GENERAL PRINCIPLES OF BRANE KINEMATICS AND DYNAMICS

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- Introduction
  - Strings, branes, geometric principle, background independence
- Brane space $\mathcal{M}$ (brane kinematics)
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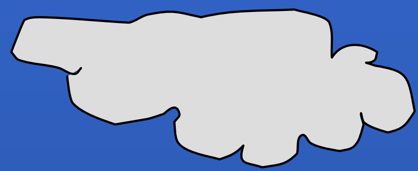
- **Introduction**

  Strings, branes

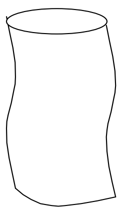
Theories of strings and higher dimensional extended objects, branes
- very promising in explaining the origin and interrelationship of the fundamental interactions,

  including gravity



But there is a cloud:
- what is a geometric principle behind string and brane theories and how to formulate them in a background independent way

$$I[g_{\mu\nu}] = \int \sqrt{-g}\, R \,\mathrm{d}^4 x$$



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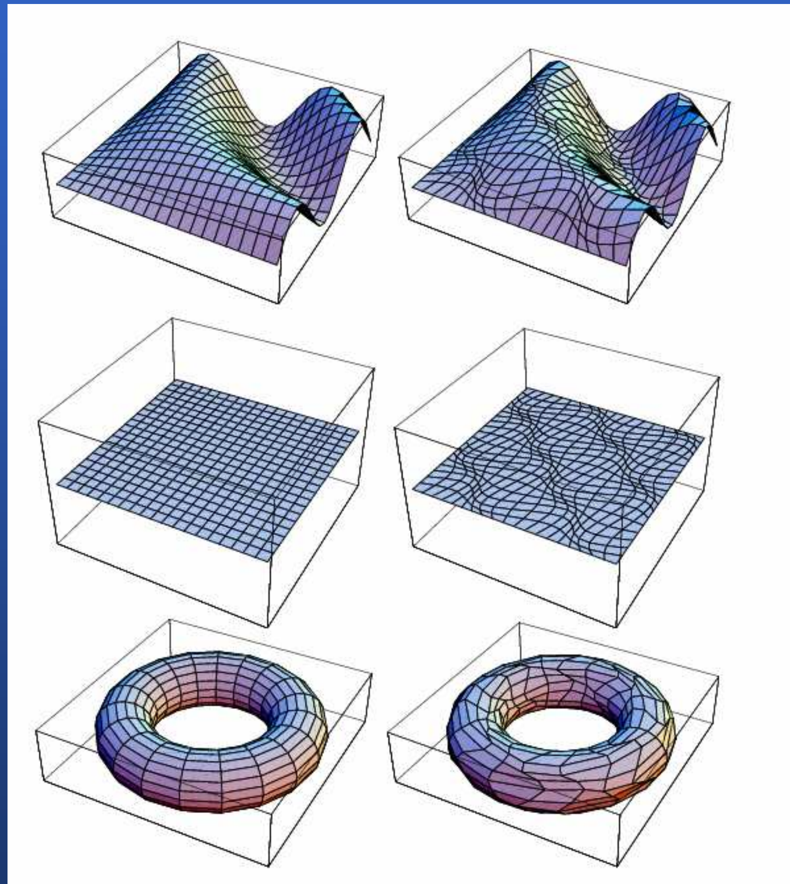
- **Brane space M (brane kinematics)**

The basic kinematically possible objects:

n-dimensional, arbitrarily deformable branes V_n living in V_N

Tangential deformations are also allowed



Mathematically the surfaces on the left and the right are the same.
Physically they are different.

We represent V_n by functions

$$X^\mu(\xi^a) \quad , \qquad \mu = 0,1,\ldots,N-1$$

where ξ^a , $a = 0,1,2,\ldots,n-1$ are parameters on V_n

According to the assumed interpretation, different functions $X^\mu(\xi)$ can represent physically different branes.

