THE EXTENDED RELATIVITY THEORY IN CLIFFORD SPACES

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Abstract

A brief review of some of the most important features of the Extended Relativity theory in Clifford-spaces (C-spaces) is presented whose "point" coordinates are non-commuting Clifford-valued quantities which incorporate lines, areas, volumes, hyper-volumes.... degrees of freedom associated with the collective particle, string, membrane p-brane,... dynamics of p-loops (closed p-branes) in target D-dimensional spacetime backgrounds. C-space Relativity naturally incorporates the ideas of an invariant length (Planck scale), maximal acceleration, non-commuting coordinates, supersymmetry, holography, higher derivative gravity with torsion and variable dimensions/signatures. It permits to study the dynamics of all (closed) p-branes, for all values of p, on a unified footing. It resolves the ordering ambiguities in QFT, the problem of time in Cosmology and admits superluminal propagation (tachyons) without violations of causality. A discussion of the maximal-acceleration Relativity principle in phase-spaces follows and the study of the invariance group of symmetry transformations in phase-space allows to show why Planck areas are *invariant* under acceleration-boosts transformations . This invariance feature suggests that a maximal-string tension principle may be operating in Nature. We continue by pointing out how the relativity of signatures of the underlying n-dimensional spacetime results from taking different n-dimensional slices through C-space. The conformal group in spacetime emerges as a natural subgroup of the Clifford group and Relativity in C-spaces involves natural *scale* changes in the sizes of physical objects without the introduction of forces nor Weyl's gauge field of dilations. We finalize by constructing the generalization of Maxwell theory of Electrodynamics of point charges to a theory in C-spaces that involves extended charges coupled to antisymmetric tensor fields of arbitrary rank. In the concluding remarks we outline briefly the current promising research programs and their plausible connections with C-space Relativity.

1 Introduction

In recent years it was argued that the underlying fundamental physical principle behind string theory, not unlike the principle of equivalence and general covariance in Einstein's general relativity, might well be related to the existence of an invariant minimal length scale (Planck scale) attainable in nature [8]. A theory involving spacetime *resolutions* was developed long ago by Nottale [23] where the Planck scale was postulated as the minimum observer independent invariant resolution [23] in Nature. Since "points" cannot be observed physically with an ultimate resolution, it is reasonable to postulate that they are smeared out into fuzzy balls. In refs.[8] it was assumed that those balls have the Planck radius and arbitrary dimension. For this reason it was argued in refs.[8] that one should construct a theory which includes all dimensions (and signatures) on the equal footing. In [8] this Extended Scale Relativity principle was applied to the quantum mechanics of p-branes which led to the construction of Clifford-space (C-space) where all p-branes were taken to be on the same footing, in the sense that the transformations in C-space reshuffled a string history for a five-brane history, a membrane history for a string history, for example.

Clifford algebras contained the appropriate algebraic-geometric features to implement this principle of polydimensional transformations [14]–[17]. In [14]–[16] it was proposed that every physical quantity is in fact a *polyvector*, that is, a Clifford number or a Clifford aggregate. Also, spinors are the members of left or right minimal ideals of Clifford algebra, which may provide the framework for a deeper understanding of sypersymmetries, i.e., the transformations relating bosons and fermions. The Fock-Stueckelberg theory of a relativistic particle can be embedded in the Clifford algebra of spacetime [15, 16]. Many important aspects of Clifford algebra are described in [1],[6], [7], [3], [15, 16, 17], [5], [48]. It is our belief that this may lead to the proper formulation of string and M theory.

A geometric approach to the physics of the Standard Model in terms of Clifford algebras was advanced by [4]. It was realized in [43] that the $Cl(8)$ Clifford algebra contains the 4 fundamental nontrivial representations of $Spin(8)$ that accomodate the chiral fermions and gauge bosons of the Standard model and which also includes gravitons via the McDowell-Mansouri-Chamseddine-West formulation of gravity, which permits to construct locally, in $D = 8$, a geometric Lagrangian for the Standard Model plus Gravity. Furthermore, discrete Clifford-algebraic methods based on hyperdiamond-lattices have been instrumental in constructing E_8 lattices and deriving the values of the force-strengths (coupling constants) and masses of the Standard model with remarkable precision by [43]. These results have recently been corroborated by [46] for Electromagnetism, and by [47], where all the Standard model parameters were obtained from first principles, despite the contrary orthodox belief that it is senseless to "derive" the values of the fundamental constants in Nature from first principles, from pure thought alone; i.e. one must invoke the Cosmological anthropic principle to explain why the constants of Nature have they values they have.

Using these methods the bosonic p-brane propagator, in the quenched mini superspace approximation, was constructed in [18, 19]; the logarithmic corrections to the black hole entropy based on the geometry of Clifford space (in short C-space) were obtained in [21];

The modified nonlinear de Broglie dispersion relations, the corresponding minimal-length stringy [11] and p-brane uncertainty relations also admitted a C-space interpretation [10], [19]. A generalization of Maxwell theory of electromagnetism in C-spaces comprised of extended charges coupled to antisymmetric tensor fields of arbitrary rank was attained recently in [75]. The resolution of the ordering ambiguities of QFT in curved spaces was resolved by using polyvectors, or Clifford-algebra valued objects [26]. One of the most remarkable features of the Extended Relativity in C-spaces is that a higher derivative Gravity with Torsion in ordinary spacetime follows naturally from the analog of the Einstein-Hlbert action in *curved* C-space [20].

In this new physical theory the arena for physics is no longer the ordinary spacetime, but a more general manifold of Clifford algebra valued objects, noncommuting polyvectors. Such a manifold has been called a pan-dimensional continuum [14] or C-space [8]. The latter describes on a unified basis the objects of various dimensionality: not only points, but also closed lines, surfaces, volumes,.., called 0-loops (points), 1-loops (closed strings) 2-loops (closed membranes), 3-loops, etc.. It is a sort of a *dimension* category, where the role of functorial maps is played by C-space transformations which reshuffles a p-brane history for a p'-brane history or a mixture of all of them, for example. The above geometric objects may be considered as to corresponding to the well-known physical objects, namely closed p-branes. Technically those transformations in C-space that reshuffle objects of different dimensions are generalizations of the ordinary Lorentz transformations to C-space.

C-space Relativity involves a generalization of Lorentz invariance (and not a deformation of such symmetry) involving superpositions of p-branes (p-loops) of all possible dimensions. The Planck scale is introduced as a natural parameter that allows us to bridge extended objects of different dimensionalities. Like the speed of light was need in Einstein Relativity to fuse space and time together in the Minkwoski spacetime interval. Another important point is that the Conformal Group of four-dimensional spacetime is a consequence of the Clifford algebra in *four-dimensions* [25] and it emphasizes thefact why the natural dilations/contractions of objects in C-space is *not* the same physical phenomenon than what occurs in Weyl's geometry which requires introducing, by hand, a gauge field of dilations. Objects move dilationally, in the absence of forces, for a different physical reasoning than in Weyl's geometry: they move dilationally because of inertia. This was discussed long ago in refs.[27, 28].

This review is organized as follows: Section **2** is dedicated to extending ordinary Special Relativity theory, from Minkowski spacetime to C-spaces, where the introduction of the invariant Planck scale is required to bridge objects, p-branes, of different dimensionality.

The generalized dynamics of particles, fields and branes in C-space is studied in section **3** . This formalism allows us to construct for the first time, to our knowledge, a *unified* action which comprises the dynamics of *all* p-branes in C-spaces, for all values of p, in one single footing (see also [15]). In particular, the polyparticle dynamics in C-space, when reduced to 4-dimensional spacetime leads to the Stuckelberg formalism and the solution to the problem of time in Cosmology [15].

In section **4** we begin by discussing the geometric Clifford calculus that allows us to reproduce all the standard results in differential and projective geometry [41]. The resolution of the ordering ambiguities of QFT in curved spaces follows next when we review how it can be resolved by using polyvectors, or Clifford-algebra valued objects [26]. Afterwards we construct the Generalized Gravitational Theories in Curved C-spaces, in particular it is shown how Higher derivative Gravity with Torsion in ordinary spacetime follows naturaly from the Geometry of C-space [20].

In section **5** we discuss the Quantization program in C-spaces, and write the C-space Klein-Gordon and Dirac equations [15]. The coresponding bosonic/fermionic p-brane loop-wave equations were studied by [12], [13] without employing Clifford algebra and the concept of C-space.

In section **6** we review the Maximal-Acceleration Relativity in Phase-Spaces [127], starting with the construction of the submaximally-accelerated particle action of [53] using Clifford algebras in phase-spaces; the $U(1,3)$ invariance transformations [74] associated with an 8-dimensional phase space, and show why the minimal Planck-Scale areas are invariant under pure acceleration boosts which suggests that there could be a principle of maximal-tension (maximal acceleration) operating in string theory [68].

In section **7** we discuss the important point that the notion of spacetime signature is relative to a chosen n-dimensional subspace of 2^n-dimensional Clifford space. Different subspaces V_n—different sections through C-space—have in general different signature [15] We show afterwards how the Conformal agebra of spacetime emerges from the Clifford algebra [25] and emphasize the physical *differences* between our model and the one based on Weyl geometry. At the end we show how Clifford algebraic methods permits one to generalize Maxwell theory of Electrodynamics (asociated with ordinary point-charges) to a generalized Maxwell theory in Clifford spaces involving *extended* charges and p-forms of arbitrary rank [75]

In the concluding remarks, we briefly discuss the possible avenues of future research in the construction of QFT in C-spaces, Quantum Gravity, Noncommutative Geometry, and other lines of current promising research in the literature.

2 Extending Relativity from Minkowski Spacetime to C-space

We embark into the construction of the extended relativity theory in C-spaces by a natural generalization of the notion of a spacetime interval in Minkwoski space to C-space [8, 14, 16, 15, 17]:

$$dX^2 = d\sigma^2 + dx_\mu dx^\mu + dx_{\mu\nu} dx^{\mu\nu} + ... \tag{1}$$

where $\mu_1 < \mu_2 < ...$ The Clifford valued polyvector:[1]

$$X = X^M E_M = \sigma \underline{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... x^{\mu_1 \mu_2 .... \mu_D} \gamma_{\mu_1} \wedge \gamma_{\mu_2} .... \wedge \gamma_{\mu_D}. \tag{2}$$

denotes the position of a point in a manifold, called Clifford space or C-space. The series of terms in (2) terminates at a *finite* grade depending on the dimension D. A Clifford

[1] If we do not restrict indices according to $\mu_1 < \mu_2 < \mu_3 < ...$, then the factors $1/2!$, $1/3!$, respectively, have to be included in front of every term in the expansion (1).

algebra $Cl(r,q)$ with $r+q=D$ has 2^D basis elements. For simplicity, the gammas γ^μ correspond to a Clifford algebra associated with a flat spacetime:

$$\frac{1}{2}\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu}. \tag{3}$$

but in general one could extend this formulation to curved spacetimes with metric $g^{\mu\nu}$ (see section 4).

The connection to strings and p-branes can be seen as follows. In the case of a closed string (a 1-loop) embedded in a target flat spacetime background of D-dimensions, one represents the projections of the closed string (1-loop) onto the embedding spacetime coordinate-planes by the variables $x^{\mu\nu}$. These variables represent the respective *areas* enclosed by the projections of the closed string (1-loop) onto the corresponding embedding spacetime planes. Similary, one can embed a closed membrane (a 2-loop) onto a D-dim flat spacetime, where the projections given by the antisymmetric variables $x^{\mu\nu\rho}$ represent the corresponding *volumes* enclosed by the projections of the 2-loop along the hyperplanes of the flat target spacetime background.

