THE LANDSCAPE OF THEORETICAL PHYSICS: A GLOBAL VIEW

From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle

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Chapter 4

GENERAL PRINCIPLES OF MEMBRANE KINEMATICS AND DYNAMICS

We are now going to extend our system. Instead of point particles we shall consider strings and higher-dimensional membranes. These objects are nowadays amongst the hottest topics in fundamental theoretical physics. Many people are convinced that strings and accompanying higher-dimensional membranes provide a clue to unify physics [51]. In spite of many spectacular successes in unifying gravity with other interactions, there still remain open problems. Amongst the most serious is perhaps the problem of a geometrical principle behind string theory [52]. The approach pursued in this book aims to shed some more light on just that problem. We shall pay much attention to the treatment of membranes as points in an infinite-dimensional space, called membrane space $\mathcal{M}$. When pursuing such an approach the researchers usually try to build in right from the beginning a complication which arises from reparametrization invariance (called also diffeomorphism invariance). Namely, the same n-dimensional membrane can be represented by different sets of parametric equations $x^\mu = X^\mu(\xi^a)$, where functions X^μ map a membrane's parameters (also called coordinates) ξ^a, $a = 1, 2, ..., n$, into spacetime coordinates x^μ, $\mu = 0, 1, 2, ..., N-1$. The problem is then what are coordinates of the membrane space $\mathcal{M}$? If there were no complication caused by reparametrization invariance, then the $X^\mu(\xi^a)$ would be coordinates of $\mathcal{M}$-space. But because of reparametrization invariance such a mapping from a point in $\mathcal{M}$ (a membrane) to its coordinates $X^\mu(\xi)$ is one-to-many. So far there is no problem: a point in any space can be represented by many different possible sets of coordinates, and all those different sets are related by coordinate transformations or diffeomorphisms. Since the latter transformations refer to the same point they are called *passive coordinate transformations* or *passive diffeomorphisms*. The problem occurs when one brings into the game *active diffeomorphisms* which refer to different points of the space in question. In the case of membrane space $\mathcal{M}$ active diffeomorphisms would imply the existence of *tangentially deformed membranes*. But such objects are not present in a relativistic theory of membranes, described by the minimal surface action which is invariant under reparametrizations of ξ^a.

The approach pursued here is the following. We shall assume that *at the kinematic level such tangentially deformed membranes do exist* [53]. When considering dynamics it may happen that a certain action and its equations of motion exclude tangential motions within the membrane. This is precisely what happens with membranes obeying the relativistic minimal surface action. But the latter dynamical principle is not the most general one. We can extend it according to geometric calculus based on Clifford algebra. We have done so in Chapter 2 for point particles, and now we shall see how the procedure can be generalized to membranes of arbitrary dimension. And we shall find a remarkable result that *the polyvector generalization* of the membrane action allows for tangential motions of membranes. Because of the presence of an extra pseudoscalar variable entering the polyvector action, the membrane variables $X^\mu(\xi)$ and the corresponding momenta become *unconstrained*; tangentially deformed membranes are thus present not only at the kinematic, but also at the dynamical level. In other words, such a generalized dynamical principle allows for tangentially deformed membranes.

In the following sections I shall put the above description into a more precise form. But in the spirit of this book I will not attempt to achieve complete mathematical rigor, because for most readers this would be at the expense of seeing the main outline of my proposal of how to formulate membrane's theory

4.1. MEMBRANE SPACE $\mathcal{M}$

The basic kinematically possible objects of the theory we are going to discuss are n-dimensional, arbitrarily deformable, and hence unconstrained, membranes $\mathcal{V}_n$ living in an N-dimensional space V_N. The dimensions n and N, as well as the corresponding signatures, are left unspecified at this stage. An unconstrained membrane $\mathcal{V}_n$ is represented by the embedding functions $X^\mu(\xi^a)$, $\mu = 0, 1, 2, ..., N-1$, where ξ^a, $a = 0, 1, 2, ..., n-1$, are local parameters (coordinates) on $\mathcal{V}_n$. The set of all possible membranes $\mathcal{V}_n$, with n fixed, forms an infinite-dimensional space $\mathcal{M}$. A membrane $\mathcal{V}_n$ can be considered as a point in $\mathcal{M}$ parametrized by coordinates $X^\mu(\xi^a) \equiv X^{\mu(\xi)}$ which bear a discrete index μ and n continuous indices ξ^a. To the discrete index μ we can ascribe arbitrary numbers: instead of $\mu = 0, 1, 2, ..., N-1$

we may set $\mu' = 1, 2, ..., N$ or $\mu' = 2, 5, 3, 1, ...$, etc.. In general,

$$\mu' = f(\mu), \tag{4.1}$$

where f is a transformation. Analogously, a continuous index ξ^a can be given arbitrary continuous values. Instead of ξ^a we may take ξ'^a which are functions of ξ^a :

$$\xi'^a = f^a(\xi). \tag{4.2}$$

As far as we consider, respectively, μ and ξ^a as a discrete and a continuous index of coordinates $X^{\mu(\xi)}$ in the infinite-dimensional space $\mathcal{M}$, reparametrization of ξ^a is analogous to a renumbering of μ. Both kinds of transformations, (4.1) and (4.2), refer to the same point of the space $\mathcal{M}$; they are *passive transformations*. For instance, under the action of (4.2) we have

$$X'^\mu(\xi') = X'^\mu\left(f(\xi)\right) = X^\mu(\xi) \tag{4.3}$$

which says that the same point $\mathcal{V}_n$ can be described either by functions $X^\mu(\xi)$ or $X'^\mu(\xi)$ (where we may write $X'^\mu(\xi)$ instead of $X'^\mu(\xi')$ since ξ' is a running parameter and can be renamed as ξ).

Then there also exist the *active transformations*, which transform one point of the space $\mathcal{M}$ into another. Given a parametrization of ξ^a and a numbering of μ, a point $\mathcal{V}_n$ of $\mathcal{M}$ with coordinates $X^\mu(\xi)$ can be transformed into another point $\mathcal{V}'_n$ with coordinates $X'^\mu(\xi)$. Parameters ξ^a are now considered as "body fixed", so that distinct functions $X^\mu(\xi)$, $X'^\mu(\xi)$ represent distinct points $\mathcal{V}_n$, $\mathcal{V}'_n$ of $\mathcal{M}$. Physically these are distinct membranes which may be deformed one with respect to the other. Such a membrane is *unconstrained*, since all coordinates $X^\mu(\xi)$ are necessary for its description [53]–[55]. In order to distinguish an unconstrained membrane $\mathcal{V}_n$ from the corresponding mathematical manifold V_n we use different symbols $\mathcal{V}_n$ and V_n.

It may happen, in particular, that two distinct membranes $\mathcal{V}_n$ and $\mathcal{V}'_n$ both lie on the same mathematical surface V_n, and yet they are physically distinct objects, represented by different points in $\mathcal{M}$.

The concept of an unconstrained membrane can be illustrated by imagining a rubber sheet spanning a surface V_2. The sheet can be deformed from one configuration (let me call it $\mathcal{V}_2$) into another configuration $\mathcal{V}'_2$ in such a way that both configurations $\mathcal{V}_2$, $\mathcal{V}'_2$ are spanning the same surface V_2. The configurations $\mathcal{V}_2$, $\mathcal{V}'_2$ are described by functions $X^i(\xi^1, \xi^2)$, $X'^i(\xi^1, \xi^2)$ $(i = 1, 2, 3)$, respectively. The latter functions, from the mathematical point of view, both represent the same surface V_2, and can be transformed one into the other by a reparametrization of ξ^1, ξ^2. But from the physical point of view, $X^i(\xi^1, \xi^2)$ and $X'^i(\xi^1, \xi^2)$ represent two different configurations of the rubber sheet (Fig. 4.1).

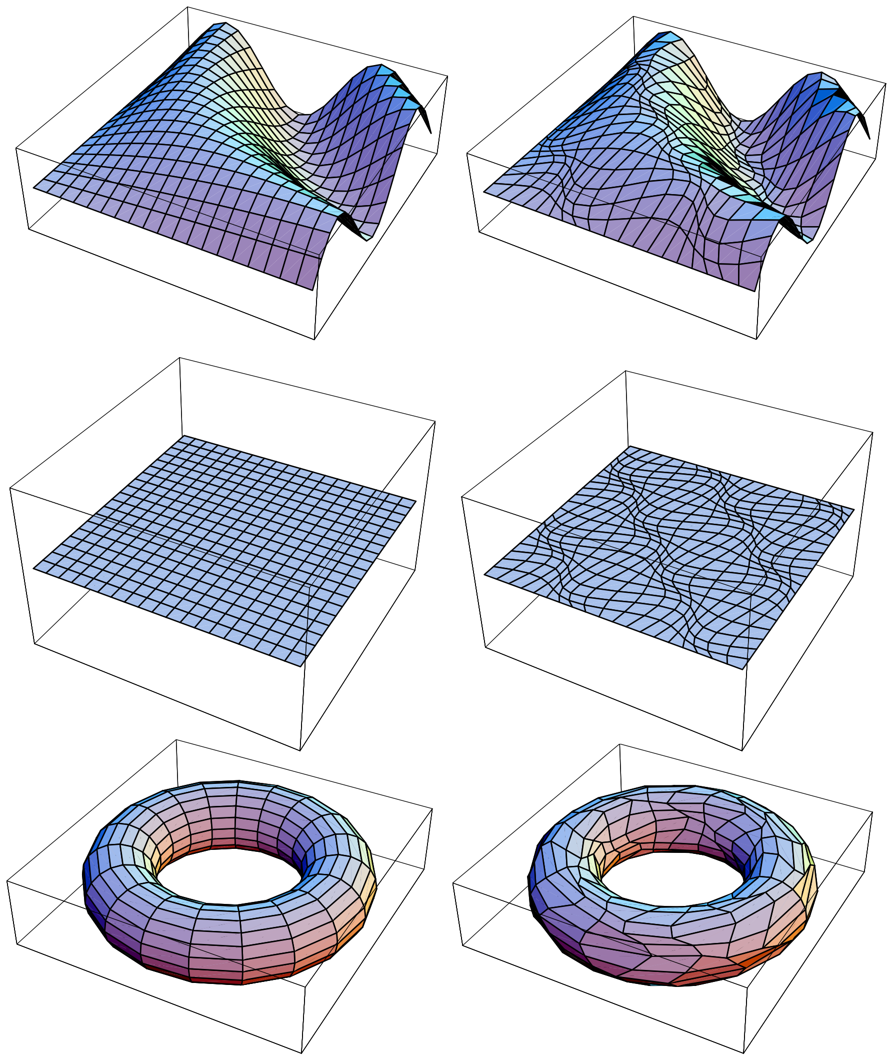

Figure 4.1. Examples of tangentially deformed membranes. Mathematically the surfaces on the right are the same as those on the left, but physically they are different.

The reasoning presented in the last few paragraphs implies that, since our membranes are assumed to be arbitrarily deformable, different functions $X^\mu(\xi)$ can always represent physically different membranes. This justifies use of the coordinates $X^\mu(\xi)$ for the description of points in $\mathcal{M}$. Later, when we consider a membrane's kinematics and dynamics we shall admit

τ-dependence of coordinates $X^\mu(\xi)$. In this section all expressions refer to a fixed value of τ, therefore we omit it from the notation.

In analogy with the finite-dimensional case we can introduce the distance $\mathrm{d}\ell$ in our infinite-dimensional space $\mathcal{M}$:

$$\begin{aligned} \mathrm{d}\ell^2 &= \int \mathrm{d}\xi \, \mathrm{d}\zeta \rho_{\mu\nu}(\xi,\zeta) \, \mathrm{d}X^\mu(\xi) \, \mathrm{d}X^\nu(\zeta) \\ &= \rho_{\mu(\xi)\nu(\zeta)} \, \mathrm{d}X^{\mu(\xi)} \, \mathrm{d}X^{\nu(\zeta)} = \mathrm{d}X^{\mu(\xi)} \mathrm{d}X_{\mu(\xi)}, \end{aligned} \tag{4.4}$$

where $\rho_{\mu\nu}(\xi,\zeta) = \rho_{\mu(\xi)\nu(\zeta)}$ is the metric in $\mathcal{M}$. In eq. (4.4) we use a notation, similar to one that is usually used when dealing with more evolved functional expressions [56], [57]. In order to distinguish continuous indices from discrete indices, the former are written within parentheses. When we write $\mu(\xi)$ as a subscript or superscript this denotes a pair of indices μ and (ξ) (and not that μ is a function of ξ). We also use the convention that summation is performed over repeated indices (such as a, b) and integration over repeated continuous indices (such as $(\xi), (\zeta)$).

The tensor calculus in $\mathcal{M}$ [54, 55] is analogous to that in a finite-dimensional space. The differential of coordinates $\mathrm{d}X^\mu(\xi) \equiv \mathrm{d}X^{\mu(\xi)}$ is a vector in $\mathcal{M}$. The coordinates $X^{\mu(\xi)}$ can be transformed into new coordinates $X'^{\mu(\xi)}$ which are functionals of $X^{\mu(\xi)}$:

$$X'^{\mu(\xi)} = F^{\mu(\xi)}[X]. \tag{4.5}$$

The transformation (4.5) is very important. It says that if functions $X^\mu(\xi)$ represent a membrane $\mathcal{V}_n$ then any other functions $X'^\mu(\xi)$ obtained from $X^\mu(\xi)$ by a functional transformation also represent the same membrane $\mathcal{V}_n$. In particular, under a reparametrization of ξ^a the functions $X^\mu(\xi)$ change into new functions; a reparametrization thus manifests itself as a special functional transformation which belongs to a subclass of the general functional transformations (4.5).

