamma^\nu$ be a 1-vector field. Then

$$\begin{aligned} \partial a &= \gamma^\mu \partial_\mu (a_\nu \gamma^\nu) = \gamma^\mu \cdot \gamma^\nu \, \partial_\mu a^\nu + \gamma^\mu \wedge \gamma^\nu \partial_\mu a_\nu \\ &= \partial_\mu a^\mu + \tfrac{1}{2}(\partial_\mu a_\nu - \partial_\nu a_\mu)\gamma^\mu \wedge \gamma^\nu \,. \end{aligned} \tag{2.33}$$

The simple expression ∂a thus contains a scalar and a bivector part, the former being the usual divergence and the latter the usual curl of a vector field.

Maxwell equations. We shall now demonstrate by a concrete physical example the usefulness of Clifford algebra. Let us consider the electromagnetic field which, in the language of Clifford algebra, is a bivector field F. The source of the field is the electromagnetic current j which is a 1-vector field. Maxwell's equations read

$$\partial F = -4\pi j. \tag{2.34}$$

The grade of the gradient operator ∂ is 1. Therefore we can use the relation (2.32) and we find that eq. (2.34) becomes

$$\partial \cdot F + \partial \wedge F = -4\pi j, \tag{2.35}$$

which is equivalent to

$$\partial \cdot F = -4\pi j, \tag{2.36}$$

$$\partial \wedge F = 0, \tag{2.37}$$

since the first term on the left of eq. (2.35) is a vector and the second term is a bivector. This results from the general relation (2.35). It can also be explicitly demonstrated. Expanding

$$F = \tfrac{1}{2} F^{\mu\nu} \, \gamma_\mu \wedge \gamma_\nu \,, \tag{2.38}$$

$$j = j^\mu \gamma_\mu \ , \tag{2.39}$$

we have

$$\begin{aligned} \partial \cdot F &= \gamma^\alpha \partial_\alpha \cdot (\tfrac{1}{2} F^{\mu\nu} \gamma_\mu \wedge \gamma_\nu) = \tfrac{1}{2} \gamma^\alpha \cdot (\gamma_\mu \wedge \gamma_\nu) \partial_\alpha F^{\mu\nu} \\ &= \tfrac{1}{2} \left((\gamma^\alpha \cdot \gamma_\mu)\gamma_\nu - (\gamma^\alpha \cdot \gamma_\nu)\gamma_\mu \right) \partial_\alpha F^{\mu\nu} = \partial_\mu F^{\mu\nu} \gamma_\nu \ , \end{aligned} \tag{2.40}$$

$$\partial \wedge F = \tfrac{1}{2} \gamma^\alpha \wedge \gamma_\mu \wedge \gamma_\nu \, \partial_\alpha F^{\mu\nu} = \tfrac{1}{2} \epsilon^\alpha{}_{\mu\nu\rho} \, \partial_\alpha F^{\mu\nu} \gamma_5 \gamma^\rho \ , \tag{2.41}$$

where we have used (2.19) and eqs.(2.27), (2.28). From the above considerations it then follows that the compact equation (2.34) is equivalent to the usual tensor form of Maxwell equations

$$\partial_\nu F^{\mu\nu} = -4\pi j^\mu \ , \tag{2.42}$$

$$\epsilon^\alpha{}_{\mu\nu\rho} \, \partial_\alpha F^{\mu\nu} = 0. \tag{2.43}$$

Applying the gradient operator ∂ to the left and to the right side of eq. (2.34) we have

$$\partial^2 F = -4\pi \, \partial j. \tag{2.44}$$

Since $\partial^2 = \partial \cdot \partial + \partial \wedge \partial = \partial \cdot \partial$ is a scalar operator, $\partial^2 F$ is a bivector. The right hand side of eq. (2.44) gives

$$\partial j = \partial \cdot j + \partial \wedge j. \tag{2.45}$$

Equating the terms of the same grade on the left and the right hand side of eq. (2.44) we obtain

$$\partial^2 F = -4\pi \, \partial \wedge j, \tag{2.46}$$

$$\partial \cdot j = 0. \tag{2.47}$$

The last equation expresses the conservation of the electromagnetic current.

Motion of a charged particle. In this example we wish to go a step forward. Our aim is not only to describe how a charged particle moves in an electromagnetic field, but also include a particle' s(classical) spin. Therefore, following Pezzaglia [23], we define the *momentum polyvector* P as the *vector momentum* p plus the bivector *spin angular momentum* S,

$$P = p + S, \tag{2.48}$$

or in components

$$P = p^\mu \gamma_\mu + \tfrac{1}{2} S^{\mu\nu}\, \gamma_\mu \wedge \gamma_\nu. \tag{2.49}$$

We also assume that the condition $p_\mu S^{\mu\nu} = 0$ is satisfied. The latter condition ensures the spin to be a simple bivector, which is purely space-like in the rest frame of the particle. The polyvector equation of motion is

$$\dot{P} \equiv \frac{\mathrm{d}P}{\mathrm{d}\tau} = \frac{e}{2m}\,[P, F], \tag{2.50}$$

where $[P, F] \equiv PF - FP$. The vector and bivector parts of eq. (2.50) are

$$\dot{p}^\mu = \frac{e}{m}\, F^\mu{}_\nu p^\nu \ , \tag{2.51}$$

$$\dot{S}^{\mu\nu} = \frac{e}{2m}(F^\mu{}_\alpha S^{\alpha\nu} - F^\nu{}_\alpha S^{\alpha\mu}). \tag{2.52}$$

These are just the equations of motion for linear momentum and spin, respectively.

2.3. PHYSICAL QUANTITIES AS POLYVECTORS

The compact equations at the end of the last section suggest a generalization that every physical quantity is a polyvector. We shall explore such an assumption and see how far we can come.

In 4-dimensional spacetime *the momentum polyvector* is

$$P = \mu + p^\mu e_\mu + S^{\mu\nu} e_\mu e_\nu + \pi^\mu e_5 e_\mu + m e_5 \ , \tag{2.53}$$

and *the velocity polyvector* is

$$\dot{X} = \dot{\sigma} + \dot{x}^\mu e_\mu + \dot{\alpha}^{\mu\nu} e_\mu e_\nu + \dot{\xi}^\mu e_5 e_\mu + \dot{s} e_5 \ , \tag{2.54}$$

where e_μ are four basis vectors satisfying

$$e_\mu \cdot e_\nu = \eta_{\mu\nu} \ , \tag{2.55}$$

and $e_5 \equiv e_0 e_1 e_2 e_3$ is the pseudoscalar. For the purposes which will become clear later we now use the symbols e_μ, e_5 instead of γ_μ and γ_5.

We associate with each particle the velocity polyvector $\dot{X}$ and its conjugate momentum polyvector P. These quantities are generalizations of the point particle 4-velocity $\dot{x}$ and its conjugate momentum p. Besides a vector part we now include the scalar part $\dot{\sigma}$, the bivector part $\dot{\alpha}^{\mu\nu} e_\mu e_\nu$, the pseudovector part $\dot{\xi}^\mu e_5 e_\mu$ and the pseudoscalar part $\dot{s} e_5$ into the definition of the particle's velocity, and analogously for the particle's momentum. We would now like to derive the equations of motion which will tell us how those quantities depend on the evolution parameter τ. For simplicity we consider a free particle.

Let the action be a straightforward generalization of the first order or phase space action (1.11) of the usual constrained point particle relativistic theory:

$$I[X,P,\lambda] = \frac{1}{2}\int \mathrm{d}\tau \, \left(P\dot{X} + \dot{X}P - \lambda(P^2 - K^2)\right), \tag{2.56}$$

where λ is a scalar Lagrange multiplier and K a polyvector constant[4]:

$$K^2 = \kappa^2 + k_\mu e_\mu + K^{\mu\nu} e_\mu e_\nu + K^\mu e_5 e_\mu + k^2 e_5. \tag{2.57}$$

It is a generalization of particle's mass squared. In the usual, unconstrained, theory, mass squared was a scalar constant, but here we admit that, in principle, mass squared is a polyvector. Let us now insert the explicit expressions (2.53),(2.54) and (2.57) into the Lagrangian

$$L = \tfrac{1}{2}\left(P\dot{X} + \dot{X}P - \lambda(P^2 - K^2)\right) = \sum_{r=0}^{4} \langle L \rangle_r \ , \tag{2.58}$$

and evaluate the corresponding multivector parts $\langle L \rangle_r$. Using

$$e_\mu \wedge e_\nu \wedge e_\rho \wedge e_\sigma = e_5 \, \epsilon_{\mu\nu\rho\sigma} \ , \tag{2.59}$$

$$e_\mu \wedge e_\nu \wedge e_\rho = (e_\mu \wedge e_\nu \wedge e_\rho \wedge e_\sigma) e^\sigma = e_5 \, \epsilon_{\mu\nu\rho\sigma} e^\sigma \ , \tag{2.60}$$

$$e_\mu \wedge e_\nu = -\tfrac{1}{2}(e_\mu \wedge e_\nu \wedge e_\rho \wedge e_\sigma)(e^\rho \wedge e^\sigma) = -\tfrac{1}{2} e_5 \, \epsilon_{\mu\nu\rho\sigma} e^\rho \wedge e_\sigma \ , \tag{2.61}$$

we obtain

$$\begin{aligned}\langle L \rangle_0 = {} & \mu\dot{\sigma} - m\dot{s} + p_\mu \dot{x}^\mu + \pi_\mu \dot{\xi}^\mu + S^{\mu\nu} \dot{\alpha}^{\rho\sigma} \eta_{\mu\sigma}\eta_{\nu\rho} \\ & - \frac{\lambda}{2}(\mu^2 + p^\mu p_\mu + \pi^\mu \pi_\mu - m^2 - 2S^{\mu\nu}S_{\mu\nu} - \kappa^2) \ ,\end{aligned} \tag{2.62}$$

$$\begin{aligned}\langle L \rangle_1 = {} & \left[\dot{\sigma} p_\sigma + \mu \dot{x}_\sigma - (\dot{\xi}^\rho S^{\mu\nu} + \pi^\rho \dot{\alpha}^{\mu\nu})\epsilon_{\mu\nu\rho\sigma}\right] e^\sigma \\ & - \lambda(\mu p_\sigma - S^{\mu\nu}\pi^\rho \epsilon_{\mu\nu\rho\sigma} - \tfrac{1}{2} k_\sigma) e^\sigma \ ,\end{aligned} \tag{2.63}$$

[4]The scalar part is not restricted to positive values, but for later convenience we write it as κ^2, on the understanding that κ^2 can be positive, negative or zero.

$$\langle L\rangle_2 = \left[\tfrac{1}{2}(\pi^\mu \dot{x}^\nu - p^\mu \dot{\xi}^\nu + \dot{s}S^{\mu\nu} + m\dot{\alpha}^{\mu\nu})\epsilon_{\mu\nu\rho\sigma} + \dot{\sigma}S_{\rho\sigma} + \mu\dot{\alpha}_{\rho\sigma} + 2S_{\rho\nu}\dot{\alpha}^\nu_\sigma\right] e^\rho \wedge e^\sigma$$
$$-\frac{\lambda}{2}\left[(\pi^\mu p^\nu + mS^{\mu\nu})\epsilon_{\mu\nu\rho\sigma} + 2\mu S_{\rho\sigma} - K_{\rho\sigma}\right] e^\rho \wedge e^\sigma \;, \qquad (2.64)$$

$$\langle L\rangle_3 = \left[\dot{\sigma}\pi^\sigma + \mu\dot{\xi}^\sigma + (S^{\mu\nu}\dot{x}^\rho + \dot{\alpha}^{\mu\nu}p^\rho)\epsilon_{\mu\nu\rho}{}^\sigma - \lambda(\mu\pi^\sigma + S^{\mu\nu}p^\rho\epsilon_{\mu\nu\rho}{}^\sigma - \tfrac{1}{2}\kappa_\sigma)\right] e_5 e^\sigma \;, \qquad (2.65)$$

$$\langle L\rangle_4 = \left[m\dot{\sigma} + \mu\dot{s} - \tfrac{1}{2}\, S^{\mu\nu}\,\dot{\alpha}^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma} - \frac{\lambda}{2}(2\mu m + S^{\mu\nu}S^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma} - k^2)\right] e_5. \qquad (2.66)$$