The set of all possible V_n forms the brane space M

A brane V_n can be considered as a point in M parametrized by coordinates $X^\mu(\xi^a) \equiv X^{\mu(\xi)}$ which bear a discrete index μ and n continuous indices ξ^a

$\mu(\xi)$ as superscript or subscript denotes pair of indices μ and (ξ)

Distance in M space

$$
\begin{aligned}
\mathrm{d}\ell^2 &= \int \mathrm{d}\xi\, \mathrm{d}\zeta\, \rho_{\mu\nu}(\xi,\zeta)\, \mathrm{d}X^\mu(\xi)\, \mathrm{d}X^\nu(\zeta) \\
&= \rho_{\mu(\xi)\nu(\zeta)}\, \mathrm{d}X^{\mu(\xi)}\, \mathrm{d}X^{\nu(\zeta)} = \mathrm{d}X^{\mu(\xi)} \mathrm{d}X_{\mu(\xi)}
\end{aligned}
$$

$\rho_{\mu(\xi)\nu(\zeta)}$ metric in M

particular choice of metric

$$\rho_{\mu(\xi)\nu(\zeta)} = \sqrt{|f|}\, \alpha\, g_{\mu\nu} \delta(\xi - \zeta)$$

$$f \equiv \det f_{ab} \quad , \qquad f_{ab} \equiv \partial_a X^\alpha \partial_b X^\beta$$

induced metric on the brane V_n

α an arbitrary constant

$g_{\mu\nu}$ metric of the embedding space V_N

$$\mathrm{d}\ell^2 = \int \mathrm{d}\xi \sqrt{|f|}\, \alpha\, g_{\mu\nu}\, \mathrm{d}X^\mu(\xi) \mathrm{d}X^\mu(\xi)$$

Invariant volume (measure) in M :

$$\sqrt{|\rho|}\, \mathcal{D}X = (|\operatorname{Det} \rho_{\mu\nu}(\xi,\varsigma)|)^{1/2} \prod_{\xi,\mu} \mathrm{d}X^\mu(\xi)$$

For the diagonal metric

$$\rho_{\mu(\xi)\nu(\zeta)} = \sqrt{|f|}\, \alpha\, g_{\mu\nu} \delta(\xi - \zeta)$$

$$\sqrt{|\rho|}\, \mathcal{D}X = \prod_{\xi,\mu} (\sqrt{|\mathrm{f}|}\, \alpha |\mathrm{g}|)^{1/2}\, \mathrm{d}X^\mu(\xi)$$

Tensor calculus in M –space: analogous to that in a finite dimensional space

Differential of coordina $\mathrm{d}X^{\mu}(\xi) \equiv \mathrm{d}X^{\mu(\xi)}$ is a vector in M

Under a general coordinate transformation a vector in M transforms according to:

$$A'^{\mu(\xi)} = \frac{\partial X'^{\mu(\xi)}}{\partial X^{\nu(\varsigma)}} A^{\nu(\varsigma)} = \int \mathrm{d}\varsigma \frac{\delta X'^{\mu}(\xi)}{\delta X^{\nu}(\varsigma)} A^{\nu(\varsigma)}$$

Such a shorthand notation for functional derivative is very effective

An arbitrary coordinate transformation in M –space:

$$X'^{\mu(\xi)} = F^{\mu(\xi)}[X]$$

If $X^{\mu(\xi)}$ represent a kinematically possible brane, then $X'^{\mu(\xi)}$ obtained from $X^{\mu(\xi)}$ by a functional transformation represent the same (kinematically possible) brane

Covariant derivative in M

acting on a ***scalar*** :

$$A_{;\mu(\xi)} = \frac{\delta A}{\delta X^{\mu}(\xi)} \equiv A_{,\mu(\xi)}$$

acting on a ***vector*** :

$$A^{\mu(\xi)}{}_{;\nu(\xi')} = A^{\mu(\xi)}{}_{,\nu(\xi')} + \Gamma^{\mu(\xi)}_{\nu(\xi')\sigma(\xi'')} A^{\sigma(\xi'')}$$

Variants of notation:

$$\frac{\delta}{\delta X^{\mu}(\xi)} \equiv \frac{\partial}{\partial X^{\mu(\xi)}} \equiv \partial_{\mu(\xi)}$$

Functional derivative

$$\frac{\mathrm{D}}{\mathrm{D}X^{\mu}(\xi)} \equiv \frac{\mathrm{D}}{\mathrm{D}X^{\mu(\xi)}} \equiv \mathrm{D}_{\mu(\xi)}$$

Covariant derivative In M

- Branedynamics

Let a brane move in the embedding space V_N. The parameter of evolution is τ.
Kinematically, every continuous trajectory $X^\mu(\tau,\xi^a) \equiv X^{\mu(\tau,\xi)}$ is possible.