This procedure can be carried to all closed p-branes (p-loops) where the values of p are $p = 0,1,2,3,....$ The $p=0$ value represents the center of mass and the coordinates $x^{\mu\nu}, x^{\mu\nu\rho}$... have been *coined* in the string-brane literature [24]. as the *holographic* areas, volumes,...projections of the nested family of p-loops (closed p-branes) onto the embedding spacetime coordinate planes/hyperplanes. In ref.[17] they were interpreted as the generalized centre of mass coordinates of an extended object. Extended objects were thus modeled in C-space.

The scalar coordinate σ entering a polyvector X is a measure associated with the p-brane's world manifold V_{p+1} (e.g., the string's 2-dimensional worldsheet V_2): it is proportional to the $(p+1)$-dimensional area/volume of V_{p+1}. In other words, σ is proportional to the areal-time parameter of the Eguchi-Schild formulation of string dynamics [126, 37, 24].

We see in this generalized scheme the objects as observed in spacetime (which is a section through C-space) need not be infinitely extended along time-like directions. They need not be infinitely long world lines, world tubes. They can be finite world lines, world tubes. The σ coordinate measures how long are world lines, world tubes. During evolution they can becomes longer and longer or shorter and shorter.

If we take the differential dX of X and compute the scalar product among two polyvectors $< dX^\dagger dX >_0 \equiv \mathrm{d}X^\dagger * \mathrm{d}X \equiv |\mathrm{d}X|^2$ we obtain the C-space extension of the particles proper time in Minkwoski space. The symbol $X^\dagger$ denotes the *reversion* operation and involves reversing the order of all the basis γ^μ elements in the expansion of X. It is the analog of the transpose (Hermitian) conjugation. The C-space proper time associated with a polyparticle motion is then the expression (1) which can be written more explicitly as:

$$\begin{aligned} |\mathrm{d}X|^2 &= G_{MN}\, \mathrm{d}X^M \mathrm{d}X^N = \mathrm{d}S^2 \\ &= \mathrm{d}\sigma^2 + L^{-2} dx_\mu dx^\mu + L^{-4} dx_{\mu\nu} dx^{\mu\nu} + ... + L^{-2D} \mathrm{d}x_{\mu_1 ... \mu_D}\, \mathrm{d}x^{\mu_1 ... \mu_D} \end{aligned} \tag{4}$$

where $G_{MN} = E_M^\dagger * E_N$ is the C-space metric.

Here we have introduced the Planck scale L since a length parameter is needed in order to tie objects of different dimensionality together: 0-loops, 1-loops,..., p-loops. Einstein introduced the speed of light as a universal absolute invariant in order to "unite" space with time (to match units) in the Minkwoski space interval:

$$ds^2 = c^2 dt^2 + dx_i dx^i.$$

A similar unification is needed here to "unite" objects of different dimensions, such as x^μ, $x^{\mu\nu}$, etc... The Planck scale then emerges as another universal invariant in constructing an extended relativity theory in C-spaces [8].

Since the D-dimensional Planck scale is given explicitly in terms of the Newton constant: $L_D = (G_N)^{1/(D-2)}$, in natural units of $\hbar = c = 1$, one can see that when $D = \infty$ the value of L_D is then $L_\infty = G^0 = 1$ (assuming a finite value of G). Hence in $D = \infty$ the Planck scale has the natural value of unity. However, if one wishes to avoid any serious algebraic divergence problems in the series of terms appearing in the expansion of the analog of proper time in C-spaces, in the extreme case when $D = \infty$, from now on we shall focus solely on a *finite* value of D. In this fashion we avoid any serious algebraic convergence problems. We shall not be concerned in this work with the representations of Clifford algebras in different dimensions and with different signatures.

The line element $\mathrm{d}S$ as defined in (4) is *dimensionless*. Alternatively, one can define [8, 9] the line element whose dimension is that of the D-volume so that:

$$\mathrm{d}\Sigma^2 = L^{2D}\mathrm{d}\sigma^2 + L^{2D-2}\mathrm{d}x_\mu \mathrm{d}^\mu + L^{2D-4}\mathrm{d}x_{\mu\nu}\mathrm{d}x^{\mu\nu} + ... + \mathrm{d}x_{\mu_1...\mu_D}\mathrm{d}x^{\mu_1...\mu_D} \tag{5}$$

Let us use the relation

$$\gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_D} = \gamma\epsilon_{\mu_1...\mu_D} \tag{6}$$

and write the *volume element* as

$$\mathrm{d}x^{\mu_1...\mu_D}\gamma_{\mu_1} \wedge ... \wedge \gamma_{\mu_D} \equiv \gamma \mathrm{d}\tilde{\sigma} \tag{7}$$

where

$$\mathrm{d}\tilde{\sigma} \equiv \mathrm{d}x^{\mu_1...\mu_D}\epsilon_{\mu_1...\mu_D} \tag{8}$$

In all expressions we assume the ordering prescription $\mu_1 < \mu_2 < ... < \mu_r$, $r = 1, 2, ..., D$. The line element can then be written in the form

$$\mathrm{d}\Sigma^2 = L^{2D}\mathrm{d}\sigma^2 + L^{2D-2}\mathrm{d}x_\mu \mathrm{d}x^\mu + L^{2D-4}\mathrm{d}x_{\mu\nu}\mathrm{d}x^{\mu\nu} + ... + |\gamma|^2 \,\mathrm{d}\tilde{\sigma}^2 \tag{9}$$

where

$$|\gamma|^2 \equiv \gamma^\dagger * \gamma \tag{10}$$

Here γ is the pseudoscalar basis element and can be writted as $\gamma_0 \wedge \gamma_1 \wedge ...\gamma_{D-1}$. In flat spacetime M_D we have that $|\gamma|^2 = +1$ or -1, depending on dimension and signature. In M_4 with signature $(+ - --)$ we have $\gamma^\dagger * \gamma = \gamma^\dagger\gamma = \gamma^2 = -1$ $(\gamma \equiv \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3)$, whilst in M_5 with signature $(+ - - - -)$ it is $\gamma^\dagger\gamma = 1$.

The analog of Lorentz transformations in C-spaces which transform a polyvector X into another poly-vector X' is given by

$$X' = RXR^{-1} \tag{11}$$

with

$$R = e^{\theta^A E_A} = exp\ [(\theta I + \theta^\mu \gamma_\mu + \theta^{\mu_1\mu_2}\gamma_{\mu_1} \wedge \gamma_{\mu_2} .....)]. \quad (12)$$

and

$$R^{-1} = e^{-\theta^A E_A} = exp\ [-(\theta I + \theta^\nu \gamma_\nu + \theta^{\nu_1\nu_2}\gamma_{\nu_1} \wedge \gamma_{\nu_2} .....)]. \quad (13)$$

where the theta parameters in (12)(13) are the components of the Clifford-value parameter $\Theta = \theta^M E_M$:

$$\theta; \theta^\mu; \theta^{\mu\nu}; .... \quad (14)$$

they are the C-space version of the Lorentz rotations/boosts parameters.

Since a Clifford algebra admits a matrix representation, one can write the norm of a poly-vectors in terms of the trace operation as: $||X||^2 = Trace\ X^2$ Hence under C-space Lorentz transformation the norms of poly-vectors behave like follows:

$$Trace\ X'^2 = Trace\ [RX^2R^{-1}] = Trace\ [RR^{-1}X^2] = Trace\ X^2. \quad (15)$$

These norms are invariant under C-space Lorentz transformations due to the cyclic property of the trace operation and $RR^{-1} = 1$.

The above transformations are *active transformations* since the transformed Clifford number X' (polyvector) is different from the "original" Clifford number X. Considering the transformations of components we have

$$X' = X'^M E_M = L^M{}_N X^N E_M \quad (16)$$

If we compare (16) with (11) we find

$$L^M{}_N E_N = R E_N R^{-1} \quad (17)$$

from which it follows that

$$L^M{}_N = \langle E^M R E_N R^{-1} \rangle_0 \equiv E^M * (R E_N R^{-1}) = E^M * E'_N. \quad (18)$$

where we have labelled E'_N as new basis element since in the active interpretation one may perform either a change of the polyvector components or a change of the basis elements. The $\langle\ \ \rangle_0$ means the scalar part of the expression and "$*$" the scalar product. Eq (18) has been obtained after multiplying (17) from the left by E^J, taking into account that $\langle E^J E_N \rangle_0 \equiv E^J * E_N = \delta^J{}_N$, and renamiming the index J into M.

3 Generalized Dynamics of Particles, Fields and Branes in C-space

An immediate application of this theory is that one may consider "strings" and "branes" in C-spaces as a unifying description of *all* branes of different dimensionality. As we have already indicated, since spinors are in left/right ideals of a Clifford algebra, a supersymmetry is then naturally incorporated into this approach as well. In particular, one can have world manifold and target space supersymmetry *simultaneously* [15]. We hope that the C-space "strings" and "branes" may lead us towards discovering the physical foundations of string and M-theory. For other alternatives to supersymmetry see the work by [50]. In particular, Z_3 generalizations of supersymmetry based on ternary algebras and Clifford algebras have been proposed by Kerner [128] in what has been called Hypersymmetry.

3.1 The Polyparticle Dynamics in C-space

We will now review the theory [15, 17] in which an extended object is modeled by the components $\sigma,\ x^\mu, x^{\mu\nu}, ...$ of the Clifford valued polyvector (2). By assumption the extended objects, as observed from Minkowski spacetime, can in general be localized not only along space-like, but also along time-like directions [15, 17]. In particular, they can be "instantonic" p-loops with either space-like or time-like orientation. Or they may be long, but finite, tube-like objetcs. The theory that we consider here goes beyond the ordinary relativity in Minkowski spacetime, therefore such localized objects in Minkowski spacetime pose no problems. They are postulated to satisfy the dynamical principle which is formulated in C-space. All conservation laws hold in C-space where we have infinitely long world "lines" or Clifford lines. In Minkowski spacetime M_4 –which is a subspace of C-space– we observe the intersections of Clifford lines with M_4. And those intersections appear as localized extended objects, p-loops, described above.

Let the motion of such an extended object be determined by the action principle

$$I = \kappa \int \mathrm{d}\tau\, (\dot{X}^\dagger * \dot{X})^{1/2} = \kappa \int \mathrm{d}\tau\, (\dot{X}^A \dot{X}_A)^{1/2} \tag{19}$$

where κ is a constant, playing the role of "mass" in C-space, and τ is an arbitrary parameter. The C-space velocities $\dot{X}^A = \mathrm{d}X^A/\mathrm{d}\tau = (\dot{\sigma}, \dot{x}^\mu, \dot{x}^{\mu\ nu}, ...)$ are also called "hollographic" velocities.

The equation of motion resulting from (19) is

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\dot{X}^A}{\sqrt{\dot{X}^B \dot{X}_B}}\right) = 0 \tag{20}$$

Taking $\dot{X}^B \dot{X}_B = \text{constant} \neq 0$ we have that $\ddot{X}^A = 0$, so that $x^A(\tau)$ is a straight worldline in C-space. The components x^A then change linearly with the parameter τ. This means that the extended object position x^μ, effective area $x^{\mu\nu}$, 3-volume $x^{\mu\nu\alpha}$, 4-volume $x^{\mu\nu\alpha\beta}$, etc., they all change with time. That is, such object experiences a sort of generalized dilational motion [17].