The equations of motion are obtained for each pure grade multivector $\langle L\rangle_r$ separately. That is, when varying the polyvector action I, we vary each of its r-vector parts separately. From the scalar part $\langle L\rangle_0$ we obtain

$$\begin{aligned}
\delta\mu &\quad : \quad \dot{\sigma} - \lambda\mu = 0, && (2.67)\\
\delta m &\quad : \quad -\dot{s} + \lambda m = 0, && (2.68)\\
\delta s &\quad : \quad \dot{m} = 0 && (2.69)\\
\delta\sigma &\quad : \quad \dot{\mu} = 0, && (2.70)\\
\delta p_\mu &\quad : \quad \dot{x}^\mu - \lambda p^\mu = 0, && (2.71)\\
\delta\pi_\mu &\quad : \quad \dot{\xi}^\mu - \lambda\pi^\mu = 0, && (2.72)\\
\delta x^\mu &\quad : \quad \dot{p}^\mu = 0 && (2.73)\\
\delta\xi^\mu &\quad : \quad \dot{\pi}_\mu = 0, && (2.74)\\
\delta\alpha^{\mu\nu} &\quad : \quad \dot{S}_{\mu\nu} = 0, && (2.75)\\
\delta S_{\mu\nu} &\quad : \quad \dot{\alpha}^{\mu\nu} - \lambda S^{\mu\nu} = 0. && (2.76)
\end{aligned}$$

From the r-vector parts $\langle L\rangle_r$ for $r = 1, 2, 3, 4$ we obtain the same set of equations (2.67)–(2.76). Each individual equation results from varying a different variable in $\langle L\rangle_0$, $\langle L\rangle_1$, etc.. Thus, for instance, the μ-equation of motion (2.67) from $\langle L\rangle_0$ is the same as the p_μ equation from $\langle L\rangle_1$ and the same as the m-equation from $\langle L\rangle_4$, and similarly for all the other equations (2.67)–(2.76). Thus, as far as the variables μ, m, s, σ, p_μ, π_μ, $S_{\mu\nu}$, $\xi^{\mu\nu}$ and $\alpha^{\mu\nu}$ are considered, the higher grade parts $\langle L\rangle_r$ of the Lagrangian L contains the same information about the equations of motion. The difference occurs if we consider the *Lagrange multiplier* λ. Then every r-vector part of L gives a different equation of motion:

$$\frac{\partial\langle L\rangle_0}{\partial\lambda} = 0 \;:\quad \mu^2 + p^\mu p_\mu + \pi^\mu\pi_\mu - m^2 - 2S^{\mu\nu}S_{\mu\nu} - \kappa^2 = 0, \qquad (2.77)$$

$$\frac{\partial \langle L \rangle_1}{\partial \lambda} = 0 : \quad \mu \pi_\sigma - S^{\mu\nu} \pi^\rho \epsilon_{\mu\nu\rho\sigma} - \tfrac{1}{2} k_\sigma = 0, \tag{2.78}$$

$$\frac{\partial \langle L \rangle_2}{\partial \lambda} = 0 : \quad (\pi^\mu \pi^\nu + m S^{\mu\nu}) \epsilon_{\mu\nu\rho\sigma} + 2\mu S_{\rho\sigma} - K_{\rho\sigma} = 0, \tag{2.79}$$

$$\frac{\partial \langle L \rangle_3}{\partial \lambda} = 0 : \quad \mu \pi_\sigma + S^{\mu\nu} p^\rho \epsilon_{\mu\nu\rho\sigma} + \tfrac{1}{2} \kappa_\sigma = 0, \tag{2.80}$$

$$\frac{\partial \langle L \rangle_4}{\partial \lambda} = 0 : \quad 2\mu m + S^{\mu\nu} S^{\rho\sigma} \epsilon_{\mu\nu\rho\sigma} - k^2 = 0. \tag{2.81}$$

The above equations represent constraints among the dynamical variables. Since our Lagrangian is not a scalar but a polyvector, we obtain more than one constraint.

Let us rewrite eqs.(2.80),(2.78) in the forms

$$\pi_\sigma - \frac{\kappa_\sigma}{2\mu} = -\frac{1}{\mu} S^{\mu\nu} p^\rho \epsilon_{\mu\nu\rho\sigma} \,, \tag{2.82}$$

$$p_\sigma - \frac{k_\sigma}{2\mu} = \frac{1}{\mu} S^{\mu\nu} \pi^\rho \epsilon_{\mu\nu\rho\sigma}. \tag{2.83}$$

We see from (2.82) that the vector momentum p_μ and its pseudovector partner π_μ are related in such a way that $\pi_\mu - \kappa_\mu/2\mu$ behaves as the well known *Pauli-Lubanski spin pseudo vector*. A similar relation (2.83) holds if we interchange p_μ and π_μ.

Squaring relations (2.82), (2.83) we find

$$(\pi_\sigma - \frac{\kappa_\sigma}{2\mu})(\pi^\sigma - \frac{\kappa^\sigma}{2\mu}) = -\frac{2}{\mu^2} p_\sigma p^\sigma S_{\mu\nu} S^{\mu\nu} + \frac{4}{\mu^2} p_\mu p^\nu S^{\mu\sigma} S_{\nu\sigma} \,, \tag{2.84}$$

$$(p_\sigma - \frac{k_\sigma}{2\mu})(p^\sigma - \frac{k^\sigma}{2\mu}) = -\frac{2}{\mu^2} \pi_\sigma \pi^\sigma S_{\mu\nu} S^{\mu\nu} + \frac{4}{\mu^2} \pi_\mu \pi^\nu S^{\mu\sigma} S_{\nu\sigma}. \tag{2.85}$$

From (2.82), (2.83) we also have

$$\left(\pi_\sigma - \frac{\kappa_\sigma}{2\mu} \right) p^\sigma = 0 \,, \quad \left(p_\sigma - \frac{k_\sigma}{2\mu} \right) \pi^\sigma = 0. \tag{2.86}$$

Additional interesting equations which follow from (2.82), (2.83) are

$$p^\rho p_\rho \left(1 + \frac{2}{\mu^2} S_{\mu\nu} S^{\mu\nu} \right) - \frac{4}{\mu^2} p_\mu p^\nu S^{\mu\sigma} S_{\nu\sigma} = -\frac{\kappa_\rho \kappa^\rho}{4\mu^2} + \frac{1}{2\mu} (p_\rho k^\rho + \pi_\rho \kappa^\rho), \tag{2.87}$$

$$\pi^\rho \pi_\rho \left(1 + \frac{2}{\mu^2} S_{\mu\nu} S^{\mu\nu} \right) - \frac{4}{\mu^2} \pi_\mu \pi^\nu S^{\mu\sigma} S_{\nu\sigma} = -\frac{k_\rho k^\rho}{4\mu^2} + \frac{1}{2\mu} (p_\rho k^\rho + \pi_\rho \kappa^\rho). \tag{2.88}$$

Contracting (2.82), (2.83) by $\epsilon^{\alpha_1\alpha_2\alpha_3\sigma}$ we can express $S^{\mu\nu}$ in terms of p^ρ and π^σ

$$S^{\mu\nu} = \frac{\mu}{2p^\alpha p_\alpha}\,\epsilon^{\mu\nu\rho\sigma} p_\rho \left(\pi_\sigma - \frac{\kappa_\sigma}{2\mu}\right) = -\,\frac{\mu}{2\pi^\alpha \pi_\alpha}\,\epsilon^{\mu\nu\rho\sigma}\pi_\rho \left(p_\sigma - \frac{k_\sigma}{2\mu}\right), \quad (2.89)$$

provided that we assume the following extra condition:

$$S^{\mu\nu} p_\nu = 0 \ , \qquad S^{\mu\nu}\pi_\nu = 0. \qquad (2.90)$$

Then for positive $p^\sigma p_\sigma$ it follows from (2.84) that $(\pi_\sigma - \kappa_\sigma/2\mu)^2$ is negative, i.e., $\pi_\sigma - \kappa_\sigma/2\mu$ are components of a space-like (pseudo-) vector. Similarly, it follows from (2.85) that when $\pi^\sigma\pi_\sigma$ is negative, $(p_\sigma - k_\sigma/2\mu)^2$ is positive, so that $p_\sigma - k_\sigma/2\mu$ is a time-like vector. Altogether we thus have that p_σ, k_σ are time-like and π_σ, κ_σ are space-like. Inserting (2.89) into the remaining constraint (2.81) and taking into account the condition (2.90) we obtain

$$2m\mu - k^2 = 0. \qquad (2.91)$$

The polyvector action (2.56) is thus shown to represent a very interesting classical dynamical system with spin. The interactions could be included by generalizing the minimal coupling prescription. *Gravitational interaction* is included by generalizing (2.55) to

$$e_\mu \cdot e_\nu = g_{\mu\nu} \ , \qquad (2.92)$$

where $g_{\mu\nu}(x)$ is the spacetime metric tensor. *A gauge interaction* is included by introducing a polyvector gauge field A, a polyvector coupling constant G, and assume an action of the kind

$$I[X,P,\lambda] = \tfrac{1}{2}\int \mathrm{d}\tau \, \left[P\dot{X} + \dot{X}P - \lambda\left((P - G\star A)^2 - K^2\right)\right], \qquad (2.93)$$

where '$\star$' means the scalar product between Clifford numbers, so that $G \star A \equiv \langle GA\rangle_0$. The polyvector equations of motion can be elegantly obtained by using the Hestenes formalism for multivector derivatives. We shall not go into details here, but merely sketch a plausible result,

$$\dot{\Pi} = \lambda[G\star\partial_X A, P] \ , \qquad \Pi \equiv P - G\star A \ , \qquad (2.94)$$

which is a generalized Lorentz force equation of motion, a more particular case of which is given in (2.50).

After this short digression let us return to our free particle case. One question immediately arises, namely, what is the physical meaning of the polyvector mass squared K^2. Literally this means that a particle is characterized not only by a scalar and/or a pseudoscalar mass squared, but also by a vector, bivector and pseudovector mass squared. To a particle are thus associated a constant vector, 2-vector, and 3-vector which point into fixed directions in spacetime, regardless of the direction of particle's motion. For a given particle the Lorentz symmetry is thus broken, since there exists a preferred direction in spacetime. This cannot be true at the fundamental level. Therefore the occurrence of the polyvector K^2 in the action must be a result of a more fundamental dynamical principle, presumably an action in a higher-dimensional spacetime without such a fixed term K^2. It is well known that the scalar mass term in 4-dimensions can be considered as coming from a massless action in 5 or more dimensions. Similarly, also the 1-vector, 2-vector, and 3-vector terms of K^2 can come from a higher-dimensional action without a K^2-term. Thus *in 5-dimensions*:

(i) *the scalar constraint* will contain the term $p^A p_A = p^\mu p_\mu + p^5 p_5$, and the constant $-p^5 p_5$ takes the role of the scalar mass term in 4-dimensions;

(ii) *the vector constraint* will contain a term like $P_{ABC} S^{AB} e^C$, $A, B = 0, 1, 2, 3, 5$, containing the term $P_{\mu\nu\alpha} S^{\mu\nu} e^\alpha$ (which, since $P_{\mu\nu\alpha} = \epsilon_{\mu\nu\alpha\beta} \pi^\beta$, corresponds to the term $S^{\mu\nu} \pi^\rho \epsilon_{\mu\nu\rho\sigma} e^\sigma$) plus an extra term $P_{5\nu\alpha} S^{5\alpha} e^\alpha$ which corresponds to the term $k^\alpha e_\alpha$.