A particular dynamical theory selects dynamically possible trajectories

Brane theory as free fall in $\mathcal{M}$ -space

Dynamically possible trajectories are geodesics in $\mathcal{M}$

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau' \left(\rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')}\right)^{1/2}$$

$$\mu \equiv \rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')}$$

$$\frac{\delta I}{\delta X^{\mu(\xi)}(\tau)} = -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\rho_{\alpha(\xi')\mu(\xi)} \dot{X}^{\alpha(\xi')}}{\mu^{1/2}}\right) + \frac{1}{2}\int \mathrm{d}\tau' \frac{1}{\mu^{1/2}} \left(\frac{\delta}{\delta X^{\mu(\xi)}(\tau)} \rho_{\alpha(\xi')\beta(\xi'')}\right) \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} = 0$$

If $\rho_{\mu(\xi)\nu(\zeta)}$ does not contain velocity

$$-\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{X}_{\mu(\xi)}\right) + \frac{1}{2}\partial_{\mu}(\xi)\rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} = 0$$

Using contravariant instead of covariant variables

$$\frac{\mathrm{d}\dot{X}^{\mu(\xi)}}{\mathrm{d}\tau} + \Gamma^{\mu(\xi)}{}_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} = 0$$

Geodesic in $\mathcal{M}$ -space

$$\frac{\delta I}{\delta X^{\mu(\xi)}(\tau)} = -\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\rho_{\alpha(\xi')\mu(\xi)}\dot{X}^{\alpha(\xi')}}{\mu^{1/2}}\right) + \frac{1}{2}\int \mathrm{d}\tau' \frac{1}{\mu^{1/2}}\left(\frac{\delta}{\delta X^{\mu(\xi)}(\tau)}\rho_{\alpha(\xi')\beta(\xi'')}\right)\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')} = 0$$

$$\rho_{\alpha(\xi')\beta(\xi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}}\delta(\xi' - \xi'')\eta_{\alpha\beta}$$

choice of metric

$$\dot{X}^2 \equiv g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu$$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{1}{\mu^{1/2}}\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\dot{X}_\mu\right) + \frac{1}{\mu^{1/2}}\partial_a\left(\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0$$

$$\mu \equiv \rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi')}$$

$\Rightarrow$ $\mathrm{d}\mu/\mathrm{d}\tau = 0$

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\dot{X}_\mu\right) + \partial_a\left(\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0$$

Equations of motion for the
Dirac-Nambu-Goto brane
(in particular gauge)

The action

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau' \left(\rho_{\alpha(\xi')\beta(\xi'')}\dot{X}^{\alpha(\xi')}\dot{X}^{\beta(\xi'')}\right)^{1/2}$$

is by definition invariant under reparametrizations of ξ^a.

In general, it is not invariant under reparametrizations of τ.

This is so when the metric contains velocity.

If metric is given by

$$\rho_{\alpha(\xi')\beta(\xi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}}\,\delta(\xi'-\xi'')\eta_{\alpha\beta}$$

then the action becomes explicitly

$$I[X^{\mu}(\xi)] = \int \mathrm{d}\tau \left(\mathrm{d}\xi\,\kappa\sqrt{|f|}\sqrt{\dot{X}^2}\right)^{1/2}$$

The equations of motion automatically contain the relation

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\dot{X}^{\mu(\xi)}\dot{X}_{\mu(\xi)}\right) \equiv \frac{\mathrm{d}}{\mathrm{d}\tau}\int \mathrm{d}\xi\,\kappa\sqrt{|f|}\sqrt{\dot{X}^2} = 0$$

which is a gauge fixing relation.