Under a general coordinate transformation (4.5) a generic vector $A^{\mu(\xi)} \equiv A^\mu(\xi)$ transforms as[1]

$$A^{\mu(\xi)} = \frac{\partial X'^{\mu(\xi)}}{\partial X^{\nu(\zeta)}} A^{\nu(\zeta)} \equiv \int \mathrm{d}\zeta \frac{\delta X'^\mu(\xi)}{\delta X^\nu(\zeta)} A^\nu(\zeta) \tag{4.6}$$

where $\delta/\delta X^\mu(\xi)$ denotes the functional derivative (see Box 4.1). Similar transformations hold for a covariant vector $A_{\mu(\xi)}$, a tensor $B_{\mu(\xi)\nu(\zeta)}$, etc..

[1] A similar formalism, but for a specific type of the functional transformations (4.5), namely the reparametrizations which functionally depend on string coordinates, was developed by Bardakci [56]

Indices are lowered and raised, respectively, by $\rho_{\mu(\xi)\nu(\zeta)}$ and $\rho^{\mu(\xi)\nu(\zeta)}$, the latter being the inverse metric tensor satisfying

$$\rho^{\mu(\xi)\alpha(\eta)}\rho_{\alpha(\eta)\nu(\zeta)} = \delta^{\mu(\xi)}{}_{\nu(\zeta)}. \tag{4.7}$$

Box 4.1: Functional derivative

Let $X^\mu(\xi)$ be a function of $\xi \equiv \xi^a$. The functional derivative of a functional $F[X^\mu(\xi)]$ is defined according to

$$\frac{\delta F}{\delta X^\nu(\xi')} = \lim_{\epsilon \to 0} \frac{F[X^\mu(\xi) + \epsilon\delta(\xi - \xi')\delta^\mu{}_\nu] - F[X^\mu(\xi)]}{\epsilon} \tag{4.8}$$

Examples

1) $F = X^\mu(\xi)$

$$\frac{\delta F}{\delta X^\nu(\xi)} = \delta(\xi - \xi')\delta^\mu{}_\nu$$

2) $F = \partial_a X^\mu(\xi)$

$$\frac{\delta F}{\delta X^\nu(\xi)} = \partial_a \delta(\xi - \xi')\delta^\mu{}_\nu$$

3) $F = \lambda(\xi)\, \partial_a X^\mu(\xi)$

$$\frac{\delta F}{\delta X^\nu(\xi)} = \lambda(\xi)\, \frac{\delta \partial_a X^\mu(\xi)}{\delta X^\nu(\xi')}$$

$$\left(\text{in general } \frac{\delta}{\delta X^\nu(\xi')}(\lambda(\xi)F[X]) = \lambda(\xi)\frac{\delta F}{\delta X^\nu(\xi)}\right)$$

4) $F = \partial_a X^\mu(\xi)\partial_b X_\mu(\xi)$

$$\frac{\delta F}{\delta X^\nu(\xi)} = \partial_a \delta(\xi - \xi')\partial_b X_\nu + \partial_b \delta(\xi - \xi')\partial_a X_\nu$$

A suitable choice of the metric — assuring the invariance of the line element (4.4) under the transformations (4.2) and (4.5) — is, for instance,

$$\rho_{\mu(\xi)\nu(\zeta)} = \sqrt{|f|}\,\alpha\, g_{\mu\nu}\delta(\xi - \zeta), \tag{4.9}$$

where $f \equiv \det f_{ab}$ is the determinant of the induced metric

$$f_{ab} \equiv \partial_a X^\alpha \partial_b X^\beta\, g_{\alpha\beta} \tag{4.10}$$

on the sheet V_n, $g_{\mu\nu}$ is the metric tensor of the embedding space V_N, and α an arbitrary function of ξ^a.

With the metric (4.9) the line element (4.4) becomes

$$\mathrm{d}\ell^2 = \int \mathrm{d}\xi \sqrt{|f|}\,\alpha\, g_{\mu\nu}\, \mathrm{d}X^{\mu(\xi)}\mathrm{d}X^{\nu(\xi)}. \tag{4.11}$$

Rewriting the abstract formulas back into the usual notation, with explicit integration, we have

$$A^{\mu(\xi)} = A^\mu(\xi), \tag{4.12}$$

$$\begin{aligned} A_{\mu(\xi)} &= \rho_{\mu(\xi)\nu(\zeta)} A^{\nu(\zeta)} \\ &= \int \mathrm{d}\zeta\, \rho_{\mu\nu}(\xi,\zeta) A^\nu(\xi) = \sqrt{|f|}\,\alpha\, g_{\mu\nu} A^\nu(\xi). \end{aligned} \tag{4.13}$$

The inverse metric is

$$\rho^{\mu(\xi)\nu(\zeta)} = \frac{1}{\alpha\sqrt{|f|}}\; g^{\mu\nu}\delta(\xi - \zeta). \tag{4.14}$$

Indeed, from (4.7), (4.9) and (4.14) we obtain

$$\delta^{\mu(\xi)}{}_{\nu(\zeta)} = \int \mathrm{d}\eta\, g^{\mu\sigma} g_{\nu\sigma}\, \delta(\xi - \eta)\delta(\zeta - \eta) = \delta^\mu{}_\nu \delta(\xi - \zeta). \tag{4.15}$$

The invariant volume element (measure) of our membrane space $\mathcal{M}$ is [58]

$$\mathcal{D}X = (\mathrm{Det}\, \rho_{\mu\nu}(\xi,\zeta))^{1/2} \prod_{\xi,\mu} \mathrm{d}X^\mu(\xi). \tag{4.16}$$

Here Det denotes a continuum determinant taken over ξ, ζ as well as over μ, ν. In the case of the diagonal metric (4.9) we have

$$\mathcal{D}X = \prod_{\xi,\mu} \left(\sqrt{|f|}\,\alpha\,|g|\right)^{1/2} \mathrm{d}X^\mu(\xi) \tag{4.17}$$

As can be done in a finite-dimensional space, we can now also define the covariant derivative in $\mathcal{M}$. For a scalar functional $A[X(\xi)]$ the covariant functional derivative coincides with the ordinary functional derivative:

$$A_{;\mu(\xi)} = \frac{\delta A}{\delta X^\mu(\xi)} \equiv A_{,\mu(\xi)}. \tag{4.18}$$

But in general a geometric object in $\mathcal{M}$ is a tensor of arbitrary rank, $A^{\mu_1(\xi_1)\mu_2(\xi_2)...}{}_{\nu_1(\zeta_1)\nu_2(\zeta_2)...}$, which is a functional of $X^\mu(\xi)$, and its covariant derivative contains the affinity $\Gamma^{\mu(\xi)}_{\nu(\zeta)\sigma(\eta)}$ composed of the metric (4.9) [54, 55]. For instance, for a vector we have

$$A^{\mu(\xi)}{}_{;\nu(\zeta)} = A^{\mu(\xi)}{}_{,\nu(\zeta)} + \Gamma^{\mu(\xi)}_{\nu(\zeta)\sigma(\eta)} A^{\sigma(\eta)}. \tag{4.19}$$

Let the alternative notations for ordinary and covariant functional derivative be analogous to those used in a finite-dimensional space:

$$\frac{\delta}{\delta X^\mu(\xi)} \equiv \frac{\partial}{\partial X^{\mu(\xi)}} \equiv \partial_{\mu(\xi)} \quad , \qquad \frac{\mathrm{D}}{\mathrm{D}X^\mu(\xi)} \equiv \frac{\mathrm{D}}{\mathrm{D}X^{\mu(\xi)}} \equiv \mathrm{D}_{\mu(\xi)}. \tag{4.20}$$

4.2. MEMBRANE DYNAMICS

In the previous section I have considered arbitrary deformable membranes as *kinematically possible objects* of a membrane theory. A membrane, in general, is not static, but is assumed to move in an embedding space V_N. The parameter of evolution ("time") will be denoted τ. Kinematically every continuous trajectory $X^\mu(\tau, \xi^a) \equiv X^{\mu(\xi)}(\tau)$ is possible in principle. A particular dynamical theory then selects which amongst those kinematically possible membranes and trajectories are also dynamically possible. In this section I am going to describe the theory in which a dynamically possible trajectory $X^{\mu(\xi)}(\tau)$ is a *geodesic* in the membrane space $\mathcal{M}$.

MEMBRANE THEORY AS A FREE FALL IN $\mathcal{M}$-SPACE

Let $X^{\alpha(\xi)}$ be τ-dependent *coordinates* of a point in $\mathcal{M}$-space and $\rho_{\alpha(\xi')\beta(\xi'')}$ an arbitrary fixed metric in $\mathcal{M}$. From the point of view of a finite-dimensional space V_N the symbol $X^{\alpha(\xi)} \equiv X^\alpha(\xi)$ represents an n-dimensional membrane embedded in V_N. We assume that every dynamically possible trajectory

$X^{\alpha(\xi)}(\tau)$ satisfies the variational principle given by the action

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau' \left(\rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')}\right)^{1/2}. \tag{4.21}$$

This is just the action for a *geodesic* in $\mathcal{M}$-space.

The equation of motion is obtained if we functionally differentiate (4.21) with respect to $X^{\alpha(\xi)}(\tau)$:

$$\frac{\delta I}{\delta X^{\mu(\xi)}(\tau)} = \int \mathrm{d}\tau' \, \frac{1}{\mu^{1/2}} \, \rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi'')} \, \frac{\mathrm{d}}{\mathrm{d}\tau'}\delta(\tau-\tau')\delta_{(\xi)}{}^{(\xi')}$$
$$+ \frac{1}{2}\int \mathrm{d}\tau' \, \frac{1}{\mu^{1/2}} \left(\frac{\delta}{\delta X^{\mu(\xi)}(\tau)} \, \rho_{\alpha(\xi')\beta(\xi'')}\right)\dot{X}^{\alpha(\xi'')}\dot{X}^{\beta(\xi'')} = 0, \tag{4.22}$$

where

$$\mu \equiv \rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')} \tag{4.23}$$

and

$$\delta_{(\xi)}{}^{(\xi')} \equiv \delta(\xi - \xi'). \tag{4.24}$$

The integration over τ in the first term of eq. (4.22) can be easily performed and eq. (4.22) becomes

$$\frac{\delta I}{\delta X^{\mu(\xi)}(\tau)} = -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\rho_{\alpha(\xi')\mu(\xi)}\dot{X}^{\alpha(\xi')}}{\mu^{1/2}}\right)$$
$$+ \frac{1}{2}\int \mathrm{d}\tau' \frac{1}{\mu^{1/2}} \left(\frac{\delta}{\delta X^{\mu(\xi)}(\tau)}\rho_{\alpha(\xi')\beta(\xi'')}\right)\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')} = 0. \tag{4.25}$$

Some exercises with such a variation are performed in Box 4.2, where we use the notation $\partial_{\mu(\tau,\xi)} \equiv \delta/\delta X^{\mu}(\tau,\xi)$.

If the expression for the metric $\rho_{\alpha(\xi')\beta(\xi'')}$ does not contain the velocity $\dot{X}^{\mu}$, then eq. (4.25) further simplifies to

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{X}_{\mu(\xi)}\right) + \frac{1}{2}\partial_{\mu}(\xi)\rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')} = 0. \tag{4.26}$$

This can be written also in the form

$$\frac{\mathrm{d}\dot{X}^{\mu(\xi)}}{\mathrm{d}\tau} + \Gamma^{\mu(\xi)}{}_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')} = 0, \tag{4.27}$$

which is a straightforward generalization of the usual geodesic equation from a finite-dimensional space to an infinite-dimensional $\mathcal{M}$-space.