In a similar manner we can generate the 2-vector term $K_{\mu\nu}$ and the 3-vector term κ_σ from 5-dimensions.

The polyvector mass term K^2 in our 4-dimensional action (2.93) is arbitrary in principle. Let us find out what happens if we set $K^2 = 0$. Then, in the presence of the condition (2.90), eqs. (2.87) or (2.88) imply

$$S_{\mu\nu} S^{\mu\nu} = -\frac{\mu^2}{2}, \tag{2.95}$$

that is $S_{\mu\nu} S^{\mu\nu} < 0$. On the other hand $S_{\mu\nu} S^{\mu\nu}$ in the presence of the condition (2.90) can only be positive (or zero), as can be straightforwardly verified. In 4-dimensional spacetime $S_{\mu\nu} S^{\mu\nu}$ were to be negative only if in the particle's rest frame the spin components S^{0r} were different from zero which would be the case if (2.90) would not hold.

Let us assume that $K^2 = 0$ and that condition (2.90) does hold. Then the constraints (2.78)–(2.81) have a solution[5]

$$S^{\mu\nu} = 0 \, , \quad \pi^\mu = 0 \, , \quad \mu = 0. \tag{2.96}$$

[5]This holds even if we keep κ^2 different from zero, but take vanishing values for k^2, κ_μ, k_μ and $K_{\mu\nu}$.

The only remaining constraint is thus

$$p^\mu p_\mu - m^2 = 0, \tag{2.97}$$

and the polyvector action (2.56) is simply

$$\begin{aligned} I[X, P, \lambda] &= I[s, m, x^\mu, p_\mu, \lambda] \\ &= \int \mathrm{d}\tau \left[-m\dot{s} + p_\mu \dot{x}^\mu - \frac{\lambda}{2}(p^\mu p_\mu - m^2) \right], \end{aligned} \tag{2.98}$$

in which the mass m is a dynamical variable conjugate to s. In the action (2.98) mass is thus just a pseudoscalar component of the polymomentum

$$P = p^\mu e_\mu + m e_5 \ , \tag{2.99}$$

and $\dot{s}$ is a pseudoscalar component of the velocity polyvector

$$\dot{X} = \dot{x}^\mu e_\mu + \dot{s} e_5. \tag{2.100}$$

Other components of the polyvectors $\dot{X}$ and P (such as $S^{\mu\nu}$, π^μ, μ), when $K^2 = 0$ (or more weakly, when $K^2 = \kappa^2$), are automatically eliminated by the constraints (2.77)–(2.81).

From a certain point of view this is very good, since our analysis of the polyvector action (2.56) has driven us close to the conventional point particle theory, with the exception that mass is now a dynamical variable. This reminds us of the Stueckelberg point particle theory [2]–[15] in which mass is a constant of motion. This will be discussed in the next section. We have here demonstrated in a very elegant and natural way that the Clifford algebra generalization of the classical point particle in four dimensions tells us that a fixed mass term in the action cannot be considered as fundamental. This is not so obvious for the scalar (or pseudoscalar) part of the polyvector mass squared term K^2, but becomes dramatically obvious for the 1-vector, 2-vector and 4-vector parts, because they imply a preferred direction in spacetime, and such a preferred direction cannot be fundamental if the theory is to be Lorentz covariant.

This is a very important point and I would like to rephrase it. We start with the well known relativistic constrained action

$$I[x^\mu, p_\mu, \lambda] = \int \mathrm{d}\tau \left(p_\mu \dot{x}^\mu - \frac{\lambda}{2}(p^2 - \kappa^2) \right). \tag{2.101}$$

Faced with the existence of the geometric calculus based on Clifford algebra, it is natural to generalize this action to polyvectors. Concerning the fixed mass constant κ^2 it is natural to replace it by a fixed polyvector or to discard it. If we discard it we find that mass is nevertheless present, because now momentum is a polyvector and as such it contains a pseudoscalar part me_5. If we keep the fixed mass term then we must also keep, in principle, its higher grade parts, but this is in conflict with Lorentz covariance. Therefore the fixed mass term in the action is not fundamental but comes, for instance, from higher dimensions. Since, without the K^2 term, in the presence of the condition $S^{\mu\nu}p_\nu = 0$ we cannot have classical spin in four dimensions (eq. (2.95) is inconsistent), this points to the existence of higher dimensions. Spacetime must have more than four dimensions, where we expect that the constraint $P^2 = 0$ (without a fixed polyvector mass squared term K) allows for nonvanishing classical spin.

The "fundamental" classical action is thus a polyvector action in higher dimensions without a fixed mass term. Interactions are associated with the metric of V_N. Reduction to four dimensions gives us gravity plus gauge interactions, such as the electromagnetic and Yang–Mills interactions, and also the classical spin which is associated with the bivector dynamical degrees of freedom sitting on the particle, for instance the particle's finite extension, magnetic moment, and similar.

There is a very well known problem with Kaluza–Klein theory, since in four dimensions a charged particle's mass cannot be smaller that the Planck mass. Namely, when reducing from five to four dimensions mass is given by $p^\mu p_\mu = \hat{m}^2 + \hat{p}_5^2$, where $\hat{m}$ is the 5-dimensional mass. Since $\hat{p}_5$ has the role of electric charge e, the latter relation is problematic for the electron: in the units in which $\hbar = c = G = 1$ the charge e is of the order of the Planck mass, so $p^\mu p_\mu$ is also of the same order of magnitude. There is no generally accepted mechanism for solving such a problem. In the polyvector generalization of the theory, the scalar constraint is (2.77) and in five or more dimensions it assumes an even more complicated form. The terms in the constraint have different signs, and the 4-dimensional mass $p^\mu p_\mu$ is not necessarily of the order of the Planck mass: there is a lot of room to "make" it small.

All those considerations clearly illustrate why the polyvector generalization of the point particle theory is of great physical interest.

2.4. THE UNCONSTRAINED ACTION FROM THE POLYVECTOR ACTION

FREE PARTICLE

In the previous section we have found that when the polyvector fixed mass squared K^2 is zero then a possible solution of the equations of motion satisfies (2.96) and the generic action (2.56) simplifies to

$$I[s, m, x^\mu, p_\mu, \lambda] = \int \mathrm{d}\tau \left[-m\dot{s} + p_\mu \dot{x}^\mu - \frac{\lambda}{2}(p^\mu p_\mu - m^2) \right]. \tag{2.102}$$

At this point let us observe that a similar action, with a scalar variable s, has been considered by DeWitt [25] and Rovelli [26]. They associate the variable s with the clock carried by the particle. We shall say more about that in Sec. 6.2.

We are now going to show that the latter action is equivalent to the Stueckelberg action discussed in Chapter 1.

The equations of motion resulting from (2.102) are

$$\begin{aligned} \delta s &: \quad \dot{m} = 0, && (2.103)\\ \delta m &: \quad \dot{s} - \lambda m = 0, && (2.104)\\ \delta x^\mu &: \quad \dot{p}_\mu = 0, && (2.105)\\ \delta p_\mu &: \quad \dot{x}^\mu - \lambda p^\mu =, 0 && (2.106)\\ \delta \lambda &: \quad p^\mu p_\mu - m^2 = 0. && (2.107)\\ & && (2.108) \end{aligned}$$

We see that in this dynamical system mass m is one of the dynamical variables; it is canonically conjugate to the variable s. From the equations of motion we easily read out that s is the proper time. Namely, from (2.104), (2.106) and (2.107) we have

$$p^\mu = \frac{\dot{x}^\mu}{\lambda} = m \, \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \, , \tag{2.109}$$

$$\dot{s}^2 = \lambda^2 m^2 = \dot{x}^2 \, , \quad \text{i.e} \qquad \mathrm{d}s^2 = \mathrm{d}x^\mu \mathrm{d}x_\mu. \tag{2.110}$$

Using eq. (2.104) we find that

$$-m\dot{s} + \frac{\lambda}{2}\kappa^2 = -\frac{m\dot{s}}{2} = -\frac{1}{2}\frac{\mathrm{d}(ms)}{\mathrm{d}\tau}. \tag{2.111}$$

The action (2.102) then becomes

$$I = \int \mathrm{d}\tau \left(\frac{1}{2}\frac{\mathrm{d}(ms)}{\mathrm{d}\tau} + p_\mu \dot{x}^\mu - \frac{\lambda}{2} \, p^\mu p_\mu \right) \, , \tag{2.112}$$

where λ should be no more considered as a quantity to be varied, but it is now fixed: $\lambda = \Lambda(\tau)$. The total derivative in (2.112) can be omitted, and the action is simply

$$I[x^\mu, p_\mu] = \int \mathrm{d}\tau (p_\mu \dot{x}^\mu - \frac{\Lambda}{2} \, p^\mu p_\mu). \tag{2.113}$$

This is just *the Stueckelberg action* (1.36) with $\kappa^2 = 0$. The equations of motion derived from (2.113) are

$$\dot{x}^\mu - \Lambda p^\mu = 0, \tag{2.114}$$

$$\dot{p}_\mu = 0. \tag{2.115}$$

From (2.115) it follows that $p_\mu p^\mu$ is a constant of motion. Denoting the latter constant of motion as m and using (2.114) we obtain that momentum can be written as

$$p^\mu = m \, \frac{\dot{x}^\mu}{\sqrt{\dot{x}^\nu \dot{x}_\nu}} = m \, \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \, , \quad \mathrm{d}s = \mathrm{d}x^\mu \mathrm{d}x_\mu \, , \tag{2.116}$$

which is the same as in eq. (2.109). The equations of motion for x^μ and p_μ derived from the Stueckelberg action (2.99) are the same as the equations of motion derived from the action (2.102). *A generic Clifford algebra action* (2.56) *thus leads directly to the Stueckleberg action.*

The above analysis can be easily repeated for a more general case where the scalar constant κ^2 is different from zero, so that instead of (2.98) or (2.102) we have

$$I[s, m, x^\mu, p_\mu, \lambda] = \int \mathrm{d}\tau \left[-m\dot{s} + p_\mu \dot{x}^\mu - \frac{\lambda}{2}(p^\mu p_\mu - m^2 - \kappa^2) \right]. \tag{2.117}$$

Then instead of (2.113) we obtain

$$I[x^\mu, p_\mu] = \int \mathrm{d}\tau \left(p_\mu \dot{x}^\mu - \frac{\Lambda}{2} \, (p^\mu p_\mu - \kappa^2) \right). \tag{2.118}$$

The corresponding Hamiltonian is

$$H = \frac{\Lambda}{2} \, (p^\mu p_\mu - \kappa^2), \tag{2.119}$$

and in the quantized theory the Schrödinger equation reads

$$i \frac{\partial \psi}{\partial \tau} = \frac{\Lambda}{2} \, (p^\mu p_\mu - \kappa^2) \psi. \tag{2.120}$$

Alternatively, in the action (2.102) or (2.117) we can first eliminate λ by the equation of motion (2.104). So we obtain

$$I[s, m, x^\mu, p_\mu] = \int \mathrm{d}\tau \left[-\frac{m\dot{s}}{2} + p_\mu \dot{x}^\mu - \frac{\dot{s}}{2m}(p^\mu p_\mu - \kappa^2) \right]. \tag{2.121}$$

The equations of motion are

$$\delta s \quad : \quad -\frac{\dot m}{2} - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{p^\mu p_\mu - \kappa^2}{2m}\right) = 0 \;\Rightarrow\; \dot m = 0, \tag{2.122}$$

$$\delta m \quad : \quad -\frac{1}{2} + \frac{1}{2m^2}\,(p^\mu p_\mu - \kappa^2) = 0 \;\Rightarrow\; p^\mu p_\mu - m^2 - \kappa^2 = 0, \tag{2.123}$$

$$\delta x^\mu \quad : \quad \dot p_\mu = 0, \tag{2.124}$$

$$\delta p_\mu \quad : \quad \dot x^\mu - \frac{\dot s}{m}\,p^\mu = 0 \;\Rightarrow\; p^\mu = \frac{m \dot x^\mu}{\dot s} = m\,\frac{\mathrm{d}x^\mu}{\mathrm{d}s}. \tag{2.125}$$

Then we can choose a "solution" for $s(\tau)$, write $\dot s/m = \Lambda$, and omit the first term, since in view of (2.122) it is a total derivative. So again we obtain the Stueckelberg action (2.118).