We shall now review the procedure exposed in ref. [17] according to which in such a generalized dynamics an object may be accelerated to faster than light speeds as viewed from a 4-dimensional Minkowski space, which is a subspace of C-space. For a different explanation of superluminal propagation based on the modified nonlinear de Broglie dispersion relations see [68].

The canonical momentum belonging to the action (19) is

$$P_A = \frac{\kappa \dot{X}_A}{(\dot{X}^B \dot{X}_B)^{1/2}} \tag{21}$$

When the denominator in eq.(21) is zero the momentum becomes infinite. We shall now calculate the speed at which this happens. This will be the *maximum speed* that an object accelerating in C-space can reach. Although an initially slow object cannot accelerate beyond that speed limit, this does not automatically exclude the possibility that fast objects traveling at a speed above that limit may exist. Such objects are C-space analog of tachyons [31, 32]. All the well known objections against tachyons should be reconsidered for the case of C-space before we could say for sure that C-space tachyons do not exist as freely propagating objects. We will leave aside this interesting possibility, and assume as a working hypothesis that there is no tachyons in C-space.

Vanishing of $\dot{X}^B \dot{X}_B$ is equivalent to vanishing of the C-space line element

$$\mathrm{d}X^A \mathrm{d}X_A = \mathrm{d}\sigma^2 + \left(\frac{\mathrm{d}x^0}{L}\right)^2 - \left(\frac{\mathrm{d}x^1}{L}\right)^2 + \left(\frac{\mathrm{d}x^{12}}{L^2}\right)^2 - \left(\frac{\mathrm{d}x^{123}}{L^3}\right)^2 - \left(\frac{\mathrm{d}x^{0123}}{L^4}\right)^2 + ... = 0 \quad (22)$$

where by "..." we mean the terms with the remaining components such as x^2, x^{01}, x^{23},..., x^{012}, etc.. The C-space line element is associated with a particular *choice* of C-space metric, namely $G_{MN} = E_M^\dagger * E_N$. If the basis E_M, $M = 1, 2, ..., 2^D$ is generated by the flat space γ^μ satisfying (3), then the C-space has the diagonal metric of eq. (22) with $+, -$ signa. In general this is not necessarily so and the C-space metric is a more complicated expression. We take now dimension of spacetime being 4, so that x^{0123} is the highest grade coordinate. In eq. (22) we introduce a length parameter L. This is necessary, since $x^0 = ct$ has dimension of length, x^{12} of length square, x^{123} of length to the third power, and x^{0123} of length to the forth power. It is natural to assume that L is the *Planck length*, that is $L = 1.6 \times 10^{-35} m$.

Let us assume that the coordinate time $t = x^0/c$ is the parameter with respect to which we define the speed V in C-space.

So we have

$$V^2 = -\left(L\frac{\mathrm{d}\sigma}{\mathrm{d}t}\right)^2 + \left(\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2 - \left(\frac{1}{L}\frac{\mathrm{d}x^{12}}{\mathrm{d}t}\right)^2 + \left(\frac{1}{L^2}\frac{\mathrm{d}x^{123}}{\mathrm{d}t}\right)^2 + \left(\frac{1}{L^3}\frac{\mathrm{d}x^{0123}}{\mathrm{d}t}\right)^2 - ... \quad (23)$$

From eqs. (22),(23) we find that the maximum speed is the maximum speed is given by

$$V^2 = c^2 \quad (24)$$

First, we see that the maximum speed squared V^2 contains not only the components of the 1-vector velocity $\mathrm{d}x^1/\mathrm{d}t$, as it is the case in the ordinary relativity, but also the multivector components such as $\mathrm{d}x^{12}/\mathrm{d}t$, $\mathrm{d}x^{123}/\mathrm{d}t$, etc..

The following special cases when only certain components of the velocity in C-space are different from zero, are of particular interest:

(i) Maximum 1-vector speed

$$\frac{\mathrm{d}x^1}{\mathrm{d}t} = c = 3.0 \times 10^8 m/s$$

(ii) Maximum 3-vector speed

$$\frac{\mathrm{d}x^{123}}{\mathrm{d}t} = L^2 c = 7.7 \times 10^{-62} m^3/s$$
$$\frac{\mathrm{d}\sqrt[3]{x^{123}}}{\mathrm{d}t} = 4.3 \times 10^{-21} m/s \quad \text{(diameter speed)}$$

(iii) Maximum 4-vector speed

$$\frac{\mathrm{d}x^{0123}}{\mathrm{d}t} = L^3 c = 1.2 \times 10^{-96}\mathrm{m}^4/\mathrm{s}$$
$$\frac{\mathrm{d}\sqrt[4]{x^{0123}}}{\mathrm{d}t} = 1.05 \times 10^{-24} m/s \quad \text{(diameter speed)}$$

Above we have also calculated the corresponding diameter speeds for the illustration of how fast the object expands or contracts.

We see that the maximum multivector speeds are very small. The diameters of objects change very slowly. Therefore we normally do not observe the dilatational motion.

Because of the positive sign in front of the σ and x^{12}, x^{012}, etc., terms in the quadratic form (22) there are no limits to correspondintg 0-vector, 2-vector and 3-vector speeds. But if we calculate, for instance, the energy necessary to excite 2-vector motion we find that it is very high. Or equivalently, to the relatively modest energies (available at the surface of the Earth), the corresponding 2-vector speed is very small. This can be seen by calculating the energy

$$p^0 = \frac{\kappa c^2}{\sqrt{1 - \frac{V^2}{c^2}}} \tag{25}$$

(a) for the case of pure 1-vector motion by taking $V = \mathrm{d}x^1/\mathrm{d}t$, and
(b) for the case of pure 2-vector motion by taking $V = \mathrm{d}x^{12}/(L\mathrm{d}t)$.

By equating the energies belonging to the cases (a) and (b we have

$$p^0 = \frac{\kappa c^2}{\sqrt{1 - \left(\frac{1}{c}\frac{\mathrm{d}x^1}{\mathrm{d}t}\right)^2}} = \frac{\kappa c^2}{\sqrt{1 - \left(\frac{1}{Lc}\frac{\mathrm{d}x^{12}}{\mathrm{d}t}\right)^2}} \tag{26}$$

which gives

$$\frac{1}{c}\frac{\mathrm{d}x^1}{\mathrm{d}t} = \frac{1}{Lc}\frac{\mathrm{d}x^{12}}{\mathrm{d}t} = \sqrt{1 - \left(\frac{\kappa c^2}{p_0}\right)^2} \tag{27}$$

Thus to the energy of an object moving translationally at $\mathrm{d}x^1/\mathrm{d}t$ =1 m/s, there corresponds the 2-vector speed $\mathrm{d}x^{12}/\mathrm{d}t = L\,\mathrm{d}x^1/\mathrm{d}t = 1.6 \times 10^{-35}$ m^2/s (diameter speed 4 $\times 10^{-18}$m/s). This would be a typical 2-vector speed of a macroscopic object. For a microscopic object, such as the electron, which can be accelerated close to the speed of light, the corresponding 2-vector speed could be of the order of 10^{-26} m^2/s (diameter speed 10^{-13}m/s). In the examples above we have provided rough estimations of possible 2-vector speeds. Exact calculations should treat concrete situations of collisions of two or more objects, assume that not only 1-vector, but also 2-vector, 3-vector and 4-vector motions are possible, and take into account the conservation of the polyvector momentum P_A.

Maximum 1-vector speed, i.e., the usual speed, can exceed the speed of light when the holographic components such as $\mathrm{d}\sigma/\mathrm{d}t$, $\mathrm{d}x^{12}/\mathrm{d}t$, $\mathrm{d}x^{012}/\mathrm{d}t$, *etc., are different from zero* [17]. This can be immediately verified from eqs. (22),(23). The speed of light is no longer such a strict barrier as it appears in the ordinary theory of relativity in M_4. In C-space a particle has extra degrees of freedom, besides the translational degrees of freedom. The scalar, σ, the bivector, x^{12} (in general, x^{rs}, $r, s = 1, 2, 3$) and the three vector, x^{012} (in general, x^{0rs}, $r, s = 1, 2, 3$), contributions to the C-space quadratic form (22) have positive sign, which is just opposite to the contributions of other components, such as x^r, x^{0r}, $x^{rst}, x^{\mu\nu\rho\sigma}$. Because some terms in the quadratic form have $+$ and some $-$ sign, the absolute value of the 3-velocity $\mathrm{d}x^r/\mathrm{d}x^0$ can be greater than c. Because the quadratic form has both $+$ and $-$ signs, the absolute value of the 3-velocity $\mathrm{d}x^r/\mathrm{d}x^0$ can be greater than c.

The issue of causality for such very peculiar sort of faster than light motion has to be reconsidered within the new framework of C-space relativity.

In the usual theory of relativity the existence of tachyons is problematic because one can arrange for situations such that tachyons are sent into the past. A tachyon T_1 is emitted from an aparatus worldline $\mathcal{C}$ at x_1^0 and a second tachyon T_2 can arrive to the same worldline $\mathcal{C}$ at an earlier time $x'^0 < x_1^0$ and trigger destruction of the aparatus. The spacetime event E' at which the aparatus is destroyed cooncides with the event E at which the aparatus by initial assumtion kept on functioning normally and later emitted T_1. So there is a paradox.

Another way of rephrasing this paradox is by saying that a breakdown of causality has occurred. The simplest way to see why causality is violated when tachyons are used to exchange signals is by writing the temporal displacements $\delta t = t^B - t^A$ between two events (in Minkowski space-time) in two different frames of reference:

$$(\delta t)' = (\delta t)cosh(\xi) + \frac{\delta x}{c}sinh(\xi) = (\delta t)[cosh(\xi) + (\frac{1}{c}\frac{\delta x}{\delta t})sinh(\xi)] = \tag{28}$$

$$(\delta t)[cosh(\xi) + (\beta_{tachyon})sinh(\xi)] \tag{29}$$

the boost parameter ξ is defined in terms of the velocity as $\beta_{frame} = v_{frame}/c = tanh(\xi)$, where v_{frame} is is the relative velocity (in the x-direction) of the two reference frames and can be written in terms of the Lorentz-boost rapidity parameter ξ by using hyperbolic functions. The Lorentz dilation factor is $cosh(\xi) = (1 - \beta^2_{frame})^{-1/2}$; whereas $\beta_{tachyon} = v_{tachyon}/c$ is the beta parameter associated with the tachyon velocity $\delta x/\delta t$. By emitting a tachyon along the *negative* x -direction one has $\beta_{tachyon} < 0$ and such that its velocity exceeds the speed of light $|\beta_{tachyon}| > 1$

A reversal in the sign of $(\delta t)' < 0$ in the above boost transformations occurs when the tachyon velocity $|\beta_{tachyon}| > 1$ and the relative velocity of the reference frames $|\beta_{frame}| < 1$ obey the inequality condition :

$$(\delta t)' = (\delta t)[cosh(\xi) - |\beta_{tachyon}|sinh(\xi)] < 0 \Rightarrow 1 < \frac{1}{tanh(\xi)} = \frac{1}{\beta_{frame}} < |\beta_{tachyon}|. \tag{30}$$

thereby resulting in a causality violation in the primed reference frame since the effect (event B) occurs $before$ the cause (event A) in the *primed* reference frame.