The metric $\rho_{\alpha(\xi')\beta(\xi'')}$ is arbitrary fixed background metric of $\mathcal{M}$-space. Choice of the latter metric determines, from the point of view of the embedding space V_N, a particular membrane theory. But from the viewpoint

Box 4.2: Excercises with variations and functional derivatives

1) $$I[X^\mu(\tau)] = \frac{1}{2}\int \mathrm{d}\tau'\, \dot{X}^\mu(\tau')\dot{X}^\nu(\tau')\eta_{\mu\nu}$$

$$\begin{aligned}\frac{\delta I}{\delta X^\alpha(\tau)} &= \int \mathrm{d}\tau'\, \dot{X}^\mu(\tau')\frac{\delta \dot{X}^\nu(\tau')}{\delta X^\alpha(\tau)}\eta_{\mu\nu}\\ &= \int \mathrm{d}\tau'\, \dot{X}_\alpha(\tau')\,\frac{\mathrm{d}}{\mathrm{d}\tau'}\delta(\tau-\tau') = -\frac{\mathrm{d}}{\mathrm{d}\tau'}\dot{X}_\alpha\end{aligned}$$

2) $$I[X^\mu(\tau,\xi)] = \frac{1}{2}\int \mathrm{d}\tau'\mathrm{d}\xi'\,\sqrt{|f(\xi')|}\,\dot{X}^\mu(\tau',\xi')\dot{X}^\nu(\tau',\xi')\eta_{\mu\nu}$$

$$\frac{\delta I}{\delta X^\alpha(\tau,\xi)} = \frac{1}{2}\int \mathrm{d}\tau'\,\mathrm{d}\xi'\,\frac{\delta\sqrt{|f(\tau',\xi')|}}{\delta X^\alpha(\tau',\xi')}\,\dot{X}^2(\tau',\xi')$$

$$+\frac{1}{2}\int \mathrm{d}\tau'\,\mathrm{d}\xi'\,\sqrt{|f(\tau',\xi')|}2\dot{X}^\mu\frac{\delta\dot{X}^\nu}{\delta\dot{X}^\alpha(\tau,\xi)}\eta_{\mu\nu}$$

$$= \frac{1}{2}\int \mathrm{d}\tau'\mathrm{d}\xi'\,\sqrt{|f(\tau',\xi')|}\partial'^a X_\alpha\partial'_a\delta(\xi-\xi')\delta(\tau-\tau')\dot{X}^2(\tau',\xi')$$

$$+\int \mathrm{d}\tau'\mathrm{d}\xi'\,\sqrt{|f(\tau',\xi')|}\dot{X}_\alpha(\tau',\xi')\,\frac{\mathrm{d}}{\mathrm{d}\tau'}\delta(\tau-\tau')\delta(\xi-\xi')$$

$$= -\frac{1}{2}\partial_a\left(\sqrt{|f|}\partial^a X_\alpha\,\dot{X}^2\right) - \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sqrt{|f|}\dot{X}_\alpha\right)$$

3) $$I = \int \mathrm{d}\tau'\,(\rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')} + K)$$

$$\frac{\delta I}{\delta X^{\mu(\xi)}(\tau)} = \frac{1}{2}\int \mathrm{d}\tau'\Big[\partial_{\mu(\tau,\xi)}\rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')}$$

$$+2\rho_{\alpha(\xi')\beta(\xi'')}\frac{\mathrm{d}}{\mathrm{d}\tau'}\delta(\tau-\tau')\delta(\xi-\xi')\delta_\mu{}^\alpha\dot{X}^{\beta(\xi'')} + \partial_{\mu(\tau,\xi)}K\Big]$$

$$= -\frac{\mathrm{d}}{\mathrm{d}\tau}(\rho_{\mu(\xi)\beta(\xi'')}\dot{X}^{\beta(\xi'')}) + \frac{1}{2}\int \mathrm{d}\tau'\Big[\partial_{\mu(\tau,\xi)}\rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')} + \partial_{\mu(\tau,\xi)}K\Big]$$

(*continued*)

Box 4.2 (*continued*)

a)
$$\rho_{\alpha(\xi')\beta(\xi'')} = \frac{\kappa\sqrt{|f(\xi')|}}{\lambda(\xi')}\delta(\xi' - \xi'')\eta_{\alpha\beta}\,, \qquad K = \int \mathrm{d}\xi \sqrt{|f|}\,\kappa\lambda$$

$$\frac{\delta I}{\delta X^{\mu(\xi)}(\tau)} = -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\kappa\sqrt{|f|}}{\lambda}\dot{X}_\mu\right) - \frac{1}{2}\,\partial_a\left(\frac{\kappa\sqrt{|f|}\partial^a X_\mu\,\dot{X}^2}{\lambda}\right) - \frac{1}{2}\,\partial_a(\kappa\sqrt{|f|}\partial^a X_\mu\,\lambda)$$

$$\frac{\delta I}{\lambda(\xi)} = 0 \qquad \Rightarrow \qquad \lambda^2 = \dot{X}^\alpha \dot{X}_\alpha$$

$$\Rightarrow \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right) + \partial_a(\kappa\,\sqrt{|f|}\partial^a X_\mu\,\sqrt{\dot{X}^2}) = 0$$

b)
$$\rho_{\alpha(\xi')\beta(\xi'')} = \frac{\kappa\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}}\,\delta(\xi' - \xi'')\eta_{\alpha\beta}\,, \qquad K = \int \mathrm{d}\xi \sqrt{|f|}\,\kappa\,\sqrt{\dot{X}^2}$$

$$\partial_{\mu(\tau,\xi)}\rho_{\alpha(\xi')\beta(\xi'')} = \kappa\,\frac{\delta\sqrt{|f(\tau',\xi')|}}{\delta X^\mu(\tau,\xi)}\,\frac{1}{\sqrt{\dot{X}^2}(\tau',\xi')}\eta_{\alpha\beta}\delta(\xi' - \xi'')$$
$$+\,\kappa\sqrt{|f(\tau',\xi')|}\,\frac{\delta}{\delta X^\mu(\tau,\xi)}\left(\frac{1}{\sqrt{\dot{X}^2}(\tau',\xi')}\right)\eta_{\alpha\beta}\delta(\xi' - \xi'')$$
$$= \kappa\,\sqrt{|f(\tau',\xi')|}\,\partial'^a X_\mu\,\frac{\partial'_a\delta(\xi - \xi')\delta(\tau - \tau')}{\sqrt{\dot{X}^2}}\,\eta_{\alpha\beta}\,\delta(\xi' - \xi'')$$
$$-\,\kappa\,\sqrt{|f(\tau',\xi')|}\,\frac{\dot{X}_\mu(\xi')}{\dot{X}^2(\xi'))^{3/2}}\delta(\xi - \xi')\delta(\xi' - \xi'')\frac{\mathrm{d}}{\mathrm{d}\tau'}\delta(\tau - \tau')\eta_{\alpha\beta}$$

$$\int \mathrm{d}\tau'\,\frac{\delta\rho_{\alpha(\xi')\beta(\xi'')}}{\delta X^{\mu(\xi)}(\tau)}\,\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')}$$
$$= -\kappa\,\partial_a(\sqrt{|f|}\partial^a X_\mu\,\sqrt{\dot{X}^2}) + \kappa\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sqrt{|f|}\,\frac{\dot{X}_\mu}{\sqrt{\dot{X}^2}}\right)$$

$$\int \mathrm{d}\tau'\,\frac{\delta K}{\delta X^{\mu(\xi)}(\tau)} = -\kappa\,\partial_a(\sqrt{|f|}\partial^a X_\mu\,\sqrt{\dot{X}^2}) - \kappa\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\sqrt{|f|}\,\frac{\dot{X}_\mu}{\sqrt{\dot{X}^2}}\right)$$

of $\mathcal{M}$-space there is just one membrane theory in a background metric $\rho_{\alpha(\xi')\beta(\xi'')}$ which is an arbitrary functional of $X^{\mu(\xi)}(\tau)$.

Suppose now that the metric is given by the following expression:

$$\rho_{\alpha(\xi')\beta(\xi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}} \, \delta(\xi' - \xi'') \eta_{\alpha\beta} \, , \tag{4.28}$$

where $\dot{X}^2(\xi') \equiv \dot{X}^\mu(\xi')\dot{X}_\mu(\xi')$, and κ is a constant. If we insert the latter expression into the equation of geodesic (4.22) and take into account the prescriptions of Boxes 4.1 and 4.2, we immediately obtain the following equations of motion:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\mu^{1/2}}\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right) + \frac{1}{\mu^{1/2}}\partial_a\left(\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0. \tag{4.29}$$

The latter equation can be written as

$$\mu^{1/2}\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\mu^{1/2}}\right)\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu + \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right) + \partial_a\left(\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0. \tag{4.30}$$

If we multiply this by $\dot{X}^\mu$ and assume that $\sqrt{|f|}\sqrt{\dot{X}^2} \neq 0$ we obtain

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}\mu}{\mathrm{d}\tau} &= \frac{1}{\sqrt{|f|}\sqrt{\dot{X}^2}}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right)\dot{X}^\mu + \partial_a(\sqrt{|f|}\partial^a X_\mu\sqrt{\dot{X}^2})\dot{X}^\mu\right] \\
&= \frac{1}{\sqrt{|f|}\sqrt{\dot{X}^2}}\left[\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right)\dot{X}^\mu - \sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu \partial_a \dot{X}^\mu\right] \\
&= \frac{1}{\sqrt{|f|}\sqrt{\dot{X}^2}}\left[\sqrt{\dot{X}^2}\,\frac{\mathrm{d}\sqrt{|f|}}{\mathrm{d}\tau} + \sqrt{|f|}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{X}_\mu}{\sqrt{\dot{X}^2}}\right)\dot{X}^\mu \right. \\
&\qquad\qquad \left. -\sqrt{\dot{X}^2}\,\frac{\mathrm{d}}{\mathrm{d}\tau}\sqrt{|f|}\right] \\
&= 0.
\end{aligned} \tag{4.31}$$

In the above calculation we have used the relations

$$\frac{\mathrm{d}\sqrt{|f|}}{\mathrm{d}\tau} = \frac{\partial\sqrt{|f|}}{\partial f_{ab}}\,\dot{f}_{ab} = \sqrt{|f|}\,f^{ab}\partial_a\dot{X}^\mu\partial_b X_\mu = \sqrt{|f|}\,\partial^a X_\mu \partial_a \dot{X}^\mu \tag{4.32}$$

and

$$\frac{\dot{X}_\mu}{\sqrt{\dot{X}^2}}\frac{\dot{X}^\mu}{\sqrt{\dot{X}^2}} = 1 \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{X}_\mu}{\sqrt{\dot{X}^2}}\right)\dot{X}^\mu = 0. \tag{4.33}$$

We have thus seen that the equations of motion (4.29) automatically imply

$$\frac{\mathrm{d}\mu}{\mathrm{d}\tau} = 0 \qquad \text{or} \qquad \frac{\mathrm{d}\sqrt{\mu}}{\mathrm{d}\tau} = 0\,, \quad \mu \neq 0. \tag{4.34}$$

Therefore, instead of (4.29) we can write

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right) + \partial_a\left(\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0. \tag{4.35}$$

This is precisely the equation of motion of the Dirac-Nambu-Goto membrane of arbitrary dimension. The latter objects are nowadays known as *p-branes*, and they include point particles (0-branes) and strings (1-branes). It is very interesting that the conventional theory of p-branes is just a particular case —with the metric (4.28)— of the membrane dynamics given by the action (4.21).

The action (4.21) is by definition invariant under reparametrizations of ξ^a. In general, it is not invariant under reparametrization of the evolution parameter τ. If the expression for the metric $\rho_{\alpha(\xi')\beta(\xi'')}$ does not contain the velocity $\dot{X}^\mu$ then the invariance of (4.21) under reparametrizations of τ is obvious. On the contrary, if $\rho_{\alpha(\xi')\beta(\xi'')}$ contains $\dot{X}^\mu$ then the action (4.21) is not invariant under reparametrizations of τ. For instance, if $\rho_{\alpha(\xi')\beta(\xi'')}$ is given by eq. (4.28), then, as we have seen, the equation of motion automatically contains the relation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{X}^{\mu(\xi)}\dot{X}_{\mu(\xi)}\right) \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}\int \mathrm{d}\xi\,\kappa\sqrt{|f|}\sqrt{\dot{X}^2} = 0. \tag{4.36}$$

The latter relation is nothing but *a gauge fixing relation*, where by "gauge" we mean here a choice of parameter τ. The action (4.21), which in the case of the metric (4.28) is not reparametrization invariant, contains the gauge fixing term. The latter term is not added separately to the action, but is implicit by the exponent $\frac{1}{2}$ of the expression $\dot{X}^{\mu(\xi)}\dot{X}_{\mu(\xi)}$.

In general the exponent in the Lagrangian is not necessarily $\frac{1}{2}$, but can be arbitrary:

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau\,\left(\rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')}\right)^a. \tag{4.37}$$

For the metric (4.28) the corresponding equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(a\mu^{a-1}\frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right) + a\mu^{a-1}\partial_a\left(\kappa\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0. \tag{4.38}$$

For any a which is different from 1 we obtain a gauge fixing relation which is equivalent to (4.34), and the same equation of motion (4.35). When

$a = 1$ we obtain directly the equation of motion (4.35), and no gauge fixing relation (4.34). For $a = 1$ and the metric (4.28) the action (4.37) is invariant under reparametrizations of τ.

We shall now focus our attention to the action

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau \, \rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi')} = \int \mathrm{d}\tau \, \mathrm{d}\xi \, \kappa \sqrt{|f|} \sqrt{\dot{X}^2} \tag{4.39}$$

with the metric (4.28). It is invariant under the transformations

$$\tau \rightarrow \tau' = \tau'(\tau), \tag{4.40}$$

$$\xi^a \rightarrow \xi'^a = \xi'^a(\xi^a) \tag{4.41}$$

in which τ and ξ^a do not mix.