The action that we started from, e.g., (2.121) or (2.102) has a constraint on the variables x^μ, s or on the p_μ, m, but the action (2.118) which we arrived at contains only the variables x^μ, p_μ and has no constraint.

In the action (2.121) we can use the relation $\dot s = \mathrm{d}s/\mathrm{d}\tau$ and write it as

$$I[m, p_\mu, x^\mu] = \int \mathrm{d}s\, \left[-\frac{m}{2} + p_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} - \frac{1}{2m}(p^\mu p_\mu - \kappa^2)\right]. \tag{2.126}$$

The evolution parameter is now s, and again variation with respect to m gives the constraint $p^\mu p_\mu - m^2 - \kappa^2 = 0$. Eliminating m from the action (2.126) by the the latter constraint, written in the form

$$m = \sqrt{p^\mu p_\mu - \kappa^2} \tag{2.127}$$

we obtain *the unconstrained action*

$$I[x^\mu, p_\mu] = \int \mathrm{d}s\, \left(p_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} - \sqrt{p^\mu p_\mu - \kappa^2}\right), \tag{2.128}$$

which is also equivalent to the original action (2.117). The Hamiltonian corresponding to (2.128) is

$$H = p_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} - L = \sqrt{p^\mu p_\mu - \kappa^2}\,. \tag{2.129}$$

Such a Hamiltonian is not very practical for quantization, since the Schrödinger equation contains the square root of operators

$$i\,\frac{\partial \psi}{\partial s} = \sqrt{p^\mu p_\mu - \kappa^2}\,\psi. \tag{2.130}$$

In order to perform the quantization properly one has to start directly from the original polyvector action (2.56). This will be discussed in Sec. 2.5.

However, in the approximation[6] $p_\mu p^\mu \ll -\kappa^2$ eq. (2.128) becomes

$$I[x^\mu, p_\mu] \approx \int \mathrm{d}s \, \left(p_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} - \frac{1}{2\sqrt{-\kappa^2}} p^\mu p_\mu - \sqrt{-\kappa^2} \right) \tag{2.131}$$

which is again the Stueckelberg action, but with $1/\sqrt{-\kappa^2} = \Lambda$. It is very interesting that on the one hand the Stueckelberg action arises exactly from the polyvector action, and on the other hand it arises as an approximation.

PARTICLE IN A FIXED BACKGROUND FIELD

Let us now consider the action (2.117) and modify it so that it will remain covariant under the transformation

$$L \to L' = L + \frac{\mathrm{d}\phi}{\mathrm{d}\tau} \, , \tag{2.132}$$

where

$$\phi = \phi(s, x^\mu) \tag{2.133}$$

For this purpose we have to introduce the gauge fields A_μ and V which transform according to

$$eA'_\mu = eA_\mu + \partial_\mu \phi, \tag{2.134}$$

$$eV' = eV + \frac{\partial \phi}{\partial s}. \tag{2.135}$$

The covariant action is then

$$I = \int \mathrm{d}\tau \left[-m\dot{s} + p_\mu \dot{x}^\mu - \frac{\lambda}{2}(\pi_\mu \pi^\mu - \mu^2 - \kappa^2) \right] , \tag{2.136}$$

where we have introduced the kinetic momentum

$$\pi_\mu = p_\mu - eA_\mu \tag{2.137}$$

and its pseudoscalar counterpart

$$\mu = m + eV. \tag{2.138}$$

The symbol 'μ' here should not be confused with the same symbol used in Sec. 2.3 for a completely different quantity.

[6]Remember that κ^2 comes from the scalar part of the polyvector mass squared term (2.57) and that it was a matter of our convention of writing it in the form κ^2. We could have used another symbol without square, e.g., α, and then it would be manifestly clear that α can be negative.

From (2.136) we derive the following equations of motion:

$$\delta x^\mu \quad : \qquad \dot{\pi}_\mu = eF_{\mu\nu}\dot{x}^\nu - \dot{s}e\left(\frac{\partial A_\mu}{\partial s} - \partial_\mu V\right), \tag{2.139}$$

$$\delta s \quad : \qquad \dot{\mu} = -\dot{x}^\nu e\left(\frac{\partial A_\nu}{\partial s} - \partial_\nu V\right), \tag{2.140}$$

$$\delta p_\mu \quad : \qquad \lambda\pi_\mu = \dot{x}_\mu\ , \tag{2.141}$$

$$\tag{2.142}$$

$$\delta m \quad : \qquad \dot{s} = \lambda\mu. \tag{2.143}$$

These equations of motion are the same as those from the Stueckelberg action (2.147).

From (2.140) and (2.143) we have

$$-m\dot{s} + \frac{\lambda}{2}\mu^2 = -\tfrac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(\mu s) + e\dot{s}V - \frac{1}{2}e\dot{x}^\nu\left(\frac{\partial A_\nu}{\partial s} - \partial_\nu V\right)s. \tag{2.144}$$

Inserting the latter expression into the action (2.136) we obtain

$$I = \int \mathrm{d}\tau\Bigg[-\tfrac{1}{2}\frac{\mathrm{d}(\mu s)}{\mathrm{d}\tau} + p_\mu\dot{x}^\mu - \frac{\lambda}{2}(\pi^\mu\pi_\mu - \kappa^2) + e\dot{s}V \\ -\frac{1}{2}e\dot{x}^\nu\left(\frac{\partial A_\nu}{\partial s} - \partial_\nu V\right)s\Bigg], \tag{2.145}$$

which is analogous to eq. (2.112). However, in general $\dot{\mu}$ is now not zero, and as a result we cannot separate the variables m, s into a total derivative term as we did in (2.117).

Let us consider a particular case when the background fields A_μ, V satisfy

$$\frac{\partial A_\mu}{\partial s} = 0\ , \qquad \partial_\mu V = 0\ . \tag{2.146}$$

Then the last term in (2.145) vanishes; in addition we may set $V = 0$. Omitting the total derivative term, eq. (2.145) becomes

$$I[x^\mu, p_\mu] = \int \mathrm{d}s\left[p_\mu\frac{\mathrm{d}x^\mu}{\mathrm{d}s} - \frac{\Lambda}{2}(\pi^\mu\pi_\mu - \kappa^2)\right]\ , \tag{2.147}$$

where $\Lambda = \lambda/\dot{s}$ is now fixed. This is precisely the Stueckelberg action in the presence of a fixed electromagnetic field, and s corresponds to the Stueckelberg Lorentz invariant parameter τ.

However, when we gauged the free particle Stueckelberg action we obtained in general a τ-dependent gauge field A_μ and also a scalar field V.

We shall now see that such a general gauged Stueckelberg action is an approximation to the action (2.136). For this purpose we shall repeat the procedure of eqs. (2.121)–(2.128). Eliminating λ from the action (2.136) by using the equation of motion (2.143) we obtain an equivalent action

$$I[x^\mu, p_\mu, s, m] = \int \mathrm{d}\tau \left[-m\dot{s} + p_\mu \dot{x}^\mu - \frac{\dot{s}}{2\mu}(\pi^\mu \pi_\mu - \mu^2 - \kappa^2) \right] \tag{2.148}$$

whose variation with respect to m again gives the constraint $\pi^\mu \pi_\mu - \mu^2 - \kappa^2 = 0$. From (2.148), using (2.138) we have

$$\begin{aligned} I &= \int \mathrm{d}\tau \left[p_\mu \dot{x}^\mu - \frac{\dot{s}}{2\mu}(\pi^\mu \pi_\mu - \kappa^2) + e\dot{s}V - \frac{\mu \dot{s}}{2} \right] \\ &= \int \mathrm{d}\tau \left[p_\mu \dot{x}^\mu - \dot{s}(\pi^\mu \pi_\mu - \kappa^2)^{1/2} + e\dot{s}V \right] \\ &\approx \int \mathrm{d}\tau \left[p_\mu \dot{x}^\mu - \frac{\dot{s}}{2\sqrt{-\kappa^2}}\, \pi^\mu \pi_\mu - \dot{s}\sqrt{-\kappa^2} + e\dot{s}V \right]. \end{aligned} \tag{2.149}$$

Thus

$$I[x^\mu, p_\mu] = \int \mathrm{d}s \left[p_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} - \frac{1}{2\sqrt{-\kappa^2}}\, \pi^\mu \pi_\mu - \sqrt{-\kappa^2} + eV \right]. \tag{2.150}$$

The last step in eq. (2.149) is valid under the approximation $\pi^\mu \pi_\mu \ll -\kappa^2$, where we assume $-\kappa^2 > 0$. In (2.150) we indeed obtain an action which is equivalent to the gauged Stueckelberg action (2.147) if we make the correspondence $1\sqrt{-\kappa^2} \rightarrow \Lambda$. The constant terms $-\sqrt{-\kappa^2}$ in (2.150) and $\Lambda\kappa^2/2$ in eq. (2.147) have no influence on the equations of motion.

We have thus found a very interesting relation between the Clifford algebra polyvector action and the Stueckelberg action in the presence of an electromagnetic and pseudoscalar field. If the electromagnetic field A_μ does not depend on the pseudoscalar parameter s and if there is no force owed to the pseudoscalar field V, then the kinetic momentum squared $\pi^\mu \pi_\mu$ is a *constant of motion*, and the gauged Clifford algebra action (2.136) is exactly equivalent to the Stueckelberg action. In the presence of a pseudoscalar force, i.e., when $\partial_\mu V \neq 0$ and/or when $\partial A_\mu / \partial s \neq 0$, the action (2.136) is approximately equivalent to the gauged Stueckelberg action (2.147) if the kinetic momentum squared $\pi^\mu \pi_\mu$ is much smaller than the scalar mass constant squared $-\kappa^2$.