In C-space the situation is different. In the extended relativity, i.e., in the relativity in C-space, the dynamics refers to a larger space. Minkowski space is just a subspace of C-space. "Wordlines" now live in C-space. In particular, a C-space worldline can be described in terms of five functions $x^\mu(\tau),\ \sigma(\tau)$ (all other C-space coordinates being kept constant). In C-space we have the *constrained action* (19), whilst in Minkowski space we have a reduced, *unconstrained* action. A reduction of variables can be done by choosing a gauge in which $\sigma(\tau) = \tau$. It was shown in ref.[16, 15, 17] that the latter unconstrained action is equivalent to the well known Stueckelberg action [33, 34]. In other words, the Stueckelberg relativistic dynamics is embedded in C-space. In Stueckelberg theory all four spacetime coordinates x^μ are independent dynamical degrees of freedom that evolve in terms of an extra parameter σ which is invariant under Lorentz transformations in M_4. From the C-space point of view, the evolution parameter σ is just one of the C-space coordintes X^M. By assumption, σ is monotonically increasing along particles' worldlines. Certain C-space worldlines may appear tachyonic from the point of view of M_4. If we now repeat the above experiment with the emission of the first and absorption of the second tachyon we find out that the second tachyon T_2 cannot reach the aparatus worldline earlier than it was emmitted from. Namely, T_2 can arrive at a C-space event E' with $x'^0 < x_1^0$, but the latter event does not coincide with the event E on the aparatus worldline, since although having the same coordinates $x'^\mu = x^\mu$, the events E and E' have different extra coordinates $\sigma' \neq \sigma$. In other words, E and E' are different points in C-space. Therefore T_2 cannot destroy the aparatus and there is no paradox.

If nature indeed obeys the dynamics in Clifford space, then a particle, as observed from the 4-dimensional Minkowski space, can be accelerated beyond the speed of light [17], provided that its extra degrees of freedom $x^{\mu\nu}$, $x^{\mu\nu\alpha}$,..., are changing simultaneously with the ordinary position x^μ. But such a particle, although moving faster than light in the subspace M_4, is moving slower than light in C-space, since its speed V, defined in eq.(23), is smaller than c. *In this respect, our particle is not tachyon at all!* In C-space we thus retain all the nice features of relativity, but in the subspace M_4 we have, as a particular case, the unconstrained Stueckelberg theory in which faster-than-light propagation is not paradoxical and is consistent with the quantum field theory as well [15]. This is so, because the Stueckelberg theory is quite different from the ordinary (constrained) theory of relativity in M_4, and faster than light motion in the former theory is of totally different nature from the faster that light motion in the latter theory. The well known objections against tachyons are not valid for our particle which moves according to the relativity in C-space.

To sum up, in the theory considered here, there are no tachyons in C-space, so physical signals in C-space are constrained to live *inside* the C-space-light cone, defined by eq. (22). However, certain worldlines in C-space, when projected onto the subspace M_4, can appear as worldlines of ordinary tachyons outside the lightcone in M_4. But since the latter tachyonic "world lines" are just projections, whilst the true world lines go out of M_4 into C-space, they cannot take part in the paradoxical arrangement in which future could influence the past.

There is also an alternative possible explanation. We can assume that the reference systems move in C-space so that they have nonvanishing components of velocity into directions orthogonal to M_4. Therefore one has to add corresponding extra terms into the C-space Lorentz boosts that transform one such reference frame into the other, with the result that the analysis leading to the relation (30) does no longer hold for such generalized case. It remains to be investigated whether there exist a subclass $\{S_0\}$ of reference frames transformed into each other by such C-space boosts that the causal order of events will be preserved in all $S_0 \in \{S_0\}$ for all C-space world lines pointing into the time-like directions of C-space (including those which appear tachyonic in M_4).

This will be the subject of future work and investigations.

The physical analog of photons in C-space corresponds to tensionless p-loops, i.e., tensionless closed branes, since the analog of mass m in C-space is the maximal p-loop tension. By 'maximal p-loop' we mean the loop with the maximum value of p associated with the hierarchy of p-loops (closed p-branes): $p = 0, 1, 2, ...$ living in the embedding target spacetime. One must not confuse the Stueckelberg parameter σ with the C-space Proper-time Σ (eq.(5)); so one could have a world line in C-space such that

$$\mathrm{d}\Sigma = 0 \leftrightarrow C\text{-space photon} \leftrightarrow \begin{array}{l}\text{tensionless branes with a monotonically increasing} \\ \text{Stueckelberg parameter } \sigma\end{array}$$

.

We have described how one can obtain faster than light motion in M_4 from the theory of relativity in C-space. There are other possible ways to achieve superluminal propagation. One such approach is described in refs. [84]

An alternative procedure In ref. [9] an alternative factorization of the C-space line element has been undertaken. Starting from the line element $\mathrm{d}\Sigma$ of eq. (5), instead of factoring out the $(dx^0)^2$ element, one may factor out the $(\mathrm{d}\Omega)^2 \equiv L^{2D}\mathrm{d}\sigma^2$ element, giving rise to the generalized "holographic " velocities measured w.r.t the Ω parameter, for example the areal-time parameter in the Eguchi-Schild formulation of string dynamics [126], [37], [24], instead of the x^0 parameter (coordinate clock). One then obtains

$$\mathrm{d}\Sigma^2 = \mathrm{d}\Omega^2 \left[1 + L^{2D-2} \frac{\mathrm{d}x_\mu}{\mathrm{d}\Omega} \frac{\mathrm{d}x^\mu}{\mathrm{d}\Omega} + L^{2D-4} \frac{\mathrm{d}x_{\mu\nu}}{\mathrm{d}\Omega} \frac{\mathrm{d}x^{\mu\nu}}{\mathrm{d}\Omega} + ... + |\gamma|^2 \left(\frac{\mathrm{d}\tilde{\sigma}}{\mathrm{d}\Omega} \right)^2 \right] \tag{31}$$

The idea of ref. [9] was to restrict the line element (31) to the non tachyonic values which imposes un upper limit on the holographic velocities. The motivation was to find a lower bound of length scale. This upper holographic-velocity bound does not necessarily translate into a lower bound on the values of lengths, areas, volumes....without the introduction of quantum mechanical considerations. One possibility could be that the upper limiting speed of light and the upper bound of the momentum $m_p c$ of a Planck-mass elementary particle (the so-called *Planckton* in the literature) generalizes now to an upper-bound in the p-loop holographic velocities and the p-loop holographic momenta associated with elementary closed p-branes whose tensions are given by powers of the Planck mass. And the latter upper bounds on the holographic p-loop momenta implies a lower-bound on the holographic areas, volumes,..., resulting from the string/brane uncertainty relations [11], [10],[19]. Thus, Quantum Mechanics is required to implement the postulated principle of minimal lengths, areas, volumes...and which cannot be derived from the classical geometry alone. The emergence of minimal Planck areas occurs also in the Loop Quantum Gravity program [111] where the expecation values of the Area operator are given by multiples of Planck area.

3.2 A Unified Theory of all p-Branes in C-Spaces

The generalization to C-spaces of string and p-brane actions as embeddings of world-manifolds onto target spacetime backgrounds involves the embeddings of polyvector-valued world-manifolds (of dimensions 2^d) onto polyvector-valued target spaces (of dimensions 2^D), given by the Clifford-valued maps $X = X(\Sigma)$ (see [15]). These are maps from the Clifford-valued world-manifold, parametrized by the polyvector-valued variables Σ, onto the Clifford-valued target space parametrized by the polyvector-valued coordinates X. Physically one envisions these maps as taking an n-dimensional simplicial cell (n-loop) of the world-manifold onto an m-dimensional simplicial cell (m-loop) of the target C-space manifold ; i.e. maps from n-dim objects onto m-dim objects generalizing the old maps of taking points onto points. One is basically dealing with a dimension-category of objects. The size of the simplicial cells (p-loops), upon quantization of a generalized harmonic oscillator, for example, are given by multiples of the Planck scale, in area, volume, hypervolume units or Clifford-bits.

In compact multi-index notation $X = X^M \Gamma_M$ one denotes for each one of the components of the target space polyvector X:

$$X^M \equiv X^{\mu_1 \mu_2 .... \mu_r}, \ \mu_1 < \mu_2 < ... < \mu_r. \tag{32}$$

and for the world-manifold polyvector $\Sigma = \Sigma^A E_A$:

$$\Sigma^A \equiv \xi^{a_1 a_2 .... a_s}, a_1 < a_2 < ... < a_s. \tag{33}$$

where $\Gamma_M = (\underline{1}, \gamma_\mu, \gamma_{\mu\nu}, ...)$ and $E_A = (\underline{1}, e_a, e_{ab}, ...)$ form the basis of the target manifold and world manifold Clifford algebra, respectively. It is very important to order the indices within each multi-index M and A as shown above. The above Clifford-valued coordinates X^M, Σ^A correspond to antisymmetric tensors of ranks r, s in the target spacetime background and in the world-manifold, respectively.

There are many different ways to construct C-space brane actions which are on-shell equivalent to the analogs of the Dirac-Nambu-Goto action for extended objects and that are given by the world-volume spanned by the branes in their motion through the target spacetime background.

One of these actions is the Polyakov-Howe-Tucker action:

$$I = \frac{T}{2} \int [D\Sigma] \sqrt{|H|} [H^{AB} \partial_A X^M \partial_B X^N G_{MN} + (2 - 2^d)]. \tag{34}$$

with the 2^d-dim world-manifold measure:

$$[D\Sigma] = (d\xi)(d\xi^a)(d\xi^{a_1 a_2})(d\xi^{a_1 a_2 a_3}).... \tag{35}$$

Upon the algebraic elimination of the auxiliary world-manifold metric H^{AB} from the action (34), via the equations of motion, yields for its on-shell solution the pullback of the target C-space metric onto the C-space world-manifold:

$$H_{AB}(on - shell) = G_{AB} = \partial_A X^M \partial_B X^N G_{MN} \tag{36}$$

upon inserting back the on-shell solutions (36) into (34) gives the Dirac-Nambu-Goto action for the C-space branes directly in terms of the C-space determinant, or measure, of the induced C-space world-manifold metric G_{AB}, as a result of the embedding:

$$I = T \int [\mathrm{D}\Sigma] \sqrt{\mathrm{Det}(\partial_A X^M \partial_B X^N G_{MN})}. \tag{37}$$

However in C-space, the Polyakov-Howe-Tucker action admits an even further generalization that is comprised of two terms $S_1 + S_2$. The first term is:

$$S_1 = \int [\mathrm{D}\Sigma] |E| E^A E^B \partial_A X^M \partial_B X^N \Gamma_M \Gamma_N. \tag{38}$$

Notice that this is a generalized action which is written in terms of the C-space coordinates $X^M(\Sigma)$ and the C-space analog of the target-spacetime vielbein/frame one-forms $e^m = e^m{}_\mu dx^\mu$ given by the Γ^M variables. The auxiliary world-manifold vielbein variables e^a, are given now by the Clifford-valued frame E^A variables.