In general, the exponent in the Lagrangian is not necessarily ½ , but can be arbitrary:

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau \, \left(\rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')}\right)^a$$

or explicitly:

$$I[X^\mu(\xi)] = \int \mathrm{d}\tau \left(\mathrm{d}\xi \, \kappa \sqrt{|f|} \sqrt{\dot{X}^2}\right)^a$$

Not invariant under reparametrizations of τ, unless a = 1

For our particular metric the corresponding equations of motion are:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(a\mu^{a-1}\frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right) + a\mu^{a-1}\partial_a\left(\kappa\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0$$

For $a \neq 1$

$$\frac{\mathrm{d}\mu}{\mathrm{d}\tau} = 0$$

Gauge fixing relation

For a = 1

No gauge fixing relation

The same equation of motion

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\sqrt{|f|}}{\sqrt{\dot{X}^2}}\,\dot{X}_\mu\right) + \partial_a\left(\sqrt{|f|}\sqrt{\dot{X}^2}\partial^a X_\mu\right) = 0$$

Let us focus our attention to the action

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau\, \rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi')} = \int \mathrm{d}\tau\, \mathrm{d}\xi\, \kappa \sqrt{|f|} \sqrt{\dot{X}^2}$$

Case a = 1

It is invariant under the transformations

$$\tau \to \tau' = \tau'(\tau)$$
$$\xi'^a \to \xi'^a = \xi'^a(\xi)$$

in which τ and ξ^a do not mix.

Invariance of the action under reparametrizations τ of implies a ***constraint*** among the canonical momenta.
Momenta are given by

$$\begin{aligned} p_{\mu(\xi)} &= \frac{\partial L}{\partial \dot{X}^{\mu(\xi)}} = 2\rho_{\mu(\xi)\nu(\xi')} \dot{X}^{\nu(\xi')} + \frac{\partial \rho_{\alpha(\xi')\beta(\xi'')}}{\partial \dot{X}^{\mu(\xi)}} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi'')} \\ &= \frac{\kappa\sqrt{|f|}}{\sqrt{\dot{X}^2}} \dot{X}_\mu . \end{aligned}$$

By distinguishing covariant and contravariant components one finds

$$p_{\mu(\xi)} = \dot{X}_{\mu(\xi)} = \rho_{\mu(\xi)\nu(\xi')} \dot{X}^{\nu(\xi')} , \qquad p^{\mu(\xi)} = \dot{X}^{\mu(\xi)}$$

We define

$$p_{\mu(\xi)} \equiv p_\mu(\xi) \equiv p_\mu , \qquad \dot{X}^{\mu(\xi)} \equiv \dot{X}^\mu$$

Hamiltonian

$$H = p_{\mu(\xi)} \dot{X}^{\mu(\xi)} - L = \tfrac{1}{2} \int \mathrm{d}\xi \, \frac{\sqrt{\dot{X}^2}}{\kappa \sqrt{|f|}} (p^\mu p_\mu - \kappa^2 |f|)$$

$$= p_{\mu(\xi)} \, p^{\mu(\xi)} - K = 0$$ Constraint

where

$$K = K[X^{\mu(\xi)}] = \int \mathrm{d}\xi \, \kappa \sqrt{|f|} \sqrt{\dot{X}^2}$$

A reparametrization of τ changes $\dot{X}^2$

Therefore $\dot{X}^2$ under the integral for H is arbitrary.
Consequently, H vanishes when the following expression under the integral vanishes:

$$(p^\mu p_\mu - \kappa^2 |f|) = 0$$ This is satisfied at every ξ^a. **"Hamilton constraint"**

This is the usual constraint for the Nambu-Goto brane (*p*-brane).

The quantity under the integral in the expression for Hamiltonian $H = \int \mathrm{d}\xi \sqrt{\dot{X}^2} \, \mathcal{H}$
is **Hamiltonian density** $\mathcal{H}$.

From the requirement that the constraint is conserved in τ we have:

$$p_\mu \partial_a X^\mu = 0$$ **"Momentum constraint"**

Both kinds of p-brane constraints are thus automatically implied by the action

$$I[X^{\alpha(\xi)}] = \int \mathrm{d}\tau \, \rho_{\alpha(\xi')\beta(\xi'')} \dot{X}^{\alpha(\xi')} \dot{X}^{\beta(\xi')} = \int \mathrm{d}\tau \, \mathrm{d}\xi \, \kappa \sqrt{|f|} \sqrt{\dot{X}^2}$$

in which the following choice of M-space metric tensor has been taken:

$$\rho_{\alpha(\xi')\beta(\xi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}} \delta(\xi' - \xi'') \eta_{\alpha\beta}$$