2.5. QUANTIZATION OF THE POLYVECTOR ACTION

We have assumed that a point particle's classical motion is governed by the polyvector action (2.56). Variation of this action with respect to λ gives the polyvector constraint

$$P^2 - K^2 = 0. \tag{2.151}$$

In the quantized theory the position and momentum polyvectors $X = X^J e_J$ and $P = P^J e_J$, where $e_J = (1, e_\mu, e_\mu e_\nu, e_5 e_\mu, e_5)$, $\mu < \nu$, become the operators

$$\widehat{X} = \widehat{X}^J e_J \ , \qquad \widehat{P} = \widehat{P}^J e_J; , \tag{2.152}$$

satisfying

$$[\widehat{X}^J, \widehat{P}_K] = i\delta^J{}_K \ . \tag{2.153}$$

Using the explicit expressions like (2.53),(2.54) the above equations imply

$$[\hat{\sigma}, \hat{\mu}] = i \ , \quad [\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu{}_\nu \ , \quad [\hat{\alpha}^{\mu\nu}, \hat{S}_{\alpha,\beta}] = i\delta^{\mu\nu}{}_{\alpha\beta} \ , \tag{2.154}$$

$$[\hat{\xi}^\mu, \hat{\pi}_\nu] = i\delta^\mu{}_\nu \ , \quad [\hat{s}, \hat{m}] = i. \tag{2.155}$$

In a particular representation in which $\widehat{X}^J$ are diagonal, the momentum polyvector operator is represented by the multivector derivative (see Sec. 6.1).

$$\widehat{P}_J = -i\frac{\partial}{\partial X^J} \tag{2.156}$$

Explicitly, the later relation means

$$\hat{m} = -i\frac{\partial}{\partial \sigma} \ , \ \ \hat{p}_\mu = -i\frac{\partial}{\partial x^\mu} \ , \ \ \hat{S}_{\mu\nu} = -i\frac{\partial}{\partial \alpha^{\mu\nu}} \ , \ \ \hat{\pi}_\mu = -i\frac{\partial}{\partial \xi^\mu} \ , \ \ \hat{m} = -i\frac{\partial}{\partial s}. \tag{2.157}$$

Let us assume that a quantum state can be represented by a polyvector-valued wave function $\Phi(X)$ of the position polyvector X. A possible physical state is a solution to the equation

$$(\widehat{P}^2 - K^2)\Phi = 0, \tag{2.158}$$

which replaces the classical constraint (2.151).

When $K^2 = \kappa^2 = 0$ eq. (2.158) becomes

$$\widehat{P}^2 \Phi = 0. \tag{2.159}$$

Amongst the set of functions $\Phi(X)$ there are some such that satisfy

$$\widehat{P}\Phi = 0. \tag{2.160}$$

Let us now consider a special case where Φ has definite values of the operators $\hat{\mu}$, $\hat{S}_{\mu\nu}$, $\hat{\pi}_\mu$:

$$\hat{\mu}\Phi = 0 \ , \quad \hat{S}_{\mu\nu}\Phi = 0 \ , \quad \hat{\pi}_\mu \Phi = 0 \tag{2.161}$$

Then

$$\widehat{P}\Phi = (\hat{p}^\mu e_\mu + \hat{m} e_5)\Phi = 0. \tag{2.162}$$

or

$$(\hat{p}^\mu \gamma_\mu - \hat{m})\Phi = 0, \tag{2.163}$$

where

$$\gamma_\mu \equiv e_5 e_\mu \ , \quad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = e_0 e_1 e_2 e_3 = e_5. \tag{2.164}$$

When Φ is an eigenstate of $\hat{m}$ with definite value m, i.e., when $\hat{m}\phi = m\Phi$, then eq. (2.163) becomes the familiar *Dirac equation*

$$(\hat{p}_\mu \gamma^\mu - m)\Phi = 0. \tag{2.165}$$

A polyvector wave function which satisfies eq. (2.165) is a *spinor*. We have arrived at the very interesting result that *spinors can be represented by particular polyvector wave functions*.

3-dimensional case

To illustrate this let us consider the 3-dimensional space V_3. Basis vectors are σ_1, σ_2, σ_3 and they satisfy the Pauli algebra

$$\sigma_i \cdot \sigma_j \equiv \tfrac{1}{2}(\sigma_i\sigma_j + \sigma_j\sigma_i) = \delta_{ij} \ , \qquad i,j = 1,2,3. \tag{2.166}$$

The unit pseudoscalar

$$\sigma_1\sigma_2\sigma_3 \equiv I \tag{2.167}$$

commutes with all elements of the Pauli algebra and its square is $I^2 = -1$. It behaves as the ordinary imaginary unit i. Therefore, in 3-space, we may identify the imaginary unit i with the unit pseudoscalar I.

An arbitrary polyvector in V_3 can be written in the form

$$\Phi = \alpha^0 + \alpha^i\sigma_i + i\beta^i\sigma_i + i\beta = \Phi^0 + \Phi^i\sigma_i \ , \tag{2.168}$$

where Φ^0, Φ^i are formally complex numbers.

We can decompose [22]:

$$\Phi = \Phi\tfrac{1}{2}(1+\sigma_3) + \Phi\tfrac{1}{2}(1-\sigma_3) = \Phi_+ + \Phi_- \ , \tag{2.169}$$

where $\Phi \in \mathcal{I}_+$ and $\Phi_- \in \mathcal{I}_-$ are independent minimal *left ideals* (see Box 3.2).

Box 3.2: Definition of ideal

A left ideal $\mathcal{I}_L$ in an algebra C is a set of elements such that if $a \in \mathcal{I}_L$ and $c \in C$, then $ca \in \mathcal{I}_L$. If $a \in \mathcal{I}_L$, $b \in \mathcal{I}_L$, then $(a+b) \in \mathcal{I}_L$. A right ideal $\mathcal{I}_R$ is defined similarly except that $ac \in \mathcal{I}_R$. A left (right) minimal ideal is a left (right) ideal which contains no other ideals but itself and the null ideal.

A basis in $\mathcal{I}_+$ is given by two polyvectors

$$u_1 = \tfrac{1}{2}(1+\sigma_3) \ , \quad u_2 = (1-\sigma_3)\sigma_1 \ , \tag{2.170}$$

which satisfy

$$\begin{aligned} \sigma_3 u_1 &= \ \ u_1 \ , & \sigma_1 u_1 &= u_2 \ , & \sigma_2 u_1 &= \ \ iu_2 \ , \\ \sigma_3 u_2 &= -u_2 \ , & \sigma_1 u_2 &= u_1 \ , & \sigma_2 u_2 &= -iu_1 . \end{aligned} \tag{2.171}$$

These are precisely the well known relations for basis spinors. Thus we have arrived at the very profound result that the polyvectors u_1, u_2 behave as basis spinors.

Similarly, a basis in $\mathcal{I}_+$ is given by

$$v_1 = \tfrac{1}{2}(1+\sigma_3)\sigma_1 \ , \quad v_2 = \tfrac{1}{2}(1-\sigma_3) \tag{2.172}$$

and satisfies

$$\begin{aligned} \sigma_3 v_1 &= \ \ v_1 \ , & \sigma_1 v_1 &= v_2 \ , & \sigma_2 v_1 &= \ \ iv_2 \ , \\ \sigma_3 v_2 &= -v_2 \ , & \sigma_1 v_2 &= v_1 \ , & \sigma_2 v_2 &= -iv_1 . \end{aligned} \tag{2.173}$$

A polyvector Φ can be written in *spinor basis* as

$$\Phi = \Phi_+^1 u_1 + \Phi_+^2 u_2 + \Phi_-^1 v_1 + \Phi_-^2 v_2 \ , \tag{2.174}$$

where

$$\begin{aligned} \Phi_+^1 &= \Phi^0 + \Phi^3 \ , \qquad & \Phi_-^1 &= \Phi^1 - i\Phi^2 \\ \Phi_+^2 &= \Phi^1 + i\Phi^2 \ , \qquad & \Phi_-^2 &= \Phi^0 - \Phi^3 \end{aligned} \tag{2.175}$$

Eq. (2.174) is an alternative expansion of a polyvector. We can expand the same polyvector Φ either according to (2.168) or according to (2.174).

Introducing the matrices

$$\xi_{ab} = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} , \qquad \Phi^{ab} = \begin{pmatrix} \Phi_+^1 & \Phi_-^1 \\ \Phi_+^2 & \Phi_-^2 \end{pmatrix} \tag{2.176}$$

we can write (2.174) as

$$\Phi = \Phi^{ab}\xi_{ab}. \tag{2.177}$$

Thus a polyvector can be represented as a *matrix* Φ^{ab}. The decomposition (2.169) then reads

$$\Phi = \Phi_+ + \Phi_- = (\Phi_+^{ab} + \Phi_-^{ab})\xi_{ab} , \tag{2.178}$$

where

$$\Phi_+^{ab} = \begin{pmatrix} \Phi_+^1 & 0 \\ \Phi_+^2 & 0 \end{pmatrix} , \tag{2.179}$$

$$\Phi_-^{ab} = \begin{pmatrix} 0 & \Phi_-^1 \\ 0 & \Phi_-^2 \end{pmatrix} . \tag{2.180}$$

From (2.177) we can directly calculate the matrix elements Φ^{ab}. We only need to introduce the new elements $\xi^{\dagger ab}$ which satisfy

$$({\xi^\dagger}^{ab}\xi_{cd})_S = \delta^a{}_c \delta^b{}_d. \tag{2.181}$$

The superscript $\dagger$ means Hermitian conjugation [22]. If

$$A = A_S + A_V + A_B + A_P \tag{2.182}$$

is a Pauli number, then

$$A^\dagger = A_S + A_V - A_B - A_P. \tag{2.183}$$

This means that the order of basis vectors σ_i in the expansion of $A^\dagger$ is reversed. Thus $u_1^\dagger = u_1$, but $u_2^\dagger = \frac{1}{2}(1 + \sigma_3)\sigma_1$. Since $(u_1^\dagger u_1)_S = \frac{1}{2}$, $(u_2^\dagger u_2)_S = \frac{1}{2}$, it is convenient to introduce ${u^\dagger}^1 = 2u_1$ and ${u^\dagger}^2 = 2u_2$ so that $({u^\dagger}^1 u_1)_S = 1$, $({u^\dagger}^2 u_2)_S = 1$. If we define similar relations for v_1, v_2 then we obtain (2.181).

From (2.177) and (2.181) we have

$$\Phi^{ab} = ({\xi^\dagger}^{ab}\Phi)_I . \tag{2.184}$$

Here the subscript I means *invariant part*, i.e., scalar plus pseudoscalar part (remember that pseudoscalar unit has here the role of imaginary unit and that Φ^{ab} are thus complex numbers).

The relation (2.184) tells us how from an arbitrary polyvector Φ (i.e., a Clifford number) can we obtain its *matrix representation* Φ^{ab}.

Φ in (2.184) is an arbitrary Clifford number. In particular, Φ may be any of the basis vectors σ_i.