In the conventional Polyakov-Howe-Tucker action, the auxiliary world-manifold metric h^{ab} associated with the standard p-brane actions is given by the usual scalar product of the frame vectors $e^a.e^b = e^a_\mu e^b_\nu g^{\mu\nu} = h^{ab}$. Hence, the C-space world-manifold metric H^{AB} appearing in (36) is given by scalar product $< (E^A)^\dagger E^B >_0 = H^{AB}$, where $(E^A)^\dagger$ denotes the reversal operation of E^A which requires reversing the orderering of the vectors present in the Clifford aggregate E^A.

Notice, however, that the form of the action (38) is far more general than the action in (34). In particular, the S_1 itself can be decomposed futher into two additional pieces by rewriting the Clifford product of two basis elements into a symmetric plus an antisymmetric piece, respectively:

$$E^A E^B = \frac{1}{2}\{E^A, E^B\} + \frac{1}{2}[E^A, E^B]. \tag{39}$$

$$\Gamma_M \Gamma_N = \frac{1}{2}\{\Gamma_M, \Gamma_N\} + \frac{1}{2}[\Gamma_M, \Gamma_N]. \tag{40}$$

In this fashion, the S_1 component has *two* kinds of terms. The first term containing the symmetric combination is just the analog of the standard non-linear sigma model action, and the second term is a Wess-Zumino-like term, containing the antisymmetric combination [15]. To extract the non-linear sigma model part of the generalized action above, we may simply take the scalar product of the vielbein-variables as follows:

$$(S_1)_{sigma} = \frac{T}{2} \int [\mathrm{D}\Sigma] |E| < (E^A \partial_A X^M \Gamma_M)^\dagger (E^B \partial_B X^N \Gamma_N) >_0 . \tag{41}$$

where once again we have made use of the reversal operation (the analog of the hermitian adjoint) before contracting multi-indices. In this fashion we recover again the Clifford-scalar valued action given by [15].

Actions like the ones presented here in terms of derivatives with respect to quantities with multi-indices can be mapped to actions involving *higher* derivatives, in the same fashion that the C-space scalar curvature, the analog of the Einstein-Hilbert action, could be recast as a higher derivative gravity with torsion (reviewed in sec. 4). Higher derivatives actions are also related to theories of Higher spin fields [117] and W-geometry, W-algebras [116], [122]. For the role of Clifford algerbras to higher spin theories see [51].

The S_2 (scalar) component of the C-space brane action is the usual cosmological constant term given by the C-space determinant $|E| = \det(H^{AB})$ based on the scalar part of the geometric product $< (E^A)^\dagger E^B >_0 = H^{AB}$

$$S_2 = \frac{T}{2} \int [\mathrm{D}\Sigma] |E| (2 - 2^d) \tag{42}$$

where the C-space determinant $|E| = \sqrt{|\det(H^{AB})|}$ of the $2^d \times 2^d$ generalized world-manifold metric H^{AB} is given by:

$$\det(H^{AB}) = \frac{1}{(2^d)!} \epsilon_{A_1 A_2 .... A_{2^d}} \epsilon_{B_1 B_2 .... B_{2^d}} H^{A_1 B_1} H^{A_2 B_2} .... H^{A_{2^d} B_{2^d}}. \tag{43}$$

The $\epsilon_{A_1 A_2 .... A_{2^d}}$ is the totally antisymmetric tensor density in C-space.

There are many different forms of p-brane actions, with and without a cosmological constant [123], and based on a new integration measure by recurring to auxiliary scalar fields [115], that one could have used to construct their C-space generalizations. Since all of them are on-shell equivalent to the Dirac-Nambu-Goto p-brane actions, we decided to focus solely on those actions having the Polyakov-Howe-Tucker form.

4 Generalized Gravitational Theories in Curved C-spaces: Higher Derivative Gravity and Torsion from the Geometry of C-Space

4.1 Ordinary space

4.1.1 Clifford algebra based geometric calculus in curved space(time)

Cliffordfd algebra is a very useful tool for description of geometry, especially of curved space V_n. Let us first review how it works in curved space(time). Later we will discuss a generalization to curved Clifford space [20].

We would like to make those techniques accessible to a wide audience of physicists who are not so familiar with the rigorous underlying mathematics, and demonstrate how Clifford algebra can be straightforwardly employed in the theory of gravity and its generalization. So we will leave aside the sophisticated mathematical approach, and rather follow as simple line of thought as possible, a praxis that is normally pursued by physicists. For instance, physicists in their works on general relativity employ a mathematical formulation and notation which is much simpler from that of purely mathematical or mathamatically oriented works. For rigorous mathematical treatment the reader is adviced to study, refs. [1, 76, 77, 78, 79].

Let the vector fields γ_μ, $\mu = 1, 2, ..., n$ be a coordinate basis in V_n satisfying the Clifford algebra relation

$$\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = g_{\mu\nu} \tag{44}$$

where $g_{\mu\nu}$ is the metric of V_n. In curved space γ_μ and $g_{\mu\nu}$ cannot be constant but necessarily depend on position x^μ. An arbitrary vector is a linear superposition [1]

$$a = a^\mu \gamma_\mu \tag{45}$$

where the components a^μ are *scalars* from the geometric point of view, whilst γ_μ are *vectors*.

Besides the basis $\{\gamma_\mu\}$ we can introduce the reciprocal basis[2] $\{\gamma^\mu\}$ satisfying

$$\gamma^\mu \cdot \gamma^\nu \equiv \tfrac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu} \tag{46}$$

where $g^{\mu\nu}$ is the covariant metric tensor such that $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu{}_\nu$, $\gamma^\mu \gamma_\nu + \gamma_\nu \gamma^\mu = 2\delta^\mu{}_\nu$ and $\gamma^\mu = g^{\mu\nu}\gamma_\nu$.

Following ref.[1] (see also [15]) we consider the *vector derivative* or *gradient* defined according to

$$\partial \equiv \gamma^\mu \partial_\mu \tag{47}$$

where ∂_μ is an operator whose action depends on the quantity it acts on [26].

Applying the vector derivative ∂ on a *scalar* field ϕ we have

$$\partial \phi = \gamma^\mu \partial_\mu \phi \tag{48}$$

where $\partial_\mu \phi \equiv (\partial / \partial x^\mu)\phi$ coincides with the partial derivative of ϕ.

But if we apply it on a *vector* field a we have

$$\partial a = \gamma^\mu \partial_\mu (a^\nu \gamma_\nu) = \gamma^\mu (\partial_\mu a^\nu \gamma_\nu + a^\nu \partial_\mu \gamma_\nu) \tag{49}$$

In general γ_ν is not constant; it satisfies the relation [1, 15]

$$\partial_\mu \gamma_\nu = \Gamma_{\mu\nu}^\alpha \gamma_\alpha \tag{50}$$

where $\Gamma_{\mu\nu}^\alpha$ is the *connection*. Similarly, for $\gamma^\nu = g^{\nu\alpha}\gamma_\alpha$ we have

$$\partial_\mu \gamma^\nu = -\Gamma_{\mu\alpha}^\nu \gamma^\alpha \tag{51}$$

The *non commuting* operator ∂_μ so defined determines *the parallel transport* of a basis vector γ^ν. Instead of the symbol ∂_μ Hestenes uses $\Box_\mu$, whilst Wheeler et. al. [36] use ∇_μ and call it "covariant derivative". In modern, mathematically opriented literature more explicit notation such as D_{γ_μ} or ∇_{γ_μ} is used. However, such a notation, although mathematically very relevant, would not be very practical in long computations. We find it very convenient to keep the symbol ∂_μ for components of the geometric operator $\partial = \gamma^\mu \partial_\mu$. When acting on a scalar field the derivative ∂_μ happens to be commuting and thus behaves as the ordinary partial derivative. When acting on a vector field, ∂_μ is a *non commuting operator*. In this respect, there can be no confusion with partial derivative, because the latter normally acts on *scalar fields*, and in such a case partial derivative and ∂_μ are one and the same thing. However, when acting on a vector field, the derivative

[2] In Appendix A of the Hesteness book [1] the frame $\{\gamma^\mu\}$ is called *dual* frame because the duality operation is used in constructing it.

∂_μ is non commuting. Our operator ∂_μ when acting on γ_μ or γ^μ should be distinguished from the ordinary—*commuting*—partial derivative, let be denoted $\gamma^\nu{}_{,\mu}$, usually used in the literature on the Dirac equation in curved spacetime. The latter derivative is not used in the present paper, so there should be no confusion.

Using (50), eq.(49) becomes

$$\partial a = \gamma^\mu \gamma_\nu (\partial_\mu a^\nu + \Gamma^\nu_{\mu\alpha} a^\alpha) \equiv \gamma^\mu \gamma_\nu \mathrm{D}_\mu a^\nu = \gamma^\mu \gamma^\nu \mathrm{D}_\mu a_\nu \tag{52}$$

where D_μ is the covariant derivative of tensor analysis..

Decomposing the Clifford product $\gamma^\mu \gamma^\nu$ into its symmetric and antisymmetric part [1]

$$\gamma^\mu \gamma^\nu = \gamma^\mu \cdot \gamma^\nu + \gamma^\mu \wedge \gamma^\nu \tag{53}$$

where

$$\gamma^\mu \cdot \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = g^{\mu\nu} \tag{54}$$

is the *inner product* and

$$\gamma^\mu \wedge \gamma^\nu \equiv \frac{1}{2}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu) \tag{55}$$

the *outer product*, we can write eq.(52) as

$$\partial a = g^{\mu\nu}\, \mathrm{D}_\mu a_\nu + \gamma^\mu \wedge \gamma^\nu \mathrm{D}_\mu a_\nu = \mathrm{D}_\mu a^\mu + \frac{1}{2}\gamma^\mu \wedge \gamma^\nu (\mathrm{D}_\mu a_\nu - \mathrm{D}_\nu a_\mu) \tag{56}$$

Without employing the expansion in terms of γ_μ we have simply

$$\partial a = \partial \cdot a + \partial \wedge a \tag{57}$$

Acting twice on a vector by the operator ∂ we have[3]

$$\begin{aligned}
\partial \partial a &= \gamma^\mu \partial_\mu (\gamma^\nu \partial_\nu)(a^\alpha \gamma_\alpha) = \gamma^\mu \gamma^\nu \gamma_\alpha \mathrm{D}_\mu \mathrm{D}_\nu a^\alpha \\
&= \gamma_\alpha \mathrm{D}_\mu \mathrm{D}^\mu a^\alpha + \frac{1}{2}(\gamma^\mu \wedge \gamma^\nu)\gamma_\alpha [\mathrm{D}_\mu, \mathrm{D}_\nu] a^\alpha \\
&= \gamma_\alpha \mathrm{D}_\mu \mathrm{D}^\mu a^\alpha + \gamma^\mu (R_{\mu\rho} a^\rho + K_{\mu\alpha}{}^\rho \mathrm{D}_\rho a^\alpha) \\
&\qquad + \frac{1}{2}(\gamma^\mu \wedge \gamma^\nu \wedge \gamma_\alpha)(R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha)
\end{aligned} \tag{58}$$

We have used

$$[\mathrm{D}_\mu, \mathrm{D}_\nu] a^\alpha = R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha \tag{59}$$

where

$$K_{\mu\nu}{}^\rho = \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \tag{60}$$

is *torsion* and $R_{\mu\nu\rho}{}^\alpha$ the *curvature tensor*. Using eq.(50) we find

$$[\partial_\alpha, \partial_\beta]\gamma_\mu = R_{\alpha\beta\mu}{}^\nu \gamma_\nu \tag{61}$$

[3]We use $(a \wedge b)c = (a \wedge b) \cdot c + a \wedge b \wedge c$ [1] and $(a \wedge b) \cdot c = (b \cdot c)a - (a \cdot c)b$.

from which we have

$$R_{\alpha\beta\mu}{}^{\nu} = ([[\partial_\alpha, \partial_\beta]\gamma_\mu) \cdot \gamma^\nu \tag{62}$$

Thus in general the commutator of derivatives ∂_μ acting on a vector does not give zero, but is given by the curvature tensor.