Invariance of the action (4.39) under reparametrizations (4.40) of the evolution parameter τ implies the existence of a constraint among the canonical momenta $p_{\mu(\xi)}$ and coordinates $X^{\mu(\xi)}$. Momenta are given by

$$\begin{aligned} p_{\mu(\xi)} &= \frac{\partial L}{\partial \dot{X}^{\mu(\xi)}} = 2\rho_{\mu(\xi)\nu(\xi')} \dot{X}^{\nu(\xi')} + \frac{\partial \rho_{\alpha(\xi')\beta(\xi'')}}{\partial \dot{X}^{\mu(\xi)}} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} \\ &= \frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}} \, \dot{X}_\mu. \end{aligned} \tag{4.42}$$

By distinsguishing covariant and contravariant components one finds

$$p_{\mu(\xi)} = \dot{X}_{\mu(\xi)} \; , \quad p^{\mu(\xi)} = \dot{X}^{\mu(\xi)}. \tag{4.43}$$

We define

$$p_{\mu(\xi)} \equiv p_\mu(\xi) \equiv p_\mu \; , \quad \dot{X}^{\mu(\xi)} \equiv \dot{X}^\mu(\xi) \equiv \dot{X}^\mu. \tag{4.44}$$

Here p_μ and $\dot{X}^\mu$ have the meaning of the usual finite dimensional vectors whose components are lowered raised by the finite-dimensional metric tensor $g_{\mu\nu}$ and its inverse $g^{\mu\nu}$:

$$p^\mu = g^{\mu\nu} p_\nu \; , \quad \dot{X}_\mu = g_{\mu\nu} \dot{X}^\nu \tag{4.45}$$

Eq.(4.42) implies

$$p^\mu p_\mu - \kappa^2 |f| = 0 \tag{4.46}$$

which is satisfied at every ξ^a.

Multiplying (4.46) by $\sqrt{\dot{X}^2}/(\kappa\sqrt{|f|}\,)$ and integrating over ξ we have

$$\frac{1}{2} \int \mathrm{d}\xi \, \frac{\sqrt{\dot{X}^2}}{\kappa\sqrt{|f|}} \, (p^\mu p_\mu - \kappa^2 |f|) = p_{\mu(\xi)} \dot{X}^{\mu(\xi)} - L = H = 0 \tag{4.47}$$

where $L = \int \mathrm{d}\xi \, \kappa \sqrt{|f|} \sqrt{\dot{X}^2}$.

We see that the Hamiltonian belonging to our action (4.39) is identically zero. This is a well known consequence of the reparametrization invariance (4.40). The relation (4.46) is a constraint at ξ^a and the Hamiltonian (4.47) is a linear superposition of the constraints at all possible ξ^a.

An action which is equivalent to (4.39) is

$$I[X^{\mu(\xi)}, \lambda] = \frac{1}{2} \int \mathrm{d}\tau \mathrm{d}\xi \, \kappa \sqrt{|f|} \left(\frac{\dot{X}^\mu \dot{X}_\mu}{\lambda} + \lambda \right), \tag{4.48}$$

where λ is a Lagrange multiplier.

In the compact notation of $\mathcal{M}$-space eq. (4.48) reads

$$I[X^{\mu(\xi)}, \lambda] = \frac{1}{2} \int \mathrm{d}\tau \left(\rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} + K \right), \tag{4.49}$$

where

$$K = K[X^{\mu(\xi)}, \lambda] = \int \mathrm{d}\xi \kappa \sqrt{|f|} \lambda \tag{4.50}$$

and

$$\rho_{\alpha(\xi')\beta(\xi'')} = \rho_{\alpha(\xi')\beta(\xi'')}[X^{\mu(\xi)}, \lambda] = \frac{\kappa \sqrt{|f(\xi')|}}{\lambda(\xi')} \delta(\xi' - \xi'') \eta_{\alpha\beta}. \tag{4.51}$$

Variation of (4.49) with respect to $X^{\mu(\xi)}(\tau)$ and λ gives

$$\begin{aligned} \frac{\delta I}{\delta X^{\mu(\xi)}(\tau)} &= -\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\kappa \sqrt{|f|}}{\lambda} \dot{X}_\mu \right) \\ &\quad - \frac{1}{2} \partial_a \left(\kappa \sqrt{|f|} \partial^a X_\mu \left(\frac{\sqrt{\dot{X}^2}}{\lambda} + \lambda \right) \right) = 0, \end{aligned} \tag{4.52}$$

$$\frac{\delta I}{\delta \lambda(\tau, \xi)} = -\frac{\dot{X}^\mu \dot{X}_\mu}{\lambda^2} + 1 = 0. \tag{4.53}$$

The system of equations (4.52), (4.53) is equivalent to (4.35). This is in agreement with the property that after inserting the λ "equation of motion" (4.53) into the action (4.48) one obtains the action (4.39) which directly leads to the equation of motion (4.35).

The invariance of the action (4.48) under reparametrizations (4.40) of the evolution parameter τ is assured if λ transforms according to

$$\lambda \to \lambda' = \frac{\mathrm{d}\tau'}{\mathrm{d}\tau} \lambda. \tag{4.54}$$

This is in agreement with the relations (4.53) which says that

$$\lambda = (\dot{X}^\mu \dot{X}_\mu)^{1/2}.$$

Box 4.3: Conservation of the constraint

Since the Hamiltonian $H = \int \mathrm{d}\xi \lambda \mathcal{H}$ in eq. (4.67) is zero for any λ, it follows that the Hamiltonian density

$$\mathcal{H}[X^\mu, p_\mu] = \frac{1}{2\kappa} \left(\frac{p_\mu p^\mu}{\sqrt{|f|}} - \kappa^2 \sqrt{|f|} \right) \tag{4.55}$$

vanishes for any ξ^a. The requirement that the constraint (6.1) is conserved in τ can be written as

$$\dot{\mathcal{H}} = \{\mathcal{H}, H\} = 0, \tag{4.56}$$

which is satisfied if

$$\{\mathcal{H}(\xi), \mathcal{H}(\xi')\} = 0. \tag{4.57}$$

That the Poisson bracket (4.57) indeed vanishes can be found as follows. Let us work in the language of the Hamilton–Jacobi functional $S[X^\mu(\xi)]$, in which one considers the momentum vector field $p_{\mu(\xi)}$ to be a function of position $X^{\mu(\xi)}$ in $\mathcal{M}$-space, i.e., a functional of $X^\mu(\xi)$ given by

$$p_{\mu(\xi)} = p_{\mu(\xi)}(X^{\mu(\xi)}) \equiv p_\mu[X^\mu(\xi)] = \frac{\delta S}{\delta X^\mu(\xi)}. \tag{4.58}$$

Therefore $\mathcal{H}[X^\mu(\xi), p_\mu(\xi)]$ is a functional of $X^\mu(\xi)$. Since $\mathcal{H} = 0$, it follows that its functional derivative also vanishes:

$$\begin{aligned} \frac{\mathrm{d}\mathcal{H}}{\mathrm{d}X^{\mu(\xi)}} &= \frac{\partial \mathcal{H}}{\partial X^{\mu(\xi)}} + \frac{\partial \mathcal{H}}{\partial p_{\nu(\xi')}} \frac{\partial p_{\nu(\xi')}}{\partial X^{\mu(\xi)}} \\ &\equiv \frac{\delta \mathcal{H}}{\delta X^\mu(\xi)} + \int \mathrm{d}\xi' \, \frac{\delta \mathcal{H}}{\delta p_\nu(\xi')} \frac{\delta p_\nu(\xi')}{\delta X^\mu(\xi)} = 0. \end{aligned} \tag{4.59}$$

Using (4.59) and (4.58) we have

$$\begin{aligned} \{\mathcal{H}(\xi), \mathcal{H}(\xi')\} &= \int \mathrm{d}\xi'' \left(\frac{\delta \mathcal{H}(\xi)}{\delta X^\mu(\xi'')} \frac{\delta \mathcal{H}(\xi')}{\delta p_\mu(\xi'')} - \frac{\delta \mathcal{H}(\xi')}{\delta X^\mu(\xi'')} \frac{\delta \mathcal{H}(\xi)}{\delta p_\mu(\xi'')} \right) \\ &= - \int \mathrm{d}\xi'' \, \mathrm{d}\xi''' \, \frac{\delta \mathcal{H}(\xi)}{\delta p_\nu(\xi''')} \frac{\delta \mathcal{H}(\xi')}{\delta p_\mu(\xi'')} \left(\frac{\delta p_\nu(\xi''')}{\delta X^\mu(\xi'')} - \frac{\delta p_\mu(\xi''')}{\delta X^\mu(\xi'')} \right) = 0. \end{aligned} \tag{4.60}$$

(*continued*)

Box 4.3 (*continued*)

Conservation of the constraint (4.55) is thus shown to be automatically sastisfied.

On the other hand, we can calculate the Poisson bracket (4.57) by using the explicit expression (4.55). So we obtain

$$\{\mathcal{H}(\xi), \mathcal{H}(\xi')\} =$$

$$-\frac{\sqrt{|f(\xi)|}}{\sqrt{|f(\xi')|}}\partial_a\delta(\xi-\xi')p_\mu(\xi')\partial^a X^\mu(\xi)+\frac{\sqrt{|f(\xi')|}}{\sqrt{|f(\xi)|}}\partial'_a\delta(\xi-\xi')p_\mu(\xi)\partial'^a X^\mu(\xi')$$

$$= -\left(p_\mu(\xi)\partial^a X^\mu(\xi) + p_\mu(\xi')\partial'^a X^\mu(\xi')\right)\partial_a\delta(\xi-\xi') = 0, \qquad (4.61)$$

where we have used the relation

$$F(\xi')\partial_a\delta(\xi-\xi') = \partial_a\left[F(\xi')\delta(\xi-\xi')\right] = \partial_a\left[F(\xi)\delta(\xi-\xi')\right]$$

$$= F(\xi)\partial_a\delta(\xi-\xi') + \partial_a F(\xi)\delta(\xi-\xi'). \qquad (4.62)$$

Multiplying (4.61) by an arbitrary "test" function $\phi(\xi')$ and integrating over ξ' we obtain

$$2p_\mu\partial^a X^\mu\partial_a\phi + \partial_a(p_\mu\partial^a X^\mu)\phi = 0. \qquad (4.63)$$

Since ϕ and $\partial_a\phi$ can be taken as independent at any point ξ^a, it follows that

$$p_\mu\partial_a X^\mu = 0. \qquad (4.64)$$

The "momentum" constraints (4.64) are thus shown to be automatically satisfied as a consequence of the conservation of the "Hamiltonian" constraint (4.55). This procedure was been discovered in ref. [59]. Here I have only adjusted it to the case of membrane theory.

If we calculate the Hamiltonian belonging to (4.49) we find

$$H = (p_{\mu(\xi)}\dot{X}^{\mu(\xi)} - L) = \tfrac{1}{2}(p_{\mu(\xi)}p^{\mu(\xi)} - K) \equiv 0, \qquad (4.65)$$

where the canonical momentum is

$$p_{\mu(\xi)} = \frac{\partial L}{\partial \dot{X}^{\mu(\xi)}} = \frac{\kappa\sqrt{|f|}}{\lambda}\dot{X}_\mu\,. \qquad (4.66)$$

Explicitly (4.65) reads

$$H = \frac{1}{2}\int \mathrm{d}\xi\,\frac{\lambda}{\kappa\sqrt{|f|}}(p^\mu p_\mu - \kappa^2|f|) \equiv 0. \qquad (4.67)$$

The Lagrange multiplier λ is arbitrary. The choice of λ determines the choice of parameter τ. Therefore (4.67) holds for every λ, which can only be satisfied if we have

$$p^\mu p_\mu - \kappa^2 |f| = 0 \tag{4.68}$$

at every point ξ^a on the membrane. Eq. (4.68) is a constraint at ξ^a, and altogether there are infinitely many constraints.

In Box 4.3 it is shown that the constraint (4.68) is conserved in τ and that as a consequence we have

$$p_\mu \partial_a X^\mu = 0. \tag{4.69}$$

The latter equation is yet are another set of constraints[2] which are satisfied at any point ξ^a of the membrane manifold V_n

First order form of the action. Having the constraints (4.68), (4.69) one can easily write the first order, or phase space action,

$$I[X^\mu, p_\mu, \lambda, \lambda^a] = \int \mathrm{d}\tau\, \mathrm{d}\xi \left(p_\mu \dot{X}^\mu - \frac{\lambda}{2\kappa\sqrt{|f|}} (p^\mu p_\mu - \kappa^2 |f|) - \lambda^a p_\mu \partial_a X^\mu \right), \tag{4.70}$$

where λ and λ^a are Lagrange multipliers.