Example $\Phi = \sigma_1$:

$$\begin{aligned}
\Phi^{11} &= ({\xi^\dagger}^{11}\sigma_1)_I = ({u^\dagger}^1\sigma_1)_I = ((1+\sigma_3)\sigma_1)_I = 0, \\
\Phi^{12} &= ({\xi^\dagger}^{12}\sigma_1)_I = ({v^\dagger}^1\sigma_1)_I = ((1-\sigma_3)\sigma_1\sigma_1)_I = 1, \\
\Phi^{21} &= ({\xi^\dagger}^{21}\sigma_1)_I = ({u^\dagger}^2\sigma_1)_I = ((1+\sigma_3)\sigma_1\sigma_1)_I = 1, \\
\Phi^{22} &= ({\xi^\dagger}^{22}\sigma_1)_I = ({v^\dagger}^2\sigma_1)_I = ((1-\sigma_3)\sigma_1)_I = 0.
\end{aligned} \tag{2.185}$$

Therefore

$$(\sigma_1)^{ab} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.186}$$

Similarly we obtain from (2.184) when $\Phi = \sigma_2$ and $\Phi = \sigma_3$, respectively, that

$$(\sigma_2)^{ab} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad , \qquad (\sigma_3)^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.187}$$

So we have obtained the matrix representation of the basis vectors σ_i. Actually (2.186), (2.187) are the well known *Pauli matrices*.

When $\Phi = u_1$ and $\Phi = u_2$, respectively, we obtain

$$(u_1)^{ab} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad , \qquad (u_2)^{ab} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{2.188}$$

which are a matrix representation of the *basis spinors* u_1 and u_2.

Similarly we find

$$(v_1)^{ab} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad , \qquad (v_2)^{ab} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.189}$$

In general a *spinor* is a superposition

$$\psi = \psi^1 u_1 + \psi^2 u_2 \,, \tag{2.190}$$

and its matrix representation is

$$\psi \rightarrow \begin{pmatrix} \psi^1 & 0 \\ \psi^2 & 0 \end{pmatrix}. \tag{2.191}$$

Another independent spinor is

$$\chi = \chi^1 v_1 + \chi^2 v_2 \,, \tag{2.192}$$

with matrix representation

$$\chi \to \begin{pmatrix} 0 & \chi^1 \\ 0 & \chi^2 \end{pmatrix} . \tag{2.193}$$

If we multiply a spinor ψ from the left by any element R of the Pauli algebra we obtain another spinor

$$\psi' = R\psi \to \begin{pmatrix} \psi'^1 & 0 \\ \psi'^2 & 0 \end{pmatrix} \tag{2.194}$$

which is an element of the same minimal left ideal. Therefore, if only multiplication from the left is considered, a spinor can be considered as a column matrix

$$\psi \to \begin{pmatrix} \psi^1 \\ \psi^2 \end{pmatrix} . \tag{2.195}$$

This is just the common representation of spinors. But it is not general enough to be valid for all the interesting situations which occur in the Clifford algebra.

We have thus arrived at a very important finding. *Spinors* are just particular *Clifford numbers*: they belong to a left or right minimal ideal. For instance, a generic *spinor* is

$$\psi = \psi^1 u_1 + \psi^2 u_2 \quad \text{with} \quad \Phi^{ab} = \begin{pmatrix} \psi^1 & 0 \\ \psi^2 & 0 \end{pmatrix} . \tag{2.196}$$

A *conjugate spinor* is

$$\psi^\dagger = \psi^{1*} u_1^\dagger + \psi^{2*} u_2^\dagger \quad \text{with} \quad (\Phi^{ab})^* = \begin{pmatrix} \psi^{1*} & \psi^{2*} \\ 0 & 0 \end{pmatrix} \tag{2.197}$$

and it is an element of a minimal *right ideal*.

4-dimensional case

The above considerations can be generalized to 4 or more dimensions. Thus

$$\psi = \psi^0 u_0 + \psi^1 u_1 + \psi^2 u_2 + \psi^3 u_3 \to \begin{pmatrix} \psi^0 & 0 & 0 & 0 \\ \psi^1 & 0 & 0 & 0 \\ \psi^2 & 0 & 0 & 0 \\ \psi^3 & 0 & 0 & 0 \end{pmatrix} \tag{2.198}$$

and

$$\psi^\dagger = \psi^{*0} u_0^\dagger + \psi^{*1} u_1^\dagger + \psi^{*2} u_2^\dagger + \psi^{*3} u_3^\dagger \rightarrow \begin{pmatrix} \psi^{*0} & \psi^{*1} & \psi^{*2} & \psi^{*3} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (2.199)$$

where u_0, u_1, u_2, u_3 are four basis spinors in spacetime, and ψ^0, ψ^1, ψ^2, ψ^3 are complex scalar coefficients.

In 3-space the pseudoscalar unit can play the role of the imaginary unit i. This is not the case of the 4-space V_4, since $e_5 = e_0 e_1 e_2 e_3$ does not commute with all elements of the Clifford algebra in V_4. Here the approaches taken by different authors differ. A straightforward possibility [37] is just to use the complex Clifford algebra with complex coefficients of expansion in terms of multivectors. Other authors prefer to consider real Clifford algebra $\mathcal{C}$ and ascribe the role of the imaginary unit i to an element of $\mathcal{C}$ which commutes with all other elements of $\mathcal{C}$ and whose square is -1. Others [22, 36] explore the possibility of using a non-commuting element as a substitute for the imaginary unit. I am not going to review all those various approaches, but I shall simply assume that the expansion coefficients are in general complex numbers. In Sec. 7.2 I explore the possibility that such complex numbers which occur in the quantized theory originate from the Clifford algebra description of the $(2 \times n)$-dimensional phase space (x^μ, p_μ). In such a way we still conform to the idea that complex numbers are nothing but special Clifford numbers.

A Clifford number ψ expanded according to (2.198) is an element of a left minimal ideal if the four elements u_0, u_1, u_2, u_3 satisfy

$$C u_\lambda = C_{0\lambda} u_0 + C_{1\lambda} u_1 + C_{2\lambda} u_2 + C_{3\lambda} u_3 \quad (2.200)$$

for an arbitrary Clifford number C. General properties of u_λ were investigated by Teitler [37]. In particular, he found the following representation for u_λ:

$$\begin{aligned} u_0 &= \tfrac{1}{4}(1 - e^0 + ie^{12} - ie^{012}), \\ u_1 &= -e^{13} u_0 = \tfrac{1}{4}(-e^{13} + e^{013} + ie^{23} - ie^{023}), \\ u_2 &= -ie^3 u_0 = \tfrac{1}{4}(-ie^3 - ie^{03} + e^{123} + e^{0123}), \\ u_3 &= -ie^1 u_0 = \tfrac{1}{4}(-ie^1 - ie^{01} - e^2 - e^{02}), \end{aligned} \quad (2.201)$$

from which we have

$$\begin{aligned} e^0 u_0 &= -u_0 \,, \\ e^1 u_0 &= i u_3 \,, \\ e^2 u_0 &= -u_3 \,, \\ e^3 u_0 &= i u_2 . \end{aligned} \quad (2.202)$$

Using the representation (2.201) we can calculate from (2.200) the matrix elements $C_{\rho\lambda}$ of any Clifford number. For the spacetime basis vectors $e^\mu \equiv (e^0, e^i)$, $i = 1, 2, 3$, we obtain

$$e^0 = \begin{pmatrix} -\mathbf{1} & 0 \\ 0 & \mathbf{1} \end{pmatrix} , \qquad e^i = \begin{pmatrix} 0 & i\sigma^{\mathbf{i}} \\ i\sigma^{\mathbf{i}} & 0 \end{pmatrix} , \tag{2.203}$$

which is one of the standard matrix representations of e^μ (the Dirac matrices).

If a spinor is multiplied from the left by an arbitrary Clifford number, it remains a spinor. But if is multiplied from the right, it in general transforms into another Clifford number which is no more a spinor. *Scalars, vectors, bivectors, etc., and spinors can be reshuffled by the elements of Clifford algebra: scalars, vectors, etc., can be transformed into spinors, and vice versa.*

Quantum states are assumed to be represented by polyvector wave functions (i.e., Clifford numbers). If the latter are pure scalars, vectors, bivectors, pseudovectors, pseudovectors, and pseudoscalars they describe *bosons*. If, on the contrary, wave functions are spinors, then they describe *fermions*. *Within Clifford algebra we have thus transformations which change bosons into fermions!* It remains to be investigated whether this kind of "supersymmetry" is related to the well known supersymmetry.

2.6. ON THE SECOND QUANTIZATION OF THE POLYVECTOR ACTION

If we first quantize the polyvector action (2.117) we obtain the wave equation

$$(\hat{p}^\mu \hat{p}_\mu - \hat{m}^2 - \kappa^2)\phi = 0, \tag{2.204}$$

where

$$\hat{p}_\mu = -\partial/\partial x^\mu \equiv \partial_\mu \, , \hat{m} = -i\partial/\partial s \, ,$$

and κ is a fixed constant. The latter wave equation can be derived from the action

$$\begin{aligned} I[\phi] &= \tfrac{1}{2} \int \mathrm{d}s \, \mathrm{d}^d x \, \phi(-\partial^\mu \partial_\mu + \frac{\partial^2}{\partial s^2} - \kappa^2)\phi \\ &= \frac{1}{2} \int \mathrm{d}s \, \mathrm{d}^d x \left(\partial_\mu \phi \partial^\mu \phi - \left(\frac{\partial \phi}{\partial s} \right)^2 - \kappa^2 \phi^2 \right) , \end{aligned} \tag{2.205}$$

where in the last step we have omitted the surface and the total derivative terms.

The canonical momentum is

$$\pi(s,x) = \frac{\partial \mathcal{L}}{\partial \partial\phi/s} = -\frac{\partial \phi}{\partial s} \tag{2.206}$$

and the Hamiltonian is

$$H[\phi,\pi] = \int \mathrm{d}^d x \left(\pi \frac{\partial \phi}{\partial s} - \mathcal{L}\right) = \tfrac{1}{2}\int \mathrm{d}^d x \,(-\pi^2 - \partial_\mu \phi \partial^\mu \phi + \kappa^2 \phi^2). \tag{2.207}$$

If instead of one field ϕ there are two fields ϕ_1, ϕ_2 we have

$$\begin{aligned} I[\phi_1,\phi_2] = \int \mathrm{d}s\, \mathrm{d}^d x \Bigg[&\partial_\mu \phi_1 \partial^\mu \phi_1 - \left(\frac{\partial \phi_1}{\partial s}\right)^2 - \kappa^2 \phi_1^2 \\ &+ \partial_\mu \phi_2 \partial^\mu \phi_2 - \left(\frac{\partial \phi_2}{\partial s}\right)^2 - \kappa^2 \phi_2^2\Bigg]. \end{aligned} \tag{2.208}$$

The canonical momenta are

$$\pi_1 = \frac{\partial \mathcal{L}}{\partial \partial\phi_1/s} = -\frac{\partial \phi_1}{\partial s}\,, \qquad \pi_2 = \frac{\partial \mathcal{L}}{\partial \partial\phi_2/s} = -\frac{\partial \phi_2}{\partial s} \tag{2.209}$$

and the Hamiltonian is

$$\begin{aligned} H[\phi_1,\phi_2,\pi_1,\pi_2] &= \int \mathrm{d}^d x \left(\pi_1 \frac{\partial \phi_1}{\partial s} + \pi_2 \frac{\partial \phi_2}{\partial s} - \mathcal{L}\right) \\ &= \tfrac{1}{2}\int \mathrm{d}^d x \,(-\pi_1^2 - \partial_\mu \phi_1 \partial^\mu \phi_1 + \kappa^2 \phi_1^2 \\ &\qquad - \pi_2^2 - \partial_\mu \phi_2 \partial^\mu \phi_2 + \kappa^2 \phi_2^2). \end{aligned} \tag{2.210}$$

Introducing the complex fields

$$\begin{aligned} \phi &= \phi_1 + i\phi_2\,, \qquad \pi = \pi_1 + i\pi_2 \\ \phi^* &= \phi_1 - i\phi_2\,, \qquad \pi^* = \pi_1 - i\pi_2 \end{aligned} \tag{2.211}$$

we have

$$I[\phi,\phi^*] = \tfrac{1}{2}\int \mathrm{d}s\, \mathrm{d}^d x \left(\frac{\partial \phi^*}{\partial s}\frac{\partial \phi}{\partial s} - \partial_\mu \phi^* \partial^\mu \phi - \kappa^2 \phi^* \phi\right) \tag{2.212}$$

and

$$H[\phi,\phi^*,\pi,\pi^*] = \tfrac{1}{2}\int \mathrm{d}^d x (-\pi^* \pi - \partial_\mu \phi^* \partial^\mu \phi + \kappa^2 \phi^* \phi). \tag{2.213}$$

Comparing the latter Hamiltonian with the one of the Stueckelberg field theory (1.161), we see that it is the same, except for the additional term

$-\pi^*\pi$ which is absent in the Stueckelberg field theory. We see also that the theory described by (2.212) and (2.213) has the same structure as the conventional field theory, except for the number of dimensions. In the conventional theory we have time t and three space-like coordinates x^i, $i = 1, 2, 3$, while here we have s and four or more coordinates x^μ, one of them being time-like.