In general, for an r-vector $A = a^{\alpha_1 ... \alpha_r} \gamma_{\alpha_1} \gamma_{\alpha_2} ... \gamma_{\alpha_r}$ we have

$$\begin{aligned} \partial\partial ... \partial A &= (\gamma^{\mu_1} \partial_{\mu_1})(\gamma^{\mu_2} \partial_{\mu_2}) ... (\gamma^{\mu_k} \partial_{\mu_k})(a^{\alpha_1 ... \alpha_r} \gamma_{\alpha_1} \gamma_{\alpha_2} ... \gamma_{\alpha_r}) \\ &= \gamma^{\mu_1} \gamma^{\mu_2} ... \gamma^{\mu_k} \gamma_{\alpha_1} \gamma_{\alpha_2} ... \gamma_{\alpha_r} \mathrm{D}_{\mu_1} \mathrm{D}_{\mu_2} ... \mathrm{D}_{\mu_k} a^{\alpha_1 ... \alpha_r} \end{aligned} \tag{63}$$

4.1.2 Clifford algebra based geometric calculus and resolution of the ordering ambiguity for the product of momentum operators

Clifford algebra is a very useful tool for description of geometry of curved space. Moreover, as shown in ref.[26] it provides a resolution of the long standing problem of the ordering ambiguity of quantum mechanics in curved space. Namely, eq.(47) for the vector derivative suggests that the momentum operator is given by

$$p = -i\,\partial = -i\gamma^\mu \partial_\mu \tag{64}$$

One can consider three distinct models:

(i) *The non relativistic particle* moving in $n$dimensional curved space. Then, $\mu = 1, 2, ..., n$, and signature is $(++++....)$.

(ii) *The relativistic particle* in curved spacetime, described by the *Schild action* [37]. Then, $\mu = 0, 1, 2, ..., n-1$ and signature is $(+---...)$.

(iii) *The Stueckelberg unconstrained particle.* [33, 34, 35, 29].

In all three cases the classical action has the form

$$I[X^\mu] = \frac{1}{2\Lambda} \int \mathrm{d}\tau \, g_{\mu\nu}(x) \dot{X}^\mu \dot{X}^\nu \tag{65}$$

and the corresponding Hamiltonian is

$$H = \frac{\Lambda}{2} g^{\mu\nu}(x) p_\mu p_\nu = \frac{\Lambda}{2}\, p^2 \tag{66}$$

If, upon quantization we take for the momentum operator $p_\mu = -i\,\partial_\mu$, then the ambiguity arises of how to write the quantum Hamilton operator. The problem occurs because the expressions $g^{\mu\nu} p_\mu p_\nu$, $p_\mu g^{\mu\nu} p_\nu$ and $p_\mu p_\nu g^{\mu\nu}$ are not equivalent.

But, if we rewrite H as

$$H = \frac{\Lambda}{2}\, p^2 \tag{67}$$

where $p = \gamma^\mu p_\mu$ is the *momentum vector* which upon quantization becomes the momentum vector operator (64), we find that there is no ambiguity in writing the square p^2. When acting with H on a *scalar* wave function ϕ we obtain the unambiguous expression

$$H\phi = \frac{\Lambda}{2} p^2 \phi = \frac{\Lambda}{2} (-i)^2 (\gamma^\mu \partial_\mu)(\gamma^\nu \partial_\nu)\phi = -\frac{\Lambda}{2} \mathrm{D}_\mu \mathrm{D}^\mu \phi \tag{68}$$

in which there is no curvature term R. We expect that a term with R will arise upon acting with H on a *spinor* field ψ.

4.2 C-space

Let us now consider C-space and review the procedure of ref. [20]. . A basis in C-space is given by

$$E_A = \{\gamma, \gamma_\mu, \gamma_\mu \wedge \gamma_\nu, \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\rho, ...\} \tag{69}$$

where in an r-vector $\gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge ... \wedge \gamma_{\mu_r}$ we take the indices so that $\mu_1 < \mu_2 < ... < \mu_r$. An element of C-space is a Clifford number, called also *Polyvector* or *Clifford aggregate* which we now write in the form

$$X = X^A E_A = s\,\gamma + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + ... \tag{70}$$

A C-space is parametrized not only by 1-vector coordinates x^μ but also by the 2-vector coordinates $x^{\mu\nu}$, 3-vector coordinates $x^{\mu\nu\alpha}$, etc., called also *holographic coordinates*, since they describe the holographic projections of 1-loops, 2-loops, 3-loops, etc., onto the coordinate planes. By p-loop we mean a closed p-brane; in particular, a 1-loop is closed string.

In order to avoid using the powers of the Planck scale length parameter L in the expansion of the polyvector X we use the dilatationally invariant units [15] in which L is set to 1. The dilation invariant physics was discussed from a different perspective also in refs. [23, 21].

In a flat C-space the basis vectors E^A are constants. In a curved C-space this is no longer true. Each E_A is a function of the C-space coordinates

$$X^A = \{s, x^\mu, x^{\mu\nu}, ...\} \tag{71}$$

which include scalar, vector, bivector,..., r-vector,..., coordinates.

Now we define the connection $\tilde{\Gamma}^C_{AB}$ in C-space according to

$$\partial_A E_B = \tilde{\Gamma}^C_{AB} E_C \tag{72}$$

where $\partial_A \equiv \partial/\partial X^A$ is the derivative in C-space. This definition is analogous to the one in ordinary space. Let us therefore define the C-space curvature as

$$\mathcal{R}_{ABC}{}^D = ([\partial_A, \partial_B] E_C) * E^D \tag{73}$$

which is a straightforward generalization of the relation (62). The 'star' means the *scalar product* between two polyvectors A and B, defined as

$$A * B = \langle A\,B \rangle_S \tag{74}$$

where 'S' means 'the scalar part' of the geometric product AB.

In the following we shall explore the above relation for curvature and see how it is related to the curvature of the ordinary space. Before doing that we shall demonstrate that the derivative with respect to the bivector coordinate $x^{\mu\nu}$ is equal to the commutator of the derivatives with respect to the vector coordinates x^μ.

Returning now to eq.(72), the differential of a C-space basis vector is given by

$$\mathrm{d}E_A = \frac{\partial E_A}{\partial X^B}\,\mathrm{d}X^B = \Gamma^C_{AB}\,E_C\,\mathrm{d}X^B \tag{75}$$

In particular, for $A = \mu$ and $E_A = \gamma_\mu$ we have

$$
\begin{aligned}
\mathrm{d}\gamma_\mu &= \frac{\partial \gamma_\mu}{\partial X^\nu}\mathrm{d}x^\nu + \frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}}\mathrm{d}x^{\alpha\beta} + ... = \tilde{\Gamma}^A_{\nu\mu} E_A \mathrm{d}x^\nu + \tilde{\Gamma}^A_{[\alpha\beta]\mu} E_A \mathrm{d}x^{\alpha\beta} + ... \\
&= (\tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha + \tilde{\Gamma}^{[\rho\sigma]}_{\nu\mu}\gamma_\rho \wedge \gamma_\sigma + ...)\mathrm{d}x^\nu \\
&\quad +(\tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho + \tilde{\Gamma}^{[\rho\sigma]}_{[\alpha\beta]\mu}\gamma_\rho \wedge \gamma_\sigma + ...)\mathrm{d}x^{\alpha\beta} + ...
\end{aligned}
\tag{76}
$$

We see that the differential $\mathrm{d}\gamma_\mu$ is in general a polyvector, i.e., a Clifford aggregate. In eq.(76) we have used

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha + \tilde{\Gamma}^{[\rho\sigma]}_{\nu\mu}\gamma_\rho \wedge \gamma_\sigma + ... \tag{77}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = \tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho + \tilde{\Gamma}^{[\rho\sigma]}_{[\alpha\beta]\mu}\gamma_\rho \wedge \gamma_\sigma + ... \tag{78}$$

Let us now consider a *restricted* space in which the derivatives of γ_μ with respect to x^ν and $x^{\alpha\beta}$ do not contain higher rank multivectors. Then eqs. (77),(78) become

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \tilde{\Gamma}^\alpha_{\nu\mu}\gamma_\alpha \tag{79}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = \tilde{\Gamma}^\rho_{[\alpha\beta]\mu}\gamma_\rho \tag{80}$$

Further we assume that

(i) the components $\tilde{\Gamma}^\alpha_{\nu\mu}$ of the C-space connection $\tilde{\Gamma}^C_{AB}$ coincide with the connection $\Gamma^\alpha_{\nu\mu}$ of an ordinary space.

(ii) the components $\tilde{\Gamma}^\rho_{[\alpha\beta]\mu}$ of the C-space connection coincide with the curvature tensor ${R_{\alpha\beta\mu}}^\rho$ of an ordinary space.

Hence, eqs.(79),(80) read

$$\frac{\partial \gamma_\mu}{\partial x^\nu} = \Gamma^\alpha_{\nu\mu}\gamma_\alpha \tag{81}$$

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = {R_{\alpha\beta\mu}}^\rho\gamma_\rho \tag{82}$$

and the differential (76) becomes

$$\mathrm{d}\gamma_\mu = (\Gamma^\rho_{\alpha\mu}\mathrm{d}x^\alpha + \tfrac{1}{2}{R_{\alpha\beta\mu}}^\rho \mathrm{d}x^{\alpha\beta})\gamma_\rho \tag{83}$$

The same relation was obtained by Pezzaglia [14] by using a different method, namely by considering how polyvectors change with position. The above relation demonstrates that a geodesic in C-space is not a geodesic in ordinary spacetime. Namely, in ordinary spacetime we obtain Papapetrou's equation. This was previously pointed out by Pezzaglia [14].

Although a C-space connection does not transform like a C-space tensor, some of its components, i.e., those of eq. (80), may have the transformation properties of a tensor in an ordinary space.