Introducing $\phi^A = (\tau, \xi^a)$ and $X^{\mu(\xi)}(\tau) \equiv X^\mu(\phi^A) \equiv X^{\mu(\phi)}$

we can write

$$I[X^{\mu(\phi)}] = \rho_{\mu(\phi)\nu(\phi')} \dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi')} = \int \mathrm{d}^{n+1}\phi \sqrt{|f|} \sqrt{\dot{X}^2}$$

where

$$\rho_{\mu(\phi')\nu(\phi'')} = \kappa \frac{\sqrt{|f(\xi')|}}{\sqrt{\dot{X}^2(\xi')}} \delta(\xi' - \xi'') \delta(\tau' - \tau'') \eta_{ab}$$

Variation of the above action with respect to X gives the geodesic equation in M-space:

$$\frac{\mathrm{d}\dot{X}^{\mu(\phi)}}{\mathrm{d}\tau} + \Gamma^{\mu(\phi)}_{\alpha(\phi')\beta(\phi'')} \dot{X}^{\alpha(\phi')} \dot{X}^{\beta(\phi'')} = 0$$

Having the constraints one can write the first order, or phase space action

$$I[X^\mu, p_\mu, \lambda, \lambda^a] = \int \mathrm{d}\tau \, \mathrm{d}\xi \left(p_\mu \dot{X}^\mu - \frac{\lambda}{2\kappa\sqrt{|f|}}(p^\mu p_\mu - \kappa^2 |f|) - \lambda^a p_\mu \partial_a X^\mu \right)$$

It is classically equivalent to the *minimal surface action* for the *(n+1)-dimensional world manifold* V_{n+1}

$$I[X^\mu] = \kappa \int \mathrm{d}^{n+1}\phi \,(\det \partial_A X^\mu \partial_B X_\mu)^{1/2}$$

This is the conventional Dirac-Nambu-Goto action, invariant under reparametrizations of ϕ^A,

• Dynamical metric field in M-space

Let us now ascribe the dynamical role to the M-space metric.
M-space perspective: motion of a point "particle" in the presence of the metric field $\rho_{\mu(\phi)\nu(\phi')}$ which is itself dynamical.

As a model let us consider

$$I[\rho] = \int \mathcal{D}X \sqrt{|\rho|} \left(\rho_{\mu(\phi)\nu(\phi')} \dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi')} + \frac{\varepsilon}{16\pi} \mathcal{R} \right)$$

$\mathcal{R}$ Ricci scalar in $\mathcal{M}$

variation with respect to $X^{\mu(\phi)}$ and $\rho_{\mu(\phi)\nu(\phi')}$

$$\frac{\mathrm{D}\dot{X}^{\mu(\phi)}}{\mathrm{D}\tau} \equiv \frac{\mathrm{d}\dot{X}^{\mu(\phi)}}{\mathrm{d}\tau} + \Gamma^{\mu(\phi)}_{\alpha(\phi')\beta(\phi'')} \dot{X}^{\alpha(\phi')} \dot{X}^{\beta(\phi'')} = 0$$