The equations of motion are

$$\delta X^\mu \quad : \quad \dot{p}_\mu + \partial_a \left(\kappa \lambda \sqrt{|f|} \partial^a X_\mu - \lambda^a p_\mu \right) = 0, \tag{4.71}$$

$$\delta p_\mu \quad : \quad \dot{X}^\mu - \frac{\lambda}{\kappa\sqrt{|f|}} p_\mu - \lambda^a \partial_a X^\mu = 0, \tag{4.72}$$

$$\delta \lambda \quad : \quad p^\mu p_\mu - \kappa^2 |f| = 0, \tag{4.73}$$

$$\delta \lambda^a \quad : \quad p_\mu \partial_a X^\mu = 0. \tag{4.74}$$

Eqs. (4.72)–(4.74) can be cast into the following form:

$$p_\mu = \frac{\kappa\sqrt{|f|}}{\lambda} (\dot{X}_\mu - \lambda^a \partial_a X^\mu), \tag{4.75}$$

$$\lambda^2 = (\dot{X}^\mu - \lambda^a \partial_a X^\mu)(\dot{X}_\mu - \lambda^b \partial_b X_\mu) \tag{4.76}$$

[2]Something similar happens in canonical gravity. Moncrief and Teitelboim [59] have shown that if one imposes the Hamiltonian constraint on the Hamilton functional then the momentum constraints are automatically satisfied.

$$\lambda_a = \dot{X}^\mu \partial_a X_\mu. \tag{4.77}$$

Inserting the last three equations into the phase space action (4.70) we have

$$I[X^\mu] = \kappa \int \mathrm{d}\tau\, \mathrm{d}\xi \sqrt{|f|}\, \left[\dot{X}^\mu \dot{X}^\nu (\eta_{\mu\nu} - \partial^a X_\mu \partial_a X_\nu)\right]^{1/2}. \tag{4.78}$$

The vector $\dot{X}(\eta_{\mu\nu} - \partial^a X_\mu \partial_a X_\nu)$ is normal to the membrane V_n; its scalar product with tangent vectors $\partial_a X^\mu$ is identically zero. The form $\dot{X}^\mu \dot{X}^\nu (\eta_{\mu\nu} - \partial^a X_\mu \partial_a X_\nu)$ can be considered as a 1-dimensional metric, equal to its determinant, on a line which is orthogonal to V_n. The product

$$f \dot{X}^\mu \dot{X}^\nu (\eta_{\mu\nu} - \partial^a X_\mu \partial_a X_\nu) = \det \partial_A X^\mu \partial_B X_\mu \tag{4.79}$$

is equal to the determinant of the induced metric $\partial_A X^\mu \partial_B X_\mu$ on the $(n+1)$-dimensional surface $X^\mu(\phi^A)$, $\phi^A = (\tau, \xi^a)$, swept by our membrane V_n. The action (4.78) is then *the minimal surface action* for the $(n+1)$-dimensional worldsheet V_{n+1}:

$$I[X^\mu] = \kappa \int \mathrm{d}^{n+1}\phi\, (\det \partial_A X^\mu \partial_B X_\mu)^{1/2}. \tag{4.80}$$

This is the conventional Dirac–Nambu–Goto action, and (4.70) is one of its equivalent forms.

We have shown that from the point of view of $\mathcal{M}$-space a membrane of any dimension is just a point moving along a geodesic in $\mathcal{M}$. The metric of $\mathcal{M}$-space is taken to be an arbitrary fixed background metric. For a special choice of the metric we obtain the conventional p-brane theory. The latter theory is thus shown to be a particular case of the more general theory, based on the concept of $\mathcal{M}$-space.

Another form of the action is obtained if in (4.70) we use the replacement

$$p_\mu = \frac{\kappa \sqrt{|f|}}{\lambda}\, (\dot{X}_\mu - \lambda^a \partial_a X_\mu) \tag{4.81}$$

which follows from "the equation of motion" (4.72). Then instead of (4.70) we obtain the action

$$I[X^\mu, \lambda, \lambda^a] = \frac{\kappa}{2} \int \mathrm{d}\tau\, \mathrm{d}^n\xi \sqrt{|f|}\, \left(\frac{(\dot{X}^\mu - \lambda^a \partial_a X^\mu)(\dot{X}_\mu - \lambda^b \partial_b X_\mu)}{\lambda} + \lambda \right). \tag{4.82}$$

If we choose a gauge such that $\lambda^a = 0$, then (4.82) coincides with the action (4.48) considered before.

The analogy with the point particle. The action (4.82), and especially (4.48), looks like the well known Howe–Tucker action [31] for a point particle, apart from the integration over coordinates ξ^a of a space-like hypersurface Σ on the worldsheet V_{n+1}. Indeed, a worldsheet can be considered as a continuum collection or a bundle of worldlines $X^\mu(\tau, \xi^a)$, and (4.82) is an action for such a bundle. Individual worldlines are distinguished by the values of parameters ξ^a.

We have found a very interesting inter-relationship between various concepts:

1) membrane as a "point particle" moving along a geodesic in an infinite-dimensional membrane space $\mathcal{M}$;

2) worldsheet swept by a membrane as a minimal surface in a finite-dimensional embedding space V_N;

3) worldsheet as a bundle of worldlines swept by point particles moving in V_N.

MEMBRANE THEORY AS A MINIMAL SURFACE IN AN EMBEDDING SPACE

In the previous section we have considered a membrane as a point in an infinite-dimensional membrane space $\mathcal{M}$. Now let us change our point of view and consider a membrane as a surface in a finite-dimensional embedding space V_D When moving, a p-dimensional membrane sweeps a ($d = p + 1$)-dimensional surface which I shall call a *worldsheet*[3]. What is an action which determines the membrane dynamics, i.e., a possible worldsheet? Again the analogy with the point particle provides a clue. Since a point particle sweeps a worldline whose action is *the minimal length action*, it is natural to postulate that a membrane's worldsheet satisfies *the minimal surface action*:

$$I[X^\mu] = \kappa \int \mathrm{d}^d\phi \, (\det \partial_A X^\mu \partial_B X_\mu)^{1/2}. \tag{4.83}$$

This action, called also the Dirac–Nambu–Goto action, is invariant under reparametrizations of the worldsheet coordinates ϕ^A, $A = 0, 1, 2, ..., d - 1$. Consequently the dynamical variables X^μ, $\mu = 0, 1, 2, ..., D$, and the corresponding momenta are subjected to d primary constraints.

Another suitable form of the action (equivalent to (4.83)) is the Howe–Tucker action [31] generalized to a membrane of arbitrary dimension p

[3] In the literature on p-branes such a surface is often called "world volume" and sometimes world surface.

(p-brane):

$$I[X^\mu, \gamma^{AB}] = \frac{\kappa_0}{2} \int \sqrt{|\gamma|}(\gamma^{AB}\partial_A X^\mu \partial_B X_\mu + 2 - d). \qquad (4.84)$$

Besides the variables $X^\mu(\phi)$, $\mu = 0, 1, 2, ..., D-1$, which denote the position of a d-dimensional ($d = p+1$) worldsheet V_d in the embedding spacetime V_D, the above action also contains the auxiliary variables γ^{AB} (with the role of Lagrange multipliers) which have to be varied independently from X^μ.

By varying (4.84) with respect to γ^{AB} we arrive at the equation for the induced metric on a worldsheet:

$$\gamma_{AB} = \partial_A X^\mu \partial_B X_\mu. \qquad (4.85)$$

Inserting (4.85) into (4.84) we obtain the Dirac–Nambu–Goto action (4.83).

In eq. (4.84) the γ^{AB} are the Lagrange multipliers, but they are not all independent. The number of worldsheet constraints is d, which is also the number of independent Lagrange multipliers. In order to separate out of γ^{AB} the independent multipliers we proceed as follows. Let Σ be a space-like hypersurface on the worldsheet, and n^A the normal vector field to Σ. Then the worldsheet metric tensor can be written as

$$\gamma^{AB} = \frac{n^A n^B}{n^2} + \bar{\gamma}^{AB} \, , \qquad \gamma_{AB} = \frac{n_A n_B}{n^2} + \bar{\gamma}_{AB} \, , \qquad (4.86)$$

where $\bar{\gamma}^{AB}$ is projection tensor, satisfying

$$\bar{\gamma}^{AB} n_B = 0, \;\; \bar{\gamma}_{AB} n^B = 0. \qquad (4.87)$$

It projects any vector into the hypersurface to which n^a is the normal. For instance, using (4.86) we can introduce the tangent derivatives

$$\bar{\partial}_A X^\mu = \bar{\gamma}_A^B \, \partial_B X^\mu = \gamma_A{}^B \partial_B X^\mu - \frac{n_A n^B}{n^2} \partial_B X^\mu. \qquad (4.88)$$

An arbitrary derivative $\partial_A X^\mu$ is thus decomposed into a normal and tangential part (relative to Σ):

$$\partial_A X^\mu = n_A \partial X^\mu + \bar{\partial}_A X^\mu \, , \qquad (4.89)$$

where

$$\partial X^\mu \equiv \frac{n^A \partial_A X^\mu}{n^2}, \qquad n^A \bar{\partial}_A X^\mu = 0. \qquad (4.90)$$

Details about using and keeping the d-dimensional covariant notation as far as possible are given in ref. [61]. Here, following ref. [62], I shall present a shorter and more transparent procedure, but without the covariant notation in d-dimensions.

Let us take such a class of coordinate systems in which covariant components of normal vectors are

$$n_A = (1, 0, 0, ..., 0). \tag{4.91}$$

From eqs. (4.86) and (4.91) we have

$$n^2 = \gamma_{AB} n^A n^B = \gamma^{AB} n_A n_B = n^0 = \gamma^{00}, \tag{4.92}$$

$$\bar{\gamma}^{00} = 0 \, , \quad \bar{\gamma}^{0a} = 0, \tag{4.93}$$

and

$$\gamma_{00} = \frac{1}{n^0} + \bar{\gamma}_{ab} \frac{n^a n^b}{(n^0)^2}, \tag{4.94}$$

$$\gamma_{0a} = -\frac{\bar{\gamma}_{ab} n^b}{n^0}, \tag{4.95}$$

$$\gamma_{ab} = \bar{\gamma}_{ab}, \tag{4.96}$$

$$\gamma^{00} = n^0, \tag{4.97}$$

$$\gamma^{0a} = n^a, \tag{4.98}$$

$$\gamma^{ab} = \bar{\gamma}^{ab} + \frac{n^a n^b}{n^0} \, , \quad a, b = 1, 2, ..., p. \tag{4.99}$$

The decomposition (4.89) then becomes

$$\partial_0 X^\mu = \partial X^\mu + \bar{\partial}_0 X^\mu, \tag{4.100}$$

$$\partial_a X^\mu = \bar{\partial}_a X^\mu, \tag{4.101}$$

where

$$\partial X^\mu = \dot{X}^\mu + \frac{n^a \partial_a X^\mu}{n^0} \, , \quad \dot{X}^\mu \equiv \partial_0 X^\mu \equiv \frac{\partial X^\mu}{\partial \xi^0} \, , \quad \partial_a X^\mu \equiv \frac{\partial x^\mu}{\partial \xi^a}, \tag{4.102}$$

$$\bar{\partial}_0 X^\mu = -\frac{n^a \partial_a X^\mu}{n^0}. \tag{4.103}$$

The $n^A = (n^0, n^a)$ can have the role of d independent Lagrange multipliers. We can now rewrite our action in terms of n^0, n^a, and $\bar{\gamma}^{ab}$ (instead of γ^{AB}). We insert (4.97)–(4.99) into (4.84) and take into account that

$$|\gamma| = \frac{\bar{\gamma}}{n^0}, \tag{4.104}$$

where $\gamma = \det \gamma_{AB}$ is the determinant of the worldsheet metric and $\bar{\gamma} = \det \bar{\gamma}_{ab}$ the determinant of the metric $\bar{\gamma}_{ab} = \gamma_{ab}\,,\ a, b = 1, 2, ..., p$ on the hypersurface Σ.

So our action (4.84) after using (4.97)–(4.99) becomes

$$[X^\mu, n^A, \bar{\gamma}^{ab}] = \frac{\kappa_0}{2} \int \mathrm{d}^d\phi \frac{\sqrt{\bar{\gamma}}}{\sqrt{n^0}}$$

$$\times \left(n^0 \dot{X}^\mu \dot{X}_\mu + 2n^a \dot{X}^\mu \partial_a X_\mu + (\bar{\gamma}^{ab} + \frac{n^a n^b}{n^0}) \partial_a X^\mu \partial_b X_\mu + 2 - d \right). \tag{4.105}$$

Variation of the latter action with respect to $\bar{\gamma}^{ab}$ gives the expression for the induced metric on the surface Σ :

$$\bar{\gamma}_{ab} = \partial_a X^\mu \partial_b X_\mu\,, \qquad \bar{\gamma}^{ab} \bar{\gamma}_{ab} = d - 1. \tag{4.106}$$

We can eliminate $\bar{\gamma}^{ab}$ from the action (4.105) by using the relation (4.106):

$$I[X^\mu, n^a] = \frac{\kappa_0}{2} \int \mathrm{d}^d\phi \frac{\sqrt{|f|}}{\sqrt{n^0}} \left(\frac{1}{n^0} (n^0 \dot{X}^\mu + n^a \partial_a X^\mu)(n^0 \dot{X}_\mu + n^b \partial_b X_\mu) + 1 \right), \tag{4.107}$$

where $\sqrt{|f|} \equiv \det \partial_a X^\mu \partial_b X_\mu$. The latter action is a functional of the worldsheet variables X^μ and d independent Lagrange multipliers $n^A = (n^0, n^a)$. Varying (4.107) with respect to n^0 and n^a we obtain the worldsheet constraints:

$$\delta n^0 \ : \quad (\dot{X}^\mu + \frac{n^b \partial_b X^\mu}{n^0}) \dot{X}_\mu = \frac{1}{n^0}, \tag{4.108}$$

$$\delta n^a \ : \quad (\dot{X}^\mu + \frac{n^b \partial_b X^\mu}{n^0}) \partial_a X_\mu = 0. \tag{4.109}$$

Using (4.102) the constraints can be written as

$$\partial X^\mu \partial X_\mu = \frac{1}{n^0}, \tag{4.110}$$

$$\partial X^\mu \partial_i X_\mu = 0. \tag{4.111}$$

The action (4.107) contains the expression for the normal derivative ∂X^μ and can be written in the form

$$I = \frac{\kappa_0}{2} \int \mathrm{d}\tau \mathrm{d}^p\xi \sqrt{|f|} \left(\frac{\partial X^\mu \partial X_\mu}{\lambda} + \lambda \right), \quad \lambda \equiv \frac{1}{\sqrt{n^0}}, \tag{4.112}$$

where we have written $\mathrm{d}^d\phi = \mathrm{d}\tau \ \mathrm{d}^p\xi$, since $\phi^A = (\tau, \xi^a)$.

So we arrived at an action which looks like the well known Howe–Tucker action for a point particle, except for the integration over a space-like hypersurface Σ, parametrized by coordinates $\xi^a, a = 1, 2, ..., p$. Introducing $\lambda^a = -n^a/n^0$ the normal derivative can be written as $\partial X^\mu \equiv \dot{X}^\mu - \lambda^a \partial_a X_\mu$. Instead of n^0, n^a we can take $\lambda \equiv 1/\sqrt{n^0}$, $\lambda^a \equiv -n^a/n^0$ as the Lagrange multipliers. In eq. (4.112) we thus recognize the action (4.82).

MEMBRANE THEORY BASED ON THE GEOMETRIC CALCULUS IN $\mathcal{M}$-SPACE

We have seen that a membrane's velocity $\dot{X}^{\mu(\xi)}$ and momentum $p_{\mu(\xi)}$ can be considered as components of vectors in an infinite-dimensional membrane space $\mathcal{M}$ in which every point can be parametrized by coordinates $X^{\mu(\xi)}$ which represent a membrane. In analogy with the finite-dimensional case considered in Chapter 2 we can introduce the concept of a vector in $\mathcal{M}$ and a set of basis vectors $e_{\mu(\xi)}$, such that any vector a can be expanded according to

$$a = a^{\mu(\xi)} e_{\mu(\xi)}. \tag{4.113}$$

From the requirement that

$$a^2 = a^{\mu(\xi)} e_{\mu(\xi)} a^{\nu(\xi')} e_{\nu(\xi')} = \rho_{\mu(\xi)\nu(\xi')} a^{\mu(\xi)} a^{\nu(\xi')} \tag{4.114}$$

we have

$$\frac{1}{2}(e_{\mu(\xi)} e_{\nu(\xi')} + e_{\nu(\xi')} e_{\mu(\xi)}) \equiv e_{\mu(\xi)} \cdot e_{\nu(\xi')} = \rho_{\mu(\xi)\nu(\xi')}. \tag{4.115}$$

This is the definition of *the inner product* and $e_{\mu(\xi)}$ are generators of *Clifford algebra* in $\mathcal{M}$-space.

A more complete elaboration of geometric calculus based on Clifford algebra in $\mathcal{M}$ will be provided in Chapter 6. Here we just use (4.113), (4.115) to extend the point particle polyvector action (2.56) to $\mathcal{M}$-space.

We shall start from the first order action (4.70). First we rewrite the latter action in terms of the compact $\mathcal{M}$-space notation:

$$I[X^\mu, p_\mu, \lambda, \lambda^a] = \int \mathrm{d}\tau \left(p_{\mu(\xi)} \dot{X}^{\mu(\xi)} - \frac{1}{2}(p_{\mu(\xi)} p^{\mu(\xi)} - K) - \lambda^a p_{\mu(\xi)} \partial_a X^{\mu(\xi)} \right). \tag{4.116}$$

In order to avoid introducing a new symbol, it is understood that the product $\lambda^a p_{\mu(\xi)}$ denotes covariant components of an $\mathcal{M}$-space vector. The Lagrange multiplier λ is included in the metric $\rho_{\mu(\xi)\nu(\xi')}$.

According to (4.113) we can write the momentum and velocity vectors as

$$p = p_{\mu(\xi)} e^{\mu(\xi)}, \tag{4.117}$$

$$\dot{X} = \dot{X}^{\mu(\xi)} e_{\mu(\xi)}, \tag{4.118}$$

where

$$e^{\mu(\xi)} = \rho^{\mu(\xi)\nu(\xi')} e_{\nu(\xi')} \tag{4.119}$$

and

$$e^{\mu(\xi)} \cdot e_{\nu(\xi')} = \delta^{\mu(\xi)}{}_{\nu(\xi')}. \tag{4.120}$$

The action (4.116) can be written as

$$I(X, p, \lambda, \lambda^a) = \int \mathrm{d}\tau \left[p \cdot \dot{X} - \frac{1}{2}(p^2 - K) - \lambda^a \partial_a X \cdot p \right], \tag{4.121}$$

where

$$\partial_a X = \partial_a X^{\mu(\xi)} e_{\mu(\xi)} \tag{4.122}$$

are tangent vectors. We can omit the dot operation in (4.121) and write the action

$$I(X, p, \lambda, \lambda^a) = \int \mathrm{d}\tau \left[p\dot{X} - \frac{1}{2}(p^2 - K) - \lambda^a p \, \partial_a X \right] \tag{4.123}$$

which contains the scalar part and the bivector part. It is straightforward to show that the bivector part contains the same information about the equations of motion as the scalar part.

Besides the objects (4.113) which are 1-vectors in $\mathcal{M}$ we can also form 2-vectors, 3-vectors, etc., according to the analogous procedures as explained in Chapter 2. For instance, a 2-vector is

$$a \wedge b = a^{\mu(\xi)} b^{\nu(\xi')} e_{\mu(\xi)} \wedge e_{\nu\xi')}, \tag{4.124}$$

where $e_{\mu(\xi)} \wedge e_{\nu\xi')}$ are basis 2-vectors. Since the index $\mu(\xi)$ has the discrete part μ and the continuous part (ξ), the wedge product

$$e_{\mu(\xi_1)} \wedge e_{\mu(\xi_2)} \wedge ... \wedge e_{\mu(\xi_k)} \tag{4.125}$$

can have any number of terms with different values of ξ and the same value of μ. The number of terms in the wedge product

$$e_{\mu_1(\xi)} \wedge e_{\mu_2(\xi)} \wedge ... \wedge e_{\mu_k(\xi)}, \tag{4.126}$$

with the same value of ξ, but with different values of μ, is limited by the number of discrete dimensions of $\mathcal{M}$-space. At fixed ξ the Clifford algebra of $\mathcal{M}$-space behaves as the Clifford algebra of a finite-dimensional space.

Let us write the pseudoscalar unit of the finite-dimensional subspace V_n of $\mathcal{M}$ as

$$I_{(\xi)} = e_{\mu_1(\xi)} \wedge e_{\mu_2(\xi)} \wedge ... \wedge e_{\mu_n(\xi)} \tag{4.127}$$

A generic polyvector in $\mathcal{M}$ is a superposition

$$A = a_0 + a^{\mu(\xi)} e_{\mu(\xi)} + a^{\mu_1(\xi_1)\mu_2(\xi_2)} e_{\mu_1(\xi_1)} \wedge e_{\mu_2(\xi_2)} + ... \\ + a^{\mu_1(\xi_1)\mu_2(\xi_2)...\mu_k(\xi_k)} e_{\mu_1(\xi_1)} \wedge e_{\mu_2(\xi_2)} \wedge ... \wedge e_{\mu_k(\xi_k)} + ... \, . \tag{4.128}$$

As in the case of the point particle I shall follow the principle that the most general physical quantities related to membranes, such as momentum P and velocity $\dot{X}$, are polyvectors in $\mathcal{M}$-space. I invite the interested reader to work out as an exercise (or perhaps as a research project) what physical interpretation[4] could be ascribed to all possible multivector terms of P and $\dot{X}$. For the finite-dimensional case I have already worked out in Chapter 2, Sec 3, to certain extent such a physical interpretation. We have also seen that at the classical level, momentum and velocity polyvectors which solve the equations of motion can have all the multivector parts vanishing except for the vector and pseudoscalar part. Let us assume a similar situation for the membrane momentum and velocity:

$$P = P^{\mu(\xi)} e_{\mu(\xi)} + m^{(\xi)} I_{(\xi)}, \tag{4.129}$$

$$\dot{X} = \dot{X}^{\mu(\xi)} e_{\mu(\xi)} + \dot{s}^{(\xi)} . I_{(\xi)} \tag{4.130}$$

In addition let us assume

$$\partial_a X = \partial_a X^{\mu(\xi)} e_{\mu(\xi)} + \partial_a s^{(\xi)} I_{(\xi)}. \tag{4.131}$$

Let us assume the following general membrane action:

$$I(X, P, \lambda, \lambda^a) = \int \mathrm{d}\tau \left[P\dot{X} - \frac{1}{2}(P^2 - K) - \lambda^a \partial_a X \, P \right]. \tag{4.132}$$

On the one hand the latter action is a generalization of the action (4.121) to arbitrary polyvectors $\dot{X}$, P, $\lambda^a \partial_a X$. On the other hand, (4.132) is a

[4]As a hint the reader is advised to look at Secs. 6.3 and 7.2.

generalization of the point particle polyvector action (2.56), where the polyvectors in a finite-dimensional space V_n are replaced by polyvectors in the infinite-dimensional space $\mathcal{M}$.

Although the polyvectors in the action (4.132) are arbitrary in principle (defined according to (4.128)), we shall from now on restrict our consideration to a particular case in which the polyvectors are given by eqs. (4.129)-(4.131). Rewriting the action action (4.132) in the component notation, that is, by inserting (4.129)-(4.131) into (4.132) and by taking into account (4.115), (4.120) and

$$e_{\mu(\xi)} \cdot e_{\nu(\xi')} = \rho_{\mu(\xi)\nu(\xi')} = \frac{\kappa\sqrt{|f|}}{\lambda}\,\delta(\xi - \xi')\eta_{\mu\nu}, \tag{4.133}$$

$$I_{(\xi)} \cdot I_{(\xi')} = \rho_{(\xi)(\xi')} = -\frac{\kappa\sqrt{|f|}}{\lambda}\,\delta(\xi - \xi'), \tag{4.134}$$

$$I_{(\xi)} \cdot e_{\nu(\xi')} = 0, \tag{4.135}$$

we obtain

$$\langle I \rangle_0 = \int \mathrm{d}\tau \Big[p_{\mu(\xi)}\dot{X}^{\mu(\xi)} - \dot{s}^{(\xi)}m_{(\xi)} - \frac{1}{2}(p^{\mu(\xi)}p_{\mu(\xi)} + m^{(\xi)}m_{(\xi)} - K) - \lambda^a(\partial_a X^{\mu(\xi)}p_{\mu(\xi)} - \partial_a s^{(\xi)}m_{(\xi)}) \Big]. \tag{4.136}$$

By the way, let us observe that using (4.133), (4.134) we have

$$-K = -\int \mathrm{d}\xi\, \kappa\sqrt{|f|}\lambda = \lambda^{(\xi)}\lambda_{(\xi)}\ , \qquad \lambda^{(\xi)} \equiv \lambda(\xi), \tag{4.137}$$

which demonstrates that the term K can also be written in the elegant tensor notation.

More explicitly, (4.136) can be written in the form

$$I[X^\mu, s, p_\mu, m, \lambda, \lambda^a] = \int \mathrm{d}\tau\, \mathrm{d}^n\xi \Big[p_\mu\dot{X}^\mu - m\dot{s} - \frac{\lambda}{2\kappa\sqrt{|f|}}\left(p^\mu p_\mu - m^2 - \kappa^2|f|\right) - \lambda^a(\partial_a X^\mu p_\mu - \partial_a\, s\, m) \Big] \tag{4.138}$$

This is a generalization of the membrane action (4.70) considered in Sec. 4.2. Besides the coordinate variables $X^\mu(\xi)$ we have now an additional variable $s(\xi)$. Besides the momentum variables $p_\mu(\xi)$ we also have the variable $m(\xi)$. We retain the same symbols p_μ and m as in the case of the point particle theory, with understanding that those variables are now ξ-dependent densities of weight 1.