Under a general coordinate transformation in C-space

$$X^A \to X'^A = X'^A(X^B) \tag{84}$$

the connection transforms according to[4]

$$\tilde{\Gamma}'^C_{AB} = \frac{\partial X'^C}{\partial X^E}\frac{\partial X^J}{\partial X'^A}\frac{\partial X^K}{\partial X'^B}\tilde{\Gamma}^E_{JK} + \frac{\partial X'^C}{\partial X^J}\frac{\partial^2 X^J}{\partial X'^A \partial X'^B} \tag{85}$$

In particular, the components which contain the bivector index $A = [\alpha\beta]$ transform as

$$\tilde{\Gamma}'^\rho_{[\alpha\beta]\mu} = \frac{\partial X'^\rho}{\partial X^E}\frac{\partial X^J}{\partial \sigma'^{\alpha\beta}}\frac{\partial X^K}{\partial x'^\mu}\tilde{\Gamma}^E_{JK} + \frac{\partial x'^\rho}{\partial X^J}\frac{\partial^2 X^J}{\partial \sigma'^{\alpha\beta}\partial x'^\mu} \tag{86}$$

Let us now consider a particular class of coordinate transformations in C-space such that

$$\frac{\partial x'^\rho}{\partial x^{\mu\nu}} = 0 \,, \qquad \frac{\partial x^{\mu\nu}}{\partial x'^\alpha} = 0 \tag{87}$$

Then the second term in eq.(86) vanishes and the transformation becomes

$$\tilde{\Gamma}'^\rho_{[\alpha\beta]\mu} = \frac{\partial X'^\rho}{\partial x^\epsilon}\frac{\partial x^{\rho\sigma}}{\partial \sigma'^{\alpha\beta}}\frac{\partial x^\gamma}{\partial x'^\mu}\tilde{\Gamma}^\epsilon_{[\rho\sigma]\gamma} \tag{88}$$

Now, for the bivector whose components are $\mathrm{d}x^{\alpha\beta}$ we have

$$\mathrm{d}\sigma'^{\alpha\beta}\gamma'_\alpha \wedge \gamma'_\beta = \mathrm{d}x^{\alpha\beta}\gamma_\alpha \wedge \gamma_\beta \tag{89}$$

Taking into account that in our particular case (87) γ_α transforms as a basis vector in an ordinary space

$$\gamma'_\alpha = \frac{\partial x^\mu}{\partial x'^\alpha}\gamma_\mu \tag{90}$$

we find that (89) and (90) imply

$$\mathrm{d}\sigma'^{\alpha\beta}\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} = \mathrm{d}x^{\mu\nu} \tag{91}$$

which means that

$$\frac{\partial x^{\mu\nu}}{\partial \sigma'^{\alpha\beta}} = \frac{1}{2}\left(\frac{\partial x^\mu}{\partial x'^\alpha}\frac{\partial x^\nu}{\partial x'^\beta} - \frac{\partial x^\nu}{\partial x'^\alpha}\frac{\partial x^\mu}{\partial x'^\beta}\right) \equiv \frac{\partial x^{[\mu}}{\partial x'^\alpha}\frac{\partial x^{\nu]}}{\partial x'^\beta} \tag{92}$$

The transformation of the bivector coordinate $x^{\mu\nu}$ is thus determined by the transformation of the vector coordinates x^μ. This is so because the basis bivectors are the wedge products of basis vectors γ_μ.

[4]This can be derived from the relation

$$\mathrm{d}E'_A = \frac{\partial E'_A}{\partial X'^B}\,\mathrm{d}X'^B \quad \text{where} \quad E'_A = \frac{\partial X^D}{\partial X'^A}E_D \quad \text{and} \quad \mathrm{d}X'^B = \frac{\partial X'^B}{\partial X^C}\mathrm{d}X^C$$

.

From (88) and (92) we see that $\tilde{\Gamma}^{\epsilon}_{[\rho\sigma]\gamma}$ transforms like a 4th-rank tensor in an ordinary space.

Comparing eq.(82) with the relation (61) we find

$$\frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} = [\partial_\alpha, \partial_\beta]\gamma_\mu \tag{93}$$

The derivative of a basis vector with respect to the bivector coordinates $x^{\alpha\beta}$ is equal to the commutator of the derivatives with respect to the vector coordinates x^α.

The above relation (93) holds for the basis vectors γ_μ. For an arbitrary polyvector

$$A = A^A E_A = s\gamma + a^\alpha \gamma_\alpha + a^{\alpha\beta}\gamma_\alpha \wedge \gamma_\beta + ... \tag{94}$$

we will assume the validity of the following relation

$$\frac{\mathrm{D}A^A}{\mathrm{D}x^{\mu\nu}} = [\mathrm{D}_\mu, \mathrm{D}_\nu]A^A \tag{95}$$

where $\mathrm{D}/\mathrm{D}x^{\mu\nu}$ is the covariant derivative, defined in analogous way as in eqs. (52):

$$\frac{\mathrm{D}A^A}{\mathrm{D}X^B} = \frac{\partial A^A}{\partial X^B} + \tilde{\Gamma}^A_{BC} A^C \tag{96}$$

From eq.(95) we obtain

$$\frac{\mathrm{D}s}{\mathrm{D}x^{\mu\nu}} = [\mathrm{D}_\mu, \mathrm{D}_\nu]s = K_{\mu\nu}{}^\rho \partial_\rho s \tag{97}$$

$$\frac{\mathrm{D}a^\alpha}{\mathrm{D}x^{\mu\nu}} = [\mathrm{D}_\mu, \mathrm{D}_\nu]a^\alpha = R_{\mu\nu\rho}{}^\alpha a^\rho + K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha \tag{98}$$

Using (96) we have that

$$\frac{\mathrm{D}s}{\mathrm{D}x^{\mu\nu}} = \frac{\partial s}{\partial x^{\mu\nu}} \tag{99}$$

and

$$\frac{\mathrm{D}a^\alpha}{\mathrm{D}x^{\mu\nu}} = \frac{\partial a^\alpha}{\partial x^{\mu\nu}} + \tilde{\Gamma}^\alpha_{[\mu\nu]\rho} a^\rho = \frac{\partial a^\alpha}{\partial x^{\mu\nu}} + R_{\mu\nu\rho}{}^\alpha a^\rho \tag{100}$$

where, according to (ii), $\tilde{\Gamma}^\alpha_{[\mu\nu]\rho}$ has been identified with curvature. So we obtain, after inserting (99),(100) into (97),(98) that

(a) the partial derivatives of the coefficients s and a^α, which are Clifford scalars[5], with respect to $x^{\mu\nu}$ are related to *torsion*:

$$\frac{\partial s}{\partial x^{\mu\nu}} = K_{\mu\nu}{}^\rho \partial_\rho s \tag{101}$$

$$\frac{\partial a^\alpha}{\partial x^{\mu\nu}} = K_{\mu\nu}{}^\rho \mathrm{D}_\rho a^\alpha \tag{102}$$

[5]In the geometric calculus based on Clifford algebra, the coefficients such as s, a^α, $a^{\alpha\beta}$, ..., are called *scalars* (although in tensor calculus they are called scalars, vectors and tensors, respectively), whilst the objects γ_α, $\gamma_\alpha \wedge \gamma_\beta$, ..., are called *vectors, bivectors*, etc..

(b) whilst the derivative of the basis vectors with respect to $x^{\mu\nu}$ are related to *curvature*:

$$\frac{\partial \gamma_\alpha}{\partial x^{\mu\nu}} = R_{\mu\nu\alpha}{}^{\beta} \gamma_\beta \tag{103}$$

In other words, the dependence of coefficients s and a^α on $x^{\mu\nu}$ indicates the presence of torsion. On the contrary, when basis vectors γ_α depend on $x^{\mu\nu}$ this indicates that the corresponding vector space has non vanishing curvature.

4.3 On the relation between the curvature of C-space and the curvature of an ordinary space

Let us now consider the C-space curvature defined in eq.(73) The indices A,B, can be of vector, bivector, etc., type. It is instructive to consider a particular example.

$A = [\mu\nu]$, $B = [\alpha\beta]$, $C = \gamma$, $D = \delta$

$$\left(\left[\frac{\partial}{\partial x^{\mu\nu}}, \frac{\partial}{\partial x^{\alpha\beta}} \right] \gamma_\gamma \right) \cdot \gamma^\delta = \mathcal{R}_{[\mu\nu][\alpha\beta]\gamma}{}^{\delta} \tag{104}$$

Using (82) we have

$$\frac{\partial}{\partial x^{\mu\nu}} \frac{\partial}{\partial x^{\alpha\beta}} \gamma_\gamma = \frac{\partial}{\partial x^{\mu\nu}} (R_{\alpha\beta\gamma}{}^{\rho} \gamma_\rho) = R_{\alpha\beta\gamma}{}^{\rho} R_{\mu\nu\rho}{}^{\sigma} \gamma_\sigma \tag{105}$$

where we have taken

$$\frac{\partial}{\partial x^{\mu\nu}} R_{\alpha\beta\gamma}{}^{\rho} = 0 \tag{106}$$

which is true in the case of vanishing torsion (see also an explanation that follows after the next paragraph). Inserting (105) into (104) we find

$$\mathcal{R}_{[\mu\nu][\alpha\beta]\gamma}{}^{\delta} = R_{\mu\nu\gamma}{}^{\rho} R_{\alpha\beta\rho}{}^{\delta} - R_{\alpha\beta\gamma}{}^{\rho} R_{\mu\nu\rho}{}^{\delta} \tag{107}$$

which is the product of two usual curvature tensors. We can proceed in analogous way to calculate the other components of $\mathcal{R}_{ABC}{}^{D}$ such as $\mathcal{R}_{[\alpha\beta\gamma\delta][\rho\sigma]\epsilon}{}^{\mu}$, $\mathcal{R}_{[\alpha\beta\gamma\delta][\rho\sigma\tau\kappa]\epsilon}{}^{[\mu\nu]}$, etc.. These contain higher powers of the curvature in an ordinary space. All this is true in our restricted C-space given by eqs.(79),(80) and the assumptions (i),(ii) bellow those equations. By releasing those restrictions we would have arrived at an even more involved situation which is beyond the scope of the present paper.

After performing the contractions of (107) and the corresponding higher order relations we obtain the expansion of the form

$$\mathcal{R} = R + \alpha_1 R^2 + \alpha_2 R_{\mu\nu} R^{\mu\nu} + ... \tag{108}$$

So we have shown that the C-space curvature can be expressed as the sum of the products of the ordinary spacetime curvature. This bears a resemblance to the string effective action in curved spacetimes given by sums of powers of the curvature tensors based on the quantization of non-linear sigma models [118].

If one sets aside the algebraic convergence problems when working with Clifford algebras in infinite dimensions, one can consider the possibility of studying Quantum Gravity in a very large number of dimensions which has been revisited recently [83] in connection to a perturbative renormalizable quantum theory of gravity in infinite dimensions. Another interesting possibility is that an infinite series expansion of the powers of the scalar curvature could yield the recently proposed modified Lagrangians $R + 1/R$ of gravity to accomodate the cosmological accelerated expansion of the Universe [131], after a judicious choice of the algebraic coefficients is taken. One may notice also that having a vanishing cosmological constant in C-space, $\mathcal{R} = \Lambda = 0$ does not necessarily imply that one has a vanishing cosmological constant in ordinary spacetime. For example, in the very special case of homogeneous symmetric spacetimes, like spheres and hyperboloids, where all the curvature tensors are proportional to suitable combinations of the metric tensor times the scalar curvature, it is possible to envision that the net combination of the sum of all the powers of the curvature tensors may cancel-out giving an overall zero value $\mathcal{R} = 0$. This possibility deserves investigation.

Let us now show that for vanishing torsion the curvature is independent of the bivector coordinates $x^{\mu\nu}$, as it was taken in eq.(106). Consider the basic relation

$$\gamma_\mu \cdot \gamma_\nu = g_{\mu\nu} \tag{109}$$

Differentiating with respect to $x^{\alpha\beta}$ we have

$$\frac{\partial}{\partial x^{\alpha\beta}}(\gamma_\mu \cdot \gamma_\nu) = \frac{\partial \gamma_\mu}{\partial x^{\alpha\beta}} \cdot \gamma_\nu + \gamma_\mu \cdot \frac{\partial \gamma_\nu}{\partial x^{\alpha\beta}} = R_{\alpha\beta\mu\nu} + R_{\alpha\beta\nu\mu} = 0 \tag{110}$$

This implies that

$$\frac{\partial g_{\mu\nu}}{\partial \sigma_{\alpha\beta}} = [\partial_\alpha, \partial_\beta] g_{\mu\nu} = 0 \tag{111}$$

Hence the metric, in this particular case, is independent of the holographic (bivector) coordinates. Since the curvature tensor —when torsion is zero— can be written in terms of the metric tensor and its derivatives, we conclude that not only the metric, but also the curvature is independent of $x^{\mu\nu}$. In general, when the metric has a dependence on the holographic coordinates one expects further corrections to eq.(107) that would include torsion.