geodesic equation in $\mathcal{M}$

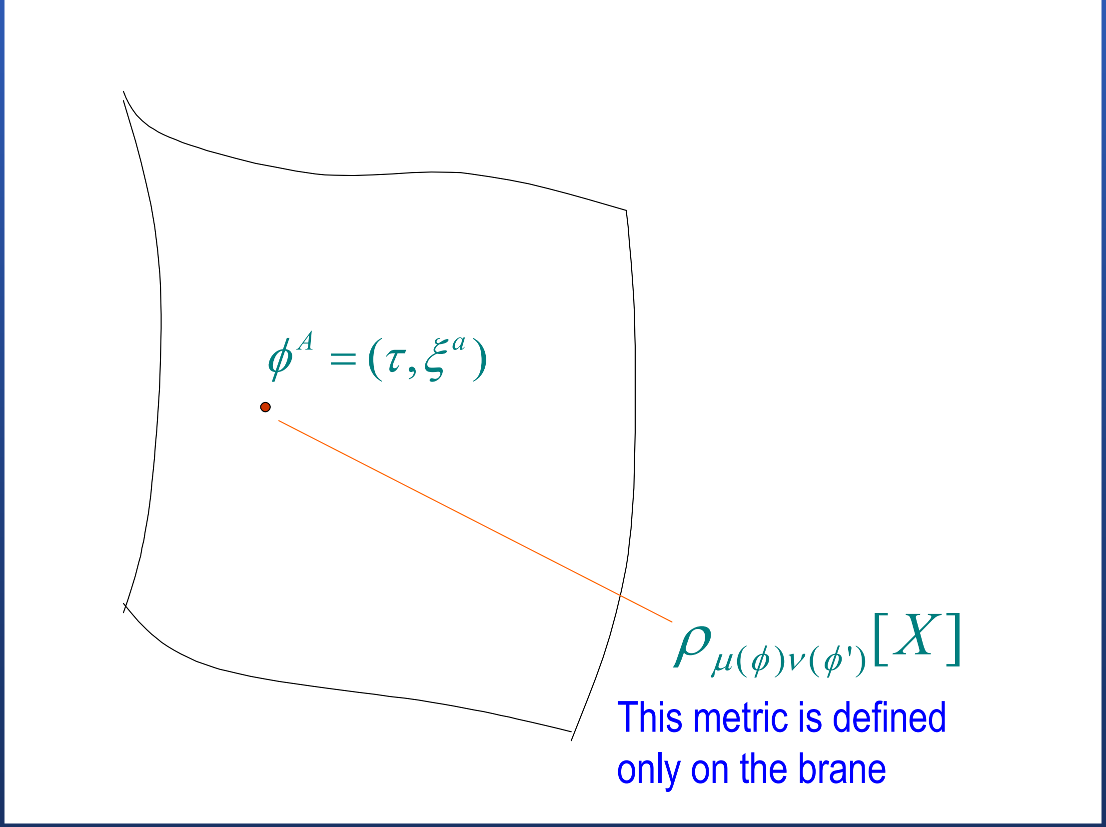
$$\dot{X}^{\mu(\phi)} \dot{X}^{\nu(\phi')} + \frac{\varepsilon}{16\pi} \mathcal{R}^{\mu(\phi)\nu(\phi')} = 0$$

Einstein's equations in $\mathcal{M}$

The metric $\rho_{\mu(\phi)\nu(\phi')}$ is a functional of the variables $X^{\mu(\phi)}$ and on the previous slide we had a system of functional differential equations which determine the set of possible solutions for $X^{\mu(\phi)}$ and $\rho_{\mu(\phi)\nu(\phi')}$.

Our brane model (including strings) is background independent:
There is no pre-existing space with a pre-existing metric, neither curved nor flat.

A model universe: a single brane

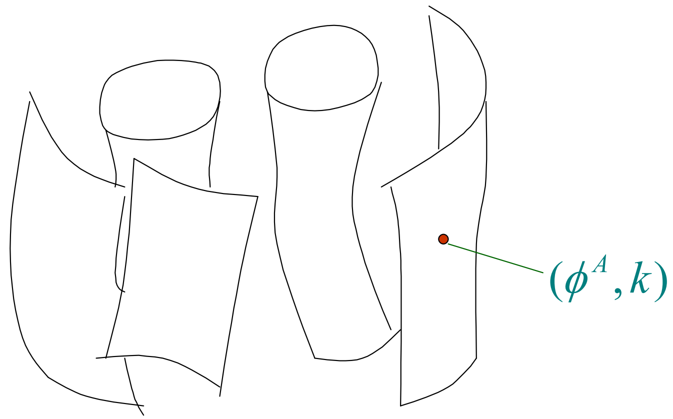


There is no metric of a space into which the brane is embedded.

Actually, there is no embedding space in this model.

All what exists is a brane configuration $X^{\mu(\phi)}$ and the corresponding metric $\rho_{\mu(\phi)\nu(\phi')}$ in M-space.