As the non-relativistic field theory is an approximation to the relativistic field theory, so the field theory derived from the Stueckelberg action is an approximation to the field theory derived from the polyvector action.

On the other hand, at the classical level (as we have seen in Sec. 2.4) Stueckelberg action, in the absence of interaction, arises exactly from the polyvector action. Even in the presence of the electromagnetic interaction both actions are equivalent, since they give the same equations of motion. However, the field theory based on the latter action differs from the field theory based on the former action by the term $\pi^*\pi$ in the Hamiltonian (2.213). While at the classical level Stueckelberg and the polyvector action are equivalent, at the first and the second quantized level differences arise which need further investigation.

Second quantization then goes along the usual lines: ϕ_i and π_i becomes operators satisfying the equal s commutation relations:

$$[\phi_i(s,x), \phi_j(s,x')] = 0 \, , \qquad [\pi_i(s,x), \pi_j(s,x')] = 0 \, ,$$
$$[\phi_i(s,x), \pi_j(s,x')] = i\delta_{ij}\delta(x - x'). \tag{2.214}$$

The field equations are then just the Heisenberg equations

$$\dot{\pi}_i = i[\pi_i, H]. \tag{2.215}$$

We shall not proceed here with formal development, since it is in many respects just a repetition of the procedure expounded in Sec. 1.4. But we shall make some remarks. First of all it is important to bear in mind that the usual arguments about causality, unitarity, negative energy, etc., do not apply anymore, and must all be worked out again. Second, whilst in the conventional quantum field theory the evolution parameter is a time-like coordinate $x^0 \equiv t$, in the field theory based on (2.212), (2.213) the evolution parameter is the pseudoscalar variable s. In even-dimensions it is invariant with respect to the Poincaré and the general coordinate transformations of x^μ, including the inversions. And what is very nice here is that the (pseudo)scalar parameter s naturally arises from the straightforward polyvector extension of the conventional reparametrization invariant theory.

Instead of the *Heisenberg picture* we can use the *Schrödinger picture* and the coordinate representation in which the operators $\phi_i(0,x) \equiv \phi_i(x)$ are diagonal, i.e., they are just ordinary functions. The momentum operator is represented by functional derivative[7]

$$\pi_j = -i \frac{\delta}{\delta \phi_j(x)}. \tag{2.216}$$

A state $|\Psi\rangle$ is represented by a *wave functional* $\Psi[\phi(x)] = \langle \phi(x)|\Psi\rangle$ and satisfies the Schrödinger equation

$$i \frac{\partial \Psi}{\partial s} = H\Psi \tag{2.217}$$

in which the evolution parameter is s. Of course s is invariant under the Lorentz transformations, and H contains all four components of the 4-momentum. Equation (2.217) is just like the Stueckelberg equation, the difference being that Ψ is now not a wave function, but a wave functional.

We started from the *constrained* polyvector action (2.117). Performing the *first quantization* we obtained the wave equation (2.204) which follows from the action (2.205) for the field $\phi(s, x^\mu)$. The latter action is *unconstrained*. Therefore we can straightforwardly quantize it, and thus perform the *second quantization*. The state vector $|\Psi\rangle$ in the Schrödinger picture evolves in s which is a *Lorentz invariant evolution parameter*. $|\Psi\rangle$ can be represented by a wave functional $\Psi[s, \phi(x)]$ which satisfied the functional Schrödinger equation. Whilst upon the first quantization the equation of motion for the field $\phi(s, x^\mu)$ contains the second order derivative with respect to s, upon the second quantization only the first order s-derivative remains in the equation of motion for the state functional $\Psi[s, \phi(x)]$.

An analogous procedure is undertaken in the usual approach to quantum field theory (see, e.g., [38]), with the difference that the evolution parameter becomes one of the space time coordinates, namely $x^0 \equiv t$. When trying to quantize the gravitational field it turns out that the evolution parameter t does not occur at all in the Wheeler–DeWitt equation! This is the well known problem of time in quantum gravity. We anticipate that a sort of polyvector generalization of the Einstein–Hilbert action should be taken, which would contain the scalar or pseudoscalar parameter s, and retain it in the generalized Wheeler–DeWitt equation. Some important research in that direction has been pioneered by Greensite and Carlini [39]

[7] A detailed discussion of the Schrödinger representation in field theory is to be found in ref. [38].

2.7. SOME FURTHER IMPORTANT CONSEQUENCES OF CLIFFORD ALGEBRA

RELATIVITY OF SIGNATURE

In previous sections we have seen how Clifford algebra can be used in the formulation of the point particle classical and quantum theory. The metric of spacetime was assumed, as usually, to have the Minkowski signature, and we have used the choice $(+---)$. We are now going to find out that within Clifford algebra the signature is a matter of choice of basis vectors amongst available Clifford numbers.

Suppose we have a 4-dimensional space V_4 with signature $(++++)$. Let e_μ, $\mu = 0,1,2,3$, be basis vectors satisfying

$$e_\mu \cdot e_\nu \equiv \tfrac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = \delta_{\mu\nu} \ , \qquad (2.218)$$

where $\delta_{\mu\nu}$ is the *Euclidean signature* of V_4. The vectors e_μ can be used as generators of Clifford algebra $\mathcal{C}$ over V_4 with a generic Clifford number (also called polyvector or Clifford aggregate) expanded in term of $e_J = (1, e_\mu, e_{\mu\nu}, e_{\mu\nu\alpha}, e_{\mu\nu\alpha\beta})$, $\mu < \nu < \alpha < \beta$,

$$A = a^J e_J = a + a^\mu e_\mu + a^{\mu\nu} e_\mu e_\nu + a^{\mu\nu\alpha} e_\mu e_\nu e_\alpha + a^{\mu\nu\alpha\beta} e_\mu e_\nu e_\alpha e_\beta. \qquad (2.219)$$

Let us consider the set of four Clifford numbers $(e_0,\ e_i e_0)$, $i = 1,2,3$, and denote them as

$$\begin{aligned} e_0 &\equiv \gamma_0, \\ e_i e_0 &\equiv \gamma_i. \end{aligned} \qquad (2.220)$$

The Clifford numbers γ_μ, $\mu = 0,1,2,3$, satisfy

$$\tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu} \ , \qquad (2.221)$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the *Minkowski tensor*. We see that the γ_μ behave as basis vectors in a 4-dimensional space $V_{1,3}$ with signature $(+---)$. We can form a Clifford aggregate

$$\alpha = \alpha^\mu \gamma_\mu \qquad (2.222)$$

which has the properties of a *vector* in $V_{1,3}$. From the point of view of the space V_4 the same object α is a linear combination of a vector and bivector:

$$\alpha = \alpha^0 e_0 + \alpha^i e_i e_0. \qquad (2.223)$$

We may use γ_μ as generators of the Clifford algebra $\mathcal{C}_{1,3}$ defined over the pseudo-Euclidean space $V_{1,3}$. The basis elements of $\mathcal{C}_{1,3}$ are $\gamma_J =$

$(1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\alpha}, \gamma_{\mu\nu\alpha\beta})$, with $\mu < \nu < \alpha < \beta$. A generic Clifford aggregate in $\mathcal{C}_{1,3}$ is given by

$$B = b^J \gamma_J = b + b^\mu \gamma_\mu + b^{\mu\nu} \gamma_\mu \gamma_\nu + b^{\mu\nu\alpha} \gamma_\mu \gamma \nu \gamma_\alpha + b^{\mu\nu\alpha\beta} \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta. \quad (2.224)$$

With suitable choice of the coefficients $b^J = (b, b^\mu, b^{\mu\nu}, b^{\mu\nu\alpha}, b^{\mu\nu\alpha\beta})$ we have that B of eq. (2.224) is equal to A of eq.(2.219). Thus the same number A can be described either within $\mathcal{C}_4$ or within $\mathcal{C}_{1,3}$. The expansions (2.224) and (2.219) exhaust all possible numbers of the Clifford algebras $\mathcal{C}_{1,3}$ and $\mathcal{C}_4$. The algebra $\mathcal{C}_{1,3}$ is isomorphic to the algebra $\mathcal{C}_4$, and actually they are just two different representations of the same set of Clifford numbers (also being called polyvectors or Clifford aggregates).

As an alternative to (2.220) we can choose

$$\begin{aligned} e_0 e_3 &\equiv \tilde{\gamma}_0, \\ e_i &\equiv \tilde{\gamma}_i, \end{aligned} \quad (2.225)$$

from which we have

$$\tfrac{1}{2}(\tilde{\gamma}_\mu \tilde{\gamma}_\nu + \tilde{\gamma}_\nu \tilde{\gamma}_\mu) = \tilde{\eta}_{\mu\nu} \quad (2.226)$$

with $\tilde{\eta}_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1)$. Obviously $\tilde{\gamma}_\mu$ are basis vectors of a pseudo-Euclidean space $\widetilde{V}_{1,3}$ and they generate the Clifford algebra over $\widetilde{V}_{1,3}$ which is yet another representation of the same set of objects (i.e., polyvectors). But the spaces V_4, $V_{1,3}$ and $\widetilde{V}_{1,3}$ are not the same and they span different subsets of polyvectors. In a similar way we can obtain spaces with signatures $(+-++)$, $(++-+)$, $(+++-)$, $(-+--)$, $(--+-)$, $(---+)$ and corresponding higher dimensional analogs. But we cannot obtain signatures of the type $(++--)$, $(+-+-)$, etc. In order to obtain such signatures we proceed as follows.

4-space. First we observe that the bivector $\bar{I} = e_3 e_4$ satisfies $\bar{I}^2 = -1$, commutes with e_1, e_2 and anticommutes with e_3, e_4. So we obtain that the set of Clifford numbers $\gamma_\mu = (e_1 \bar{I}, e_2 \bar{I}, e_3, e_3)$ satisfies

$$\gamma_\mu \cdot \gamma_\nu = \bar{\eta}_{\mu\nu} \ , \quad (2.227)$$

where $\bar{\eta} = \mathrm{diag}(-1, -1, 1, 1)$.