5 On the Quantization in C-spaces

5.1 The momentum constraint in C-space

A detailed discussion of the physical properties of all the components of the polymomentum P in four dimensions and the emergence of the physical mass in Minkowski spacetime has been provided in the book [15]. The polymomentum in $D = 4$, canonically conjugate to the position polyvector

$$X = \sigma + x^\mu \gamma_\mu + \gamma^{\mu\nu}\, \gamma_\mu \wedge \gamma_\nu + \xi^\mu \gamma_5 \gamma_\mu + s\gamma_5 \tag{112}$$

can be written as:

$$P = \mu + p^\mu \gamma_\mu + S^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + \pi^\mu \gamma_5 \gamma_\mu + m\gamma_5. \tag{113}$$

where besides the vector components p^μ we have the scalar component μ, the 2-vector components $S^{\mu\nu}$, that are connected to the spin as shown by [14]; the pseudovector components π^μ and the pseudoscalar component m.

The most salient feature of the polyparticle dynamics in C-spaces [15] is that one can start with a *constrained* action in C-space and arrive, nevertheless, at an *unconstrained* Stuckelberg action in Minkowski space (a subspace of C-space) in which $p_\mu p^\mu$ is a constant of motion. The true constraint in C-space is:

$$P_A P^A = \mu^2 + p_\mu p^\mu - 2S^{\mu\nu} S_{\mu\nu} + \pi_\mu \pi^\mu - m^2 = M^2. \tag{114}$$

where M is a *fixed* constant, the mass in C-space. The pseudoscalar component m is a variable, like μ, p_μ, $S^{\mu\nu}$, and π^μ, which altogether are constrained according to eq.(114). It becomes the physical mass in Minkwoski spacetime in the special case when other extra components vanish, i.e., when $\mu = 0$, $S^{\mu\nu} = 0$ and $\pi^\mu = 0$. This justifies using the notation m for mass. This is basically the distinction between the mass in Minkowski space which is a constant of motion $p_\mu p^\mu$ and the fixed mass M in C-space. The variable m is canonically conjugate to s which acquires the role of the Stuckelberg evolution parameter s that allowed ref.[29, 15] to propose a natural solution of the problem of time in quantum gravity.

A derivation of a charge, mass, and spin relationship of a polyparticle can be obtained from the above polymomentum constraint in C-space if one relates the norm of the axial-momentum component π^μ of the polymomentum P to the charge [80]. It agrees exactly with the recent charge-mass-spin relationship obtained by [44] based on the Kerr-Newman black hole metric solutions of the Einstein-Maxwell equations. The naked singularity Kerr-Newman solutions have been interpreted by [45] as Dirac particles. Further investigation is needed to understand better these relationships, in particular, the deep reasons behind the *charge* assignment to the norm of the axial-vector π^μ component of the polymomentum which suggests that mass has a gravitational, electromagnetic and rotational aspects to it. In a Kaluza-Klein reduction from $D = 5$ to $D = 4$ it is well known that the electric charge is related to the p_5 component of the momentum. Hence, charge bears a connection to an internal momentum.

5.2 C-space Klein-Gordon and Dirac Wave Equations

The ordinary Klein-Gordon equation can be easily obtained by implementing the on-shell constraint $p^2 - m^2 = 0$ as an operator constraint on the physical states after replacing p_μ for $-i\partial/\partial x^\mu$ (we use units in which $\hbar = 1, c = 1$):

$$\left(\frac{\partial^2}{\partial x^\mu \partial x_\mu} + m^2 \right) \phi = 0. \tag{115}$$

The C-space generalization follows from the $P^2 - M^2 = 0$ condition by replacing

$$P_A \to -i \frac{\partial}{\partial X^A} = -i \left(\frac{\partial}{\partial \sigma}, \frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^{\mu\nu}}, ... \right) \tag{116}$$

$$\left(\frac{\partial^2}{\partial \sigma^2} + \frac{\partial^2}{\partial x^\mu \partial x_\mu} + \frac{\partial^2}{\partial x^{\mu\nu} \partial x_{\mu\nu}} + ... + M^2 \right) \Phi = 0 \tag{117}$$

where we have set $L = \hbar = c = 1$ for convenience purposes and the C-space scalar field $\Phi(\sigma, x^\mu, x^{\mu\nu}, ...)$ is a polyvector-valued *scalar* function of *all* the C-space variables. This is the Klein-Gordon equation associated with a free scalar polyparticle in C-space.

A wave equation for a generalized C-space harmonic oscillator requires to introduce the potential of the form $V = \kappa X^2$ that admits straightforward solutions in terms of Gaussians and Hermite polynomials similar to the ordinary point-particle oscillator. There are now collective excitations of the Clifford-oscillator in terms of the number of Clifford-bits and which represent the quanta of areas, volumes, hypervolumes,..., associated with the p-loops oscillations in Planck scale units. The logarithm of the degeneracy of the first collective state of the C-space oscillator, as a function of the number of bits, bears the same functional form as the Bekenstein-Hawking black hole entropy, with the upshot that one recovers, in a natural way, the logarithmic corrections to the black-hole entropy as well, if one identifies the number of Clifford-bits with the number of area-quanta of the black hole horizon. For further details about this derivation and the emergence of the Schwarzschild horizon radius relation, the Hawking temperature, the maximal Planck temperature condition, etc., we refer to [21]. Perhaps the most important consequence of this latter view of black hole entropy is the possibility that there is a ground state of quantum spacetime, resulting from of a Bose-Einstein condensate of the C-space harmonic oscillator.

A C-space version of the Dirac Equation, representing the dynamics of spinning-polyparticles (theories of extended-spin, extended charges) is obtained via the square-root procedure of the Klein-Gordon equation:

$$-i \left(\frac{\partial}{\partial \sigma} + \gamma^\mu \frac{\partial}{\partial x_\mu} + \gamma^\mu \wedge \gamma^\nu \frac{\partial}{\partial x_{\mu\nu}} + ... \right) \Psi = M \Psi \tag{118}$$

where $\Psi(\sigma, x^\mu, x^{\mu\nu}, ...)$ is a polyvector-valued function, a Clifford-number, $\Psi = \Psi^A E_A$ of *all* the C-space variables. For simplicity we consider here a *flat C*-space in which the metric $G_{AB} = E_A^\dagger * E_B = \eta_{AB}$ is diagonal, η_{AB} being the C-space analog of Minkowski tensor. In curved C-space the equation (118) should be properly generalized. This goes beyond the scope of the present paper.

Ordinary spinors are nothing but elements of the left/right ideals of a Clifford algebra. So they are automatically contained in the polyvector valued wave function Ψ. The ordinary Dirac equation can be obtained when Ψ is independent of the extra variables associated with a polyvector-valued coordinates X (i.e., of $x^{\mu\nu},\ x^{\mu\nu\rho}, ...$). For details see [15].

Thus far we have written ordinary wave equations in C-space, that is, we considered the wave equations for a "point particle" in C-space. From the perspective of the 4-dimensional Minkowski spacetime the latter "point particle" has, of course, a much richer structure then a mere point: it is an extended object, modeled by coordinates $x^\mu,\ x^{\mu\nu}, ...$ But such modeling does not embrace all the details of an extended object. In order to provide a description with more details, one can considere not the "point particles" in C-space, but *branes* in C-space. They are described by the embeddings $X = X(\Sigma)$, that is $X^M = X^M(\Sigma^A)$, considered in sec.3.2. Quantization of such branes can employ wave functional equation, or other methods, including the second quantization formalism. For a more detailed study detailed study of the second quantization of extended objects using the tools of Clifford algebra see [15].

Without emplying Clifford algebra a lot of illuminating work has been done in relation to description of branes in terms of p-loop coordinates [132]. A bosonic/fermionic p-brane wave-functional equation was presented in [12], generalizing the closed-string(loop) results in [13] and the the quantum bosonic p-brane propagator, in the quenched-reduced minisuperspace approximation, was attained by [18]. In the latter work branes are described in terms of the collective coordinates which are just the highest grade components in the expansion of a poplyvector X given in eq (2). This work thus paved the way for the next logical step, that is, to consider other multivector components of X in a unified description of all branes.

Notice that the approach based on eqs.(117),(118) is different from that by Hestenes [1] who proposed an equation which is known as the Dirac-Hestenes equation. As far as we know the first person to derive Dirac's equation using quaternions (related to Clifford algebra) was Lanczos [91]. Later on the Dirac-Lanczos equation was rediscovered by many people, in particular by Hestenes and Gursey [92]. The former Dirac-Lanczos equation is Lorentz covariant despite the fact that it singles out an arbitrary but unique direction in ordinary space: the *spin* quantization axis. Lanczos, without knowing, had anticipated the existence of isospin as well.

In the Dirac-Hestenes equation instead of the imaginary unit i there occurs the bivector $\gamma_1\gamma_2$. Its square is -1 and it commutes with all the elements of the Dirac algebra which is just a desired property. But on the other hand, the introduction of a bivector into an equation of motion implies a selection of a preferred orientation in spacetime. A question arises of why the preferred orientation defined by $\gamma_1\gamma_2$ and not some other preferred orientation $a \wedge b$ defined by orthonormal vectors a and b which are linear superpositions of γ_μ? How such a preferred orientation is determined? Is there some hidden dynamical principle which determines the preferred orientation? What is the more fundamental action which incorporates a kinetic term for the vectors a and b?

Many subtleties of the Dirac-Hesteness equation and its relation to the ordinary Dirac equation are investigated from the rigorous mathematical point of view in refs. [93]. The approach in refs. [16, 15, 17, 8], reviewed here, is different. We start from the usual formulation of quantum theory and extend it to C-space. We retain the imaginary unit i. Next step is to give a geometric interpretation to i. Instead of trying to find a geometric origin of i in *spacetime* we adopt the interpretation proposed in [15] according to which the i is the bivector of the 2-dimensional *phase space* (whose direct product with the n-dimensional configuration space gives the $2n$-dimensional phase space). [6] This appears to be a natural assumption due to the fact that complex valued quantum mechanical wave functions involve momenta p_μ and coordinates x^μ (e.g., a plane wave is given by $\exp[ip_\mu x^\mu]$, and arbitrary wave packet is a superposition of plane waves).

[6] Yet another interpretation of the imaginary unit i present in the Heisenberg uncertainty relations has been undertaken by Finkelstein and collaborators [96].

6 Maximal-Acceleration Relativity in Phase-Spaces

In this section we shall discuss the maximal acceleration Relativity principle [68] based on Finsler geometry which does not destroy, nor deform, Lorentz invariance. Our discussion differs from the pseudo-complex Lorentz group description by Schuller [61] related to the effects of maximal acceleration in Born-Infeld models that also maintains Lorentz invariance, in contrast to the