The equations of motion derived from (4.138) are:

$$\delta s \quad : \quad -\dot{m} + \partial_a(\lambda^a m) = 0, \tag{4.139}$$

$$\delta X^\mu \quad : \quad \dot{p}_\mu + \partial_a\left(\kappa\lambda\sqrt{|f|}\partial^a X_\mu - \lambda^a p_\mu\right) = 0, \tag{4.140}$$

$$\delta m \quad : \quad -\dot{s} + \lambda^a \partial_a s + \frac{\lambda}{\kappa\sqrt{|f|}} m = 0, \tag{4.141}$$

$$\delta p_\mu \quad : \quad \dot{X}^\mu - \lambda^a \partial_a X^\mu - \frac{\lambda}{\kappa\sqrt{|f|}}\, p_\mu = 0, \tag{4.142}$$

$$\delta\lambda \quad : \quad p^\mu p_\mu - m^2 - \kappa^2 |f| = 0, \tag{4.143}$$

$$\delta\lambda^a \quad : \quad \partial_a X^\mu p_\mu - \partial_a s\, m = 0. \tag{4.144}$$

Let us collect in the action those term which contain s and m and re-express them by using the equations of motion (4.141). We obtain

$$-m\dot{s} + \frac{\lambda}{2\sqrt{|f|}}\, m^2 + \lambda^a \partial_a s\, m = -\frac{\lambda}{2\kappa\sqrt{|f|}} m^2 \tag{4.145}$$

Using again (4.141) and also (4.139) we have

$$-\frac{\lambda}{2\kappa\sqrt{|f|}} m^2 = \frac{1}{2}(-m\dot{s} + \lambda^a \partial_a s\, m) = \frac{1}{2}\left(-\frac{\mathrm{d}(ms)}{\mathrm{d}\tau} + \partial_a(\lambda^a s\, m)\right). \tag{4.146}$$

Inserting (4.145) and (4.146) into the action (4.138) we obtain

$$I = \int \mathrm{d}\tau\, \mathrm{d}^n\xi \left[-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\tau}(ms) + \frac{1}{2}\partial_a(ms\lambda^a) \right.$$
$$\left. + \, p_\mu \dot{X}^\mu - \frac{\lambda}{2\kappa\sqrt{|f|}}(p^\mu p_\mu - \kappa^2|f|) - \lambda^a \partial_a X^\mu p_\mu \right]. \tag{4.147}$$

We see that the extra variables s, m occur only in the terms which are total derivatives. Those terms have no influence on the equations of motion, and can be omitted, so that

$$I[X^\mu, p_\mu] = \int \mathrm{d}\tau\, \mathrm{d}^n\xi \left[p_\mu \dot{X}^\mu - \frac{\lambda}{2\kappa\sqrt{|f|}}(p^\mu p_\mu - \kappa^2|f|) - \lambda^a \partial_a X^\mu p_\mu\right]. \tag{4.148}$$

This action action looks like the action (4.70) considered in Sec 2.2. However, now λ and λ^a are no longer Lagrange multipliers. They should be considered as fixed since they have already been "used" when forming the terms $-(\mathrm{d}/\mathrm{d}\tau)(ms)$ and $\partial_a(ms\lambda^a)$ in (4.147). Fixing of λ, λ^a means fixing the gauge, that is the choice of parameters τ and ξ^a. In (4.148) we have thus obtained a *reduced action* which is a functional of the reduced number of variables X^μ, p_μ. All X^μ or all p_μ are independent; there are no more constraints.

However, a choice of gauge (the fixing of λ, λ^a) must be such that the equations of motion derived from the reduced action are consistent with the equations of motion derived from the original constrained action. In our case we find that an admissible choice of gauge is given by

$$\frac{\lambda}{\kappa\sqrt{|f|}} = \Lambda \ , \quad \lambda^a = \Lambda^a, \tag{4.149}$$

where Λ, Λ^a are arbitrary fixed functions of τ, ξ^a. So we obtained the following *unconstrained action*:

$$I[X^\mu, p_\mu] = \int \mathrm{d}\tau \, \mathrm{d}^n\xi \left[p_\mu \dot{X}^\mu - \frac{\Lambda}{2}(p^\mu p_\mu - \kappa^2 |f|) - \Lambda^a \partial_a X^\mu p_\mu \right]. \tag{4.150}$$

The fixed function Λ does not transform as a scalar under reparametrizations of ξ^a, but a scalar density of weight -1, whereas Λ^a transforms as a vector. Under reparametrizations of τ they are assumed to transform according to $\Lambda' = (\mathrm{d}\tau/\mathrm{d}\tau')\Lambda$ and $\Lambda'^a = (\mathrm{d}\tau/\mathrm{d}\tau')\Lambda^a$. The action (4.150) is then *covariant* under reparametrizations of τ and ξ^a, i.e., it retains the same form. However, it is not *invariant* (Λ and Λ^a in a new parametrization are different functions of the new parameters), therefore there are no constraints.

Variation of the action (4.150) with respect to X^μ and p_μ gives

$$\delta X^\mu \quad : \quad \dot{p}_\mu + \partial_a \left(\Lambda \kappa^2 |f| \partial^a X_\mu - \Lambda^a p_\mu \right) = 0, \tag{4.151}$$

$$\delta p_\mu \quad : \quad \dot{X}^\mu - \Lambda^a \partial_a X^\mu - \Lambda p^\mu = 0. \tag{4.152}$$

The latter equations of motion are indeed equal to the equations of motion (4.140),(4.142) in which gauge is fixed according to (4.149).

Eliminating p_μ in (4.150) by using eq (4.152), we obtain

$$I[X^\mu] = \frac{1}{2} \mathrm{d}\tau \, \mathrm{d}^n\xi \left[\frac{(\dot{X}^\mu - \Lambda^a \partial_a X^\mu)(\dot{X}_\mu - \Lambda^b \partial_b X^\mu)}{\Lambda} + \Lambda \kappa^2 |f| \right]. \tag{4.153}$$

If $\Lambda^a = 0$ this simplifies to

$$I[X^\mu] = \frac{1}{2} \int \mathrm{d}\tau \, \mathrm{d}^n\xi \left(\frac{\dot{X}^\mu \dot{X}_\mu}{\Lambda} + \Lambda \kappa^2 |f| \right). \tag{4.154}$$

In the static case, i.e., when $\dot{X}^\mu = 0$, we have

$$I[X^\mu] = \frac{1}{2} \int \mathrm{d}\tau \, \mathrm{d}^n \xi \, \Lambda \kappa^2 |f|, \tag{4.155}$$

which is the well known Schild action [63].

Alternative form of the $\mathcal{M}$-space metric. Let us now again consider the action (4.136). Instead of (4.133), (4.134) let us now take the following form of the metric:

$$\rho_{\mu(\xi)\nu(\xi')} = \frac{1}{\tilde{\lambda}} \delta(\xi - \xi') \eta_{\mu\nu} \, , \tag{4.156}$$

$$\rho_{(\xi)(\xi')} = -\frac{1}{\tilde{\lambda}} \delta(\xi - \xi') \, , \tag{4.157}$$

and insert it into (4.136). Then we obtain the action

$$I[X^\mu, s, p_\mu, m, \lambda', \lambda^a] = \tag{4.158}$$

$$\int \mathrm{d}\tau \, \mathrm{d}^n \xi \, \left[-m \dot{s} + p_\mu \dot{X}^\mu - \frac{\tilde{\lambda}}{2} (p^\mu p_\mu - m^2 - \kappa^2 |f|) - \lambda^a (\partial_a X^\mu p_\mu - \partial_a s \, m) \right],$$

which is equivalent to (4.138). Namely, we can easily verify that the corresponding equations of motion are equivalent to the equations of motion (4.139)–(4.144). From the action (4.158) we then obtain the unconstrained action (4.150) by fixing $\tilde{\lambda} = \Lambda$ and $\lambda^a = \Lambda^a$.

We have again found (as in the case of a point particle) that the polyvector generalization of the action naturally contains "time" and evolution of the membrane variables X^μ. Namely, in the theory there occurs an extra variable s whose derivative $\dot{s}$ with respect to the worldsheet parameter τ is the pseudoscalar part of the velocity polyvector. This provides a mechanism of obtaining the Stueckelberg action from a more basic principle.

Alternative form of the constrained action. Let us again consider the constrained action (4.138) which is a functional of the variables X^μ, s and the canonical momenta p_μ, m. We can use equation of motion (4.141) in order to eliminate the Lagrange multipliers λ from the action. By doing so we obtain

$$I = \int \mathrm{d}\tau \, \mathrm{d}^n \xi \Big[-m(\dot{s} - \lambda^a \partial_a s) + p_\mu (\dot{X}^\mu - \lambda^a \partial_a X^\mu) - \frac{\dot{s} - \lambda^a \partial_a s}{2m} \, (p^2 - m^2 - \kappa^2 |f|) \Big]. \tag{4.159}$$

We shall now prove that

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \dot{s} - \lambda^a \partial_a s \quad \text{and} \quad \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \dot{X}^\mu - \lambda^a \partial_a X^\mu, \tag{4.160}$$

where $\dot{s} \equiv \partial s/\partial\tau$ and $\dot{X}^\mu = \partial X^\mu/\partial\tau$ are partial derivatives. The latter relations follow from the definitions of the total derivatives

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \frac{\partial s}{\partial\tau} + \frac{\mathrm{d}\xi^a}{\mathrm{d}\tau} \quad \text{and} \quad \frac{\mathrm{d}X^\mu}{\mathrm{d}\tau} = \frac{\partial X^\mu}{\partial\tau} + \partial_a X^\mu \frac{\mathrm{d}\xi^a}{\mathrm{d}\tau}, \tag{4.161}$$

and the relation $\lambda^a = -\mathrm{d}\xi^a/\mathrm{d}\tau$, which comes from the momentum constraint $p_\mu \partial_a X^\mu = 0$.

Inserting (4.160) into eq. (4.159) we obtain yet another equivalent classical action

$$I[X^\mu, p_\mu, m] = \int \mathrm{d}s\, \mathrm{d}^n\xi \left[p_\mu \frac{\mathrm{d}X^\mu}{\mathrm{d}s} - \frac{m}{2} - \frac{1}{2m}(p^2 - \kappa^2|f|) \right], \tag{4.162}$$

in which the variable s has disappeared from the Lagrangian, and it has instead become the evolution parameter. Alternatively, if in the action (4.159) we choose a gauge such that $\dot{s} = 1$, $\lambda^a = 0$, then we also obtain the same action (4.162).

The variable m in (4.162) acquired the status of a Lagrange multiplier leading to the constraint

$$\delta m: \quad p^2 - \kappa^2|f| - m^2 = 0. \tag{4.163}$$

Using the constraint (4.163) we can eliminate m from the action (4.162) and we obtain the following reduced action

$$I[X^\mu, p_\mu] = \int \mathrm{d}s\, \mathrm{d}^n\xi \left(p_\mu \frac{\mathrm{d}X^\mu}{\mathrm{d}s} - \sqrt{p^2 - \kappa^2|f|} \right) \tag{4.164}$$

which, of course, is unconstrained. It is straightforward to verify that the equations of motion derived from the unconstrained action (4.164) are the same as the ones derived from the original constrained action (4.138).

The extra variable s in the reparametrization invariant constrained action (4.138), after performing reduction of variables by using the constraints, has become the evolution parameter.

There is also a more direct derivation of the unconstrained action which will be provided in the next section.

Conclusion. Geometric calculus based on Clifford algebra in a finite-dimensional space can be generalized to the infinite-dimensional membrane space $\mathcal{M}$. Mathematical objects of such an algebra are Clifford numbers, also called Clifford aggregates, or polyvectors. It seems natural to assume that physical quantities are in general polyvectors in $\mathcal{M}$. Then, for instance, the membrane velocity $\dot{X}^\mu$ in general is not a vector, but a polyvector, and hence it contains all other possible r-vector parts, including a scalar and a pseudoscalar part. As a preliminary step I have considered here a model in which velocity is the sum of a vector and a pseudoscalar. The pseudoscalar component is $\dot{s}$, i.e., the derivative of an extra variable s. Altogether we thus have the variables X^μ and s, and the corresponding canonically conjugate momenta p_μ and m. The polyvector action is reparametrization invariant, and as a consequence there are constraints on those variables. Therefore we are free to choose appropriate number of extra relations which eliminate the redundancy of variables. We may choose relations such that we get rid of the extra variables s and m, but then the remaining variables X^μ, p_μ are *unconstrained*, and they evolve in the evolution parameter τ which, by choice of a gauge, can be made proportional to s.

Our model with the polyvector action thus allows for *dynamics* in spacetime. *It resolves the old problem of the conflict between our experience of the passage of time on the one hand, and the fact that the theory of relativity seems incapable of describing the flow of time at all:* past, present and future coexist in a four- (or higher-dimensional) "block" spacetime, with objects corresponding to worldlines (or worldsheets) within this block. And what, in my opinion, is very nice, the resolution is not a result of an *ad hoc procedure*, but is a necessary consequence of the existence of Clifford algebra as a general tool for the description of the geometry of spacetime!

Moreover, when we shall also consider dynamics of spacetime itself, we shall find out that the above model with the polyvector action, when suitably generalized, will provide a natural resolution of the notorious "problem of time" in quantum gravity.

4.3. MORE ABOUT THE INTERCONNECTIONS AMONG VARIOUS MEMBRANE ACTIONS

In the previous section we have considered various membrane actions. One action was just that of a *free fall in $\mathcal{M}$-space* (eq. (4.21)). For a special metric (4.28) which contains the membrane velocity we have obtained the equation of motion (4.35) which is identical to that of the *Dirac–Nambu–Goto membrane* described by *the minimal surface action* (4.80).

Instead of the free fall action in $\mathcal{M}$-space we have considered some *equivalent forms* such as *the quadratic actions* (4.39), (4.49) and the corresponding *first order* or *phase space action* (4.70).

Then we have brought into the play *the geometric calculus based on Clifford algebra* and applied it to $\mathcal{M}$-space. The membrane velocity and momentum are promoted to *polyvectors*. The latter variables were then used to construct *the polyvector phase space action* (4.132), and its more restricted form in which the polyvectors contain the vector and the pseudoscalar parts only.