A system branes (brane configuration)



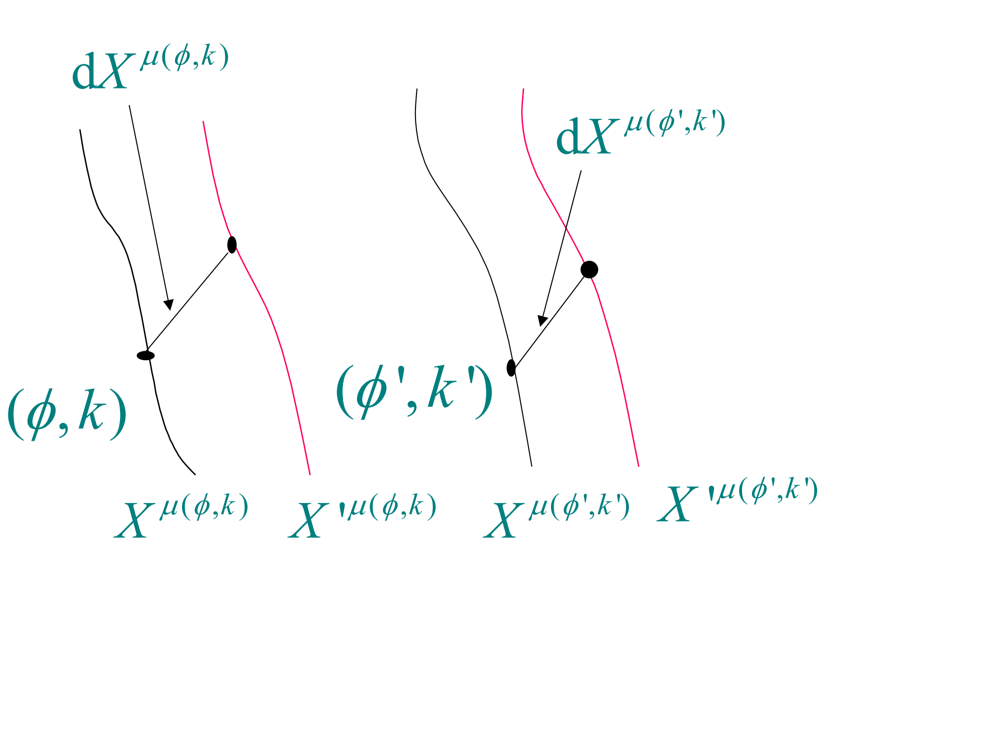
Metric is defined only at the points situated on the branes

In the limit of infinitely many densely packed branes, the set of points (ϕ^A, k) is supposed to become a continuous, finite dimensional metric space V_N.

If we replace (ϕ) with (ϕ, k), or, alternatively, if we interpret (ϕ) to include the index k, then the previous equations are also valid for a system of branes.

A brane configuration is all what exists in such a model.
It is identified with the embedding space.

From $\mathcal{M}$-space to spacetime



$$\mathrm{d}\ell^2 = \rho_{\mu(\phi,k)\nu(\phi',k')}\,\mathrm{d}X^{\mu(\phi,k)}\,\mathrm{d}X^{\nu(\phi',k')}$$

The metric ρ determines the *distance* between the points belonging to two different brane configurations

Brane configuration is a skeleton *S* of a target space V_N

Let us now introduce

$$\tilde{\Delta}X^{\mu}(\phi,k) \equiv X^{\mu(\phi',k')} - X^{\mu(\phi,k)}$$

and define

$$\Delta s^2 = \rho_{\mu(\phi,k)\nu(\phi',k')}\,\tilde{\Delta}X^{\mu}(\phi,k)\,\tilde{\Delta}X^{\nu}(\phi,k)$$

Distance between the points within a given brane configuration

This is the quadratic form in the skeleton space *S*

The metric ρ in the skeleton space *S* is the prototype for the metric in V_N

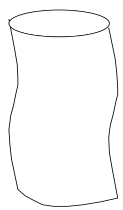
• Conclusion

We have taken the brane space *M* seriously as an arena for physics.

The arena itself is also a part of the dynamical system, it is not prescribed in advance.

The theory is thus background independent. It is based on the geometric principle which has its roots in the brane space *M*

$$I[g_{\mu\nu}] = \int \mathrm{d}^4 x \sqrt{|g|}\, R$$



$$I[\rho_{\mu(\phi)\nu(\phi')}] = \int \mathcal{D}X \sqrt{|\rho|}\, \mathcal{R}$$

We have formulated a theory in which an embedding space *per se* does not exist, but is intimately connected to the existence of branes (including strings).

Without branes there is no embedding space.

There is no pre-existing space and metric: they appear dynamically as solutions to the equations of motion.