8-space. Let e_A be basis vectors of 8-dimensional vector space with signature $(+ + + + + + + +)$. Let us decompose

$$e_A = (e_\mu, e_{\bar{\mu}}) \ , \qquad \begin{aligned} \mu &= 0, 1, 2, 3, \\ \bar{\mu} &= \bar{0}, \bar{1}, \bar{2}, \bar{3}. \end{aligned} \quad (2.228)$$

The inner product of two basis vectors

$$e_A \cdot e_B = \delta_{AB}, \quad (2.229)$$

then splits into the following set of equations:

$$\begin{aligned} e_\mu \cdot e_\nu &= \delta_{\mu\nu} \ , \\ e_{\bar\mu} \cdot e_{\bar\nu} &= \delta_{\bar\mu\bar\nu} \ , \\ e_\mu \cdot e_{\bar\nu} &= 0. \end{aligned} \tag{2.230}$$

The number $\bar I = e_{\bar 0} e_{\bar 1} e_{\bar 2} e_{\bar 3}$ has the properties

$$\begin{aligned} \bar I^2 &= 1, \\ \bar I e_\mu &= e_\mu \bar I, \\ \bar I e_{\bar\mu} &= -e_{\bar\mu} \bar I. \end{aligned} \tag{2.231}$$

The set of numbers

$$\begin{aligned} \gamma_\mu &= e_\mu \ , \\ \gamma_{\bar\mu} &= e_{\bar\mu} \bar I \end{aligned} \tag{2.232}$$

satisfies

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &= \delta_{\mu\nu} \ , \\ \gamma_{\bar\mu} \cdot \gamma_{\bar\nu} &= -\delta_{\bar\mu\bar\nu} \ , \\ \gamma_\mu \cdot \gamma_{\bar\mu} &= 0. \end{aligned} \tag{2.233}$$

The numbers $(\gamma_\mu, \, \gamma_{\bar\mu})$ thus form a set of basis vectors of a vector space $V_{4,4}$ with signature $(+ + + + - - - -)$.

10-space. Let $e_A = (e_\mu, e_{\bar\mu})$, $\mu = 1,2,3,4,5$; $\bar\mu = \bar 1, \bar 2, \bar 3, \bar 4, \bar 5$ be basis vectors of a 10-dimensional Euclidean space V_{10} with signature $(+ + + \ ....)$. We introduce $\bar I = e_{\bar 1} e_{\bar 2} e_{\bar 3} e_{\bar 4} e_{\bar 5}$ which satisfies

$$\begin{aligned} \bar I^2 &= 1 \ , \\ e_\mu \bar I &= -\bar I e_\mu \ , \\ e_{\bar\mu} \bar I &= \bar I e_{\bar\mu}. \end{aligned} \tag{2.234}$$

Then the Clifford numbers

$$\begin{aligned} \gamma_\mu &= e_\mu \bar I \ , \\ \gamma_{\bar\mu} &= e_\mu \end{aligned} \tag{2.235}$$

satisfy

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &= -\delta_{\mu\nu} \ , \\ \gamma_{\bar\mu} \cdot \gamma_{\bar\nu} &= \delta_{\bar\mu\bar\nu} \ , \\ \gamma_\mu \cdot \gamma_{\bar\mu} &= 0. \end{aligned} \tag{2.236}$$

The set $\gamma_A = (\gamma_\mu, \gamma_{\bar{\mu}})$ therefore spans the vector space of signature $(- - - - - + + + + +)$.

The examples above demonstrate how vector spaces of various signatures are obtained within a given set of polyvectors. Namely, vector spaces of different signature are different subsets of polyvectors within the same Clifford algebra.

This has important physical implications. We have argued that physical quantities are polyvectors (Clifford numbers or Clifford aggregates). Physical space is then not simply a vector space (e.g., Minkowski space), but a space of polyvectors. The latter is a pandimensional continuum $\mathcal{P}$ [23] of points, lines, planes, volumes, etc., altogether. Minkowski space is then just a subspace with pseudo-Euclidean signature. Other subspaces with other signatures also exist within the pandimensional continuum $\mathcal{P}$ and they all have physical significance. If we describe a particle as moving in Minkowski spacetime $V_{1,3}$ we consider only certain physical aspects of the object considered. We have omitted its other physical properties like spin, charge, magnetic moment, etc.. We can as well describe the same object as moving in an Euclidean space V_4. Again such a description would reflect only a part of the underlying physical situation described by Clifford algebra.

GRASSMAN NUMBERS FROM CLIFFORD NUMBERS

In Sec. 2.5 we have seen that certain Clifford aggregates are spinors. Now we shall find out that also Grassmann (anticommuting) numbers are Clifford aggregates. As an example let us consider 8-dimensional space $V_{4,4}$ with signature $(+ - - - - + + +)$ spanned by basis vectors $\gamma_A = (\gamma_\mu, \gamma_{\bar{\mu}})$. The numbers

$$\begin{aligned} \theta_\mu &= \tfrac{1}{2}(\gamma_\mu + \gamma_{\bar{\mu}}), \\ \theta_\mu^\dagger &= \tfrac{1}{2}(\gamma_\mu - \gamma_{\bar{\mu}}) \end{aligned} \tag{2.237}$$

satisfy

$$\{\theta_\mu, \theta_\nu\} = \{\theta_\mu^\dagger, \theta_\nu^\dagger\} = 0 \ , \tag{2.238}$$

$$\{\theta_\mu, \theta_\nu^\dagger\} = \eta_{\mu\nu} \ , \tag{2.239}$$

where $\{A, B\} \equiv AB + BA$. From (2.238) we read out that θ_μ anticommute among themselves and are thus Grassmann numbers. Similarly $\theta_\mu^\dagger$ form a set of Grassmann numbers. But, because of (2.239), θ_μ and $\theta_\mu^\dagger$ altogether do not form a set of Grassmann numbers. They form yet another set of basis elements which generate Clifford algebra.

A Clifford number in $V_{4,4}$ can be expanded as

$$C = c + c^{A_1}\gamma_{A_1} + c^{A_1A_2}\gamma_{A_1}\gamma_{A_2} + ... + c^{A_1A_2...A_8}\gamma_{A_1}\gamma_{A_2}...\gamma_{A_8}. \qquad (2.240)$$

Using (2.237), the same Clifford number C can be expanded in terms of θ_μ, $\theta_\mu^\dagger$:

$$\begin{aligned} C = c \quad &+ \quad a^\mu\theta_\mu + a^{\mu\nu}\theta_\mu\theta_\nu + a^{\mu\nu\alpha}\theta_\mu\theta_\nu\theta_\alpha + a^{\mu\nu\alpha\beta}\theta_\mu\theta_\nu\theta_\alpha\theta_\beta \\ &+ \quad \bar{a}^\mu\theta_\mu^\dagger + \bar{a}^{\mu\nu}\theta_\mu^\dagger\theta_\nu^\dagger + \bar{a}^{\mu\nu\alpha}\theta_\mu^\dagger\theta_\nu^\dagger\theta_\alpha^\dagger + \bar{a}^{\mu\nu\alpha\beta}\theta_\mu^\dagger\theta_\nu^\dagger\theta_\alpha^\dagger\theta_\beta^\dagger \\ &+ \quad (\text{mixed terms like } \theta_\mu^\dagger\theta_\nu, \text{ etc.}) \end{aligned} \qquad (2.241)$$

where the coefficients a^μ, $a^{\mu\nu}$, ..., $\bar{a}^\mu, \bar{a}^{\mu\nu}$... are linear combinations of coefficients c^{A_i}, $c^{A_iA_j}$,...

In a particular case, coefficients c, $\bar{a}^\mu$, $\bar{a}^{\mu\nu}$, etc., can be zero and our Clifford number is then a *Grassmann number* in 4-space:

$$\xi = a^\mu\theta_\mu + a^{\mu\nu}\theta_\mu\theta_\nu + a^{\mu\nu\alpha}\theta_\mu\theta_\nu\theta_\alpha + a^{\mu\nu\alpha\beta}\theta_\mu\theta_\nu\theta_\alpha\theta_\beta. \qquad (2.242)$$

Grassmann numbers expanded according to (2.242), or analogous expressions in dimensions other than 4, are much used in contemporary theoretical physics. Recognition that Grassmann numbers can be considered as particular numbers within a more general set of numbers, namely Clifford numbers (or polyvectors), leads in my opinion to further progress in understanding and development of the currently fashionable supersymmetric theories, including superstrings, D-branes and M-theory.

We have seen that a Clifford number C in 8-dimensional space can be expanded in terms of the basis vectors $(\gamma_\mu, \gamma_{\bar{\mu}})$ or $(\theta_\mu, \theta_\mu^\dagger)$. Besides that, one can expand C also in terms of (γ_μ, θ_μ):

$$\begin{aligned} C = c \quad &+ \quad b^\mu\gamma_\mu + b^{\mu\nu}\gamma_\mu\gamma_\nu + b^{\mu\nu\alpha}\gamma_\mu\gamma_\nu\gamma_\alpha + b^{\mu\nu\alpha\beta}\gamma_\mu\gamma_\nu\gamma_\alpha\gamma_\beta \\ &+ \quad \beta^\mu\theta_\mu + \beta^{\mu\nu}\theta_\mu\theta_\nu + \beta^{\mu\nu\alpha}\theta_\mu\theta_\nu\theta_\alpha + \beta^{\mu\nu\alpha\beta}\theta_\mu\theta_\nu\theta_\alpha\theta_\beta \\ &+ \quad (\text{mixed terms such as } \theta_\mu\gamma_\nu, \text{ etc.}). \end{aligned} \qquad (2.243)$$

The basic vectors γ_μ span the familiar 4-dimensional spacetime, while θ_μ span an extra space, often called Grassmann space. Usually it is stated that besides four spacetime coordinates x^μ there are also four extra Grassmann coordinates θ_μ and their conjugates $\theta_\mu^\dagger$ or $\bar{\theta}_\mu = \gamma_0\theta_\mu^\dagger$. This should be contrasted with the picture above in which θ_μ are basis *vectors* of an extra space, and not *coordinates*.

2.8. THE POLYVECTOR ACTION AND DE WITT–ROVELLI MATERIAL REFERENCE SYSTEM

Following an argument by Einstein [42], that points of spacetime are not *a priori* distinguishable, DeWitt [25] introduced a concept of reference fluid. Spacetime points are then defined with respect to the reference fluid. The idea that we can localize points by means of some matter has been further elaborated by Rovelli [26]. As a starting model he considers a reference system consisting of a single particle and a clock attached to it. Besides the particle coordinate variables $X^\mu(\tau)$ there is also the clock variable $T(\tau)$, attached to the particle, which grows monotonically along the particle trajectory. Rovelli then assumes the following action for the variables $X^\mu(\tau)$, $T(\tau)$:

$$I[X^\mu, T] = m \int \mathrm{d}\tau \left(\frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}X_\mu}{\mathrm{d}\tau} - \frac{1}{\omega^2} \left(\frac{\mathrm{d}T}{\mathrm{d}\tau} \right)^2 \right)^{1/2} . \tag{2.244}$$

If we make replacement $m \to \kappa$ and $T/\omega \to s$ the latter action reads

$$I[X^\mu, s] = \int \mathrm{d}\tau \kappa \left(\dot{X}^\mu \dot{X}_\mu - \dot{s}^2 \right)^{1/2} . \tag{2.245}$$

If, on the other hand, we start from the polyvector action (2.117) and eliminate m, p_μ, λ by using the equations of motion, we again obtain the action (2.245). *Thus the pseudoscalar variable $s(\tau)$ entering the polyvector action may be identified with Rovelli's clock variable.* Although Rovelli starts with a single particle and clock, he later fills space with these objects. We shall return to Rovelli's reference systems when we discuss extended objects.