Whilst all the actions described in the first two paragraphs were equivalent to the usual minimal surface action which describes *the constrained membrane*, we have taken with the polyvectors a step beyond the conventional membrane theory. We have seen that the presence of a pseudoscalar variable results in *unconstraining* the rest of the membrane's variables which are $X^\mu(\tau, \xi)$. This has important consequences.

If momentum and velocity polyvectors are given by expressions (4.129)–(4.131), then the polyvector action (4.132) becomes (4.136) whose more explicit form is (4.138). Eliminating from the latter phase space action the variables P_μ and m by using their equations of motion (4.139), (4.142), we obtain

$$
\begin{aligned}
I[X^\mu, s, \lambda, \lambda^a] \\
= \; & \frac{\kappa}{2} \int \mathrm{d}\tau \, \mathrm{d}^n \xi \, \sqrt{|f|} \qquad (4.165) \\
& \times \left(\frac{(\dot{X}^\mu - \lambda^a \partial_a X^\mu)(\dot{X}_\mu - \lambda^b \partial_b X_\mu) - (\dot{s} - \lambda^a \partial_a s)^2}{\lambda} + \lambda \right).
\end{aligned}
$$

The choice of the Lagrange multipliers λ, λ^a fixes the parametrization τ and ξ^a. We may choose $\lambda^a = 0$ and action (4.165) simplifies to

$$
I[X^\mu, s, \lambda] = \frac{\kappa}{2} \int \mathrm{d}\tau \, \mathrm{d}^n \xi \, \sqrt{|f|} \left(\frac{\dot{X}^\mu \dot{X}_\mu - \dot{s}^2}{\lambda} + \lambda \right), \qquad (4.166)
$$

which is an extension of the Howe–Tucker-like action (4.48) or (2.31) considered in the first two sections.

Varying (4.166) with respect to λ we have

$$
\lambda^2 = \dot{X}^\mu \dot{X}_\mu - \dot{s}^2. \qquad (4.167)
$$

Using relation (4.167) in eq. (4.166) we obtain

$$
I[X^\mu, s] = \kappa \int \mathrm{d}\tau \, \mathrm{d}^n \xi \, \sqrt{|f|} \sqrt{\dot{X}^\mu \dot{X}_\mu - \dot{s}^2}. \qquad (4.168)
$$

This reminds us of the relativistic point particle action (4.21). The difference is in the extra variable s and in that the variables depend not only on the parameter τ but also on the parameters ξ^a, hence the integration over ξ^a with the measure $\mathrm{d}^n\xi\sqrt{|f|}$ (which is invariant under reparametrizations of ξ^a).

Bearing in mind $\dot{X} = \partial X^\mu/\partial\tau$, $\dot{s} = \partial s/\partial\tau$, and using the relations (4.160), (4.161), we can write (4.168) as

$$I[X^\mu] = \kappa \int \mathrm{d}s\, \mathrm{d}^n\xi\, \sqrt{|f|}\sqrt{\frac{\mathrm{d}X^\mu}{\mathrm{d}s}\frac{\mathrm{d}X_\mu}{\mathrm{d}s} - 1}. \tag{4.169}$$

The step from (4.168) to (4.169) is equivalent to choosing the parametrization of τ such that $\dot{s} = 1$ for any ξ^a, which means that $\mathrm{d}s = \mathrm{d}\tau$.

We see that in (4.169) the extra variable s takes the role of the evolution parameter and that the variables $X^\mu(\tau, \xi)$ and the conjugate momenta $p_\mu(\tau, \xi) = \partial\mathcal{L}/\partial\dot{X}^\mu$ are *unconstrained* [5].

In particular, a membrane $\mathcal{V}_n$ which solves the variational principle (4.169) can have vanishing velocity

$$\frac{\mathrm{d}X^\mu}{\mathrm{d}s} = 0. \tag{4.170}$$

Inserting this back into (4.169) we obtain the action[6]

$$I[X^\mu] = i\kappa \int \mathrm{d}s\, \mathrm{d}^n\xi\, \sqrt{|f|}, \tag{4.171}$$

which governs the shape of such a *static* membrane $\mathcal{V}_n$.

In the action (4.168) or (4.169) the dimensions and signatures of the corresponding manifolds V_n and V_N are left unspecified. So action (4.169) contains many possible particular cases. Especially interesting are the following cases:

Case 1. The manifold V_n belonging to an unconstrained membrane $\mathcal{V}_n$ has the signature $(+ - - - ...)$ and corresponds to an *n-dimensional worldsheet* with one time-like and $n-1$ space-like dimensions. The index of the worldsheet coordinates assumes the values $a = 0, 1, 2, ..., n-1$.

Case 2. The manifold V_n belonging to our membrane $\mathcal{V}_n$ has the signature $(- - - - ...)$ and corresponds to a space-like *p-brane*; therefore we take $n = p$. The index of the membrane's coordinates ξ^a assumes the values $a = 1, 2, ..., p$.

Throughout the book we shall often use the single formalism and apply it, when convenient, either to the *Case 1* or to the *Case 2*.

[5] The invariance of action (4.169) under reparametrizations of ξ^a brings no constraints amongst the dynamical variables $X^\mu(\tau, \xi)$ and $p_\mu(\tau, \xi)$ which are related to motion in τ (see also [53]-[55]).

[6] The factor i comes from our inclusion of a *pseudoscalar* in the velocity polyvector. Had we instead included a scalar, the corresponding factor would then be 1.

When the dimension of the manifold V_n belonging to $\mathcal{V}_n$ is $n = p + 1$ and the signature is $(+ - - - ...)$, i.e. when we consider *Case 1*, then the action (4.171) is just that of the usual *Dirac–Nambu–Goto p-dimensional membrane* (well known under the name *p-brane*)

$$I = i\tilde{\kappa} \int \mathrm{d}^n \xi \, \sqrt{|f|} \tag{4.172}$$

with $\tilde{\kappa} = \kappa \int \mathrm{d}s$.

The usual p-brane is considered here as a particular case of a more general membrane[7] which can *move* in the embedding spacetime (target space) according to the action (4.168) or (4.169). Bearing in mind two particular cases described above, our action (4.169) describes either

(i) *a moving worldsheet*, in the *Case I*; or
(ii) *a moving space like membrane*, in the *Case II*.

Let us return to the action (4.166). We can write it in the form

$$I[X^\mu, s, \lambda] = \frac{\kappa}{2} \int \mathrm{d}\tau \, \mathrm{d}^n \xi \left[\sqrt{|f|} \left(\frac{\dot{X}^\mu \dot{X}_\mu}{\lambda} + \lambda \right) - \frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\kappa \sqrt{|f|} \dot{s} s}{\lambda} \right) \right], \tag{4.173}$$

where by the equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\kappa \sqrt{|f|}}{\lambda} \, \dot{s} \right) = 0 \tag{4.174}$$

we have

$$\frac{\mathrm{d}}{\mathrm{d}\tau} \left(\frac{\kappa \sqrt{|f|} \dot{s} s}{\lambda} \right) = \frac{\kappa \sqrt{|f|} \dot{s}^2}{\lambda} \, . \tag{4.175}$$

The term with the total derivative does not contribute to the equations of motion and we may omit it, provided that we fix λ in such a way that the X^μ-equations of motion derived from the reduced action are consistent with those derived from the original constrained action (4.165). This is indeed the case if we choose

$$\lambda = \Lambda \kappa \sqrt{|f|}, \tag{4.176}$$

where Λ is arbitrary fixed function of τ.

Using (4.167) we have

$$\sqrt{\dot{X}^\mu \dot{X}_\mu - \dot{s}^2} = \Lambda \kappa \sqrt{|f|}. \tag{4.177}$$

---

[7]By using the name *membrane* we distinguish our moving extended object from the static object, which is called *p-brane*.

Inserting into (4.177) the relation

$$\frac{\kappa\sqrt{|f|}\dot{s}}{\sqrt{\dot{X}^\mu \dot{X}_\mu - \dot{s}^2}} = \frac{1}{C} = \text{constant}, \tag{4.178}$$

which follows from the equation of motion (4.139), we obtain

$$\frac{\Lambda}{C} = \frac{\mathrm{d}s}{\mathrm{d}\tau} \quad \text{or} \quad \Lambda\,\mathrm{d}\tau = C\,\mathrm{d}s \tag{4.179}$$

where the differential $\mathrm{d}s = (\partial s/\partial\tau)\mathrm{d}\tau + \partial_a s \mathrm{d}\xi^a$ is taken along a curve on the membrane along which $\mathrm{d}\xi^a = 0$ (see also eqs. (4.160), (4.161)). Our choice of parameter τ (given by a choice of λ in eq. (4.176)) is related to the variable s by the simple proportionality relation (4.179).

Omitting the total derivative term in action (4.173) and using the gauge fixing (4.176) we obtain

$$I[X^\mu] = \frac{1}{2}\int \mathrm{d}\tau\,\mathrm{d}^n\xi \left(\frac{\dot{X}^\mu \dot{X}_\mu}{\Lambda} + \Lambda\kappa^2|f|\right). \tag{4.180}$$

This is the *unconstrained* membrane action that was already derived in previous section, eq. (4.154).

Using (4.179) we find that action (4.180) can be written in terms of s as the evolution parameter:

$$I[X^\mu] = \frac{1}{2}\int \mathrm{d}s\,\mathrm{d}^n\xi \left(\frac{\overset{\circ}{X}{}^\mu \overset{\circ}{X}_\mu}{C} + C\kappa^2|f|\right) \tag{4.181}$$

where $\overset{\circ}{X}{}^\mu \equiv \mathrm{d}X^\mu/\mathrm{d}s$.

The equations of motion derived from the *constrained action* (4.166) are

$$\delta X^\mu : \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\kappa\sqrt{|f|}\dot{X}_\mu}{\sqrt{\dot{X}^2 - \dot{s}^2}}\right) + \partial_a\left(\kappa\sqrt{|f|}\sqrt{\dot{X}^2 - \dot{s}^2}\,\partial^a X_\mu\right) = 0, \tag{4.182}$$

$$\delta s : \qquad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\kappa\sqrt{|f|}\dot{s}}{\sqrt{\dot{X}^2 - \dot{s}^2}}\right) = 0, \tag{4.183}$$

whilst those from the *reduced* or *unconstrained action* (4.180) are

$$\delta X^\mu : \quad \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{X}_\mu}{\Lambda}\right) + \partial_a\left(\kappa^2|f|\Lambda\partial^a X_\mu\right) = 0. \tag{4.184}$$

By using the relation (4.177) we verify the equivalence of (4.184) and (4.182).

The original, constrained action (4.168) implies the constraint

$$p^\mu p_\mu - m^2 - \kappa^2 |f| = 0, \tag{4.185}$$

where

$$p_\mu - \kappa \sqrt{|f|} \dot{X}_\mu / \lambda \,, \quad m = \kappa \sqrt{|f|} \dot{s} / \lambda \,, \quad \lambda = \sqrt{\dot{X}^\mu \dot{X}_\mu - \dot{s}^2}.$$

According to the equation of motion (4.174) $\dot{m} = 0$, therefore

$$p^\mu p_\mu - \kappa^2 |f| = m^2 = \text{constant}. \tag{4.186}$$

The same relation (4.186) also holds in the reduced, *unconstrained theory* based on the action (4.180). If, in particular, $m = 0$, then the corresponding solution $X^\mu(\tau, \xi)$ is identical with that for the ordinary Dirac–Nambu–Goto membrane described by the minimal surface action which, in a special parametrization, is

$$I[X^\mu] = \kappa \int \mathrm{d}\tau \mathrm{d}^n \, \xi \sqrt{|f|} \sqrt{\dot{X}^\mu \dot{X}_\mu} \tag{4.187}$$

This is just a special case of (4.168) for $\dot{s} = 0$.

To sum up, *the constrained action* (4.168) has the two limits:

(i) *Limit* $\dot{X}^\mu = 0$. Then

$$I[X^\mu(\xi)] = i \tilde{\kappa} \int \mathrm{d}^n \xi \sqrt{|f|}. \tag{4.188}$$

This is the minimal surface action. Here the n-dimensional membrane (or the worldsheet in the *Case I*) is *static* with respect to the evolution[8] parameter τ.

(ii) *Limit* $\dot{s} = 0$. Then

$$I[X^\mu(\tau, \xi)] = \kappa \int \mathrm{d}\tau \, \mathrm{d}^n \xi \, \sqrt{|f|} \sqrt{\dot{X}^\mu \dot{X}_\mu}. \tag{4.189}$$

This is an action for a *moving* n-dimensional membrane which sweeps an $(n{+}1)$-dimensional surface $X^\mu(\tau, \xi)$ subject to the constraint $p^\mu p_\mu - \kappa^2 |f| = 0$. Since the latter constraint is conserved in τ we have automatically also the constraint $p_\mu \dot{X}^\mu = 0$ (see Box 4.3). Assuming the *Case II* we have thus the motion of a conventional constrained p-brane, with $p = n$.

In general none of the limits (i) or (ii) is satisfied, and our membrane moves according to the action (4.168) which involves the constraint (4.185).

[8] The evolution parameter τ should not be confused with one of the worldsheet parameters ξ^a.

From the point of view of the variables X^μ and the conjugate momenta p_μ there is no constraint, and instead of (4.168) we can use *the unconstrained action* (4.180) or (4.181), where *the extra variable s has become the parameter of evolution* .