All this was just an introduction. Much more can be found in a book

M. Pavsic: The Landscape of Theoretical Physics: A Global view;
From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle
(Kluwer Academic, 2001)

where the description with a metric tensor has been surpassed.

Very promising is the description in terms of the Clifford algebra equivalent of the tetrad field which simplifies calculations significantly.

Possible connections to other topics:

- How to identify spacetime points (famous Einstein's "hole argument")
- DeWitt-Rovelli reference fluid (with respect to which the points of the target space are defined)
- Mach principle

The system, or condensate of branes represents a reference system or reference fluid with respect to which the points of the target space are defined.

Motion of matter at a given location is determined by all the matter In the universe.

Such a situation is implemented in the model of a universe consisting of a system of branes: the motion of a *k*-the brane, including its inertia (metric), is determined by the presence of all the other branes.

$$I[X^\mu, p_\mu, \lambda, \lambda^a] = \int \mathrm{d}\tau\, \mathrm{d}\xi \left(p_\mu \dot{X}^\mu - \frac{\lambda}{2\kappa\sqrt{|f|}}(p^\mu p_\mu - \kappa^2 |f|) - \lambda^a p_\mu \partial_a X^\mu \right), \tag{4.70}$$

where λ and λ^a are Lagrange multipliers.

The equations of motion are

$$\delta X^\mu \quad : \quad \dot{p}_\mu + \partial_a \left(\kappa\lambda\sqrt{|f|}\partial^a X_\mu - \lambda^a p_\mu \right) = 0, \tag{4.71}$$

$$\delta p_\mu \quad : \quad \dot{X}^\mu - \frac{\lambda}{\kappa\sqrt{|f|}} p_\mu - \lambda^a \partial_a X^\mu = 0, \tag{4.72}$$

$$\delta\lambda \quad : \quad p^\mu p_\mu - \kappa^2 |f| = 0, \tag{4.73}$$

$$\delta\lambda^a \quad : \quad p_\mu \partial_a X^\mu = 0. \tag{4.74}$$

Eqs. (4.72)–(4.74) can be cast into the following form:

$$p_\mu = \frac{\kappa\sqrt{|f|}}{\lambda}(\dot{X}_\mu - \lambda^a \partial_a X^\mu), \tag{4.75}$$

$$\lambda^2 = (\dot{X}^\mu - \lambda^a \partial_a X^\mu)(\dot{X}_\mu - \lambda^b \partial_b X_\mu) \tag{4.76}$$

$$\lambda_a = \dot{X}^\mu \partial_a X_\mu. \tag{4.77}$$

Inserting the last three equations into the phase space action (4.70) we have

$$I[X^\mu] = \kappa \int \mathrm{d}\tau \, \mathrm{d}\xi \sqrt{|f|} \left[\dot{X}^\mu \dot{X}^\nu (\eta_{\mu\nu} - \partial^a X_\mu \partial_a X_\nu) \right]^{1/2} . \tag{4.78}$$

The vector $\dot{X}(\eta_{\mu\nu} - \partial^a X_\mu \partial_a X_\nu)$ is normal to the membrane V_n; its scalar product with tangent vectors $\partial_a X^\mu$ is identically zero. The form $\dot{X}^\mu \dot{X}^\nu (\eta_{\mu\nu} - \partial^a X_\mu \partial_a X_\nu)$ can be considered as a 1-dimensional metric, equal to its determinant, on a line which is orthogonal to V_n. The product

$$f \dot{X}^\mu \dot{X}^\nu (\eta_{\mu\nu} - \partial^a X_\mu \partial_a X_\nu) = \det \partial_A X^\mu \partial_B X_\mu \tag{4.79}$$

is equal to the determinant of the induced metric $\partial_A X^\mu \partial_B X_\mu$ on the $(n+1)$-dimensional surface $X^\mu(\phi^A)$, $\phi^A = (\tau, \xi^a)$, swept by our membrane V_n. The action (4.78) is then *the minimal surface action* for the $(n+1)$-dimensional worldsheet V_{n+1}:

$$I[X^\mu] = \kappa \int \mathrm{d}^{n+1}\phi \, (\det \partial_A X^\mu \partial_B X_\mu)^{1/2} . \tag{4.80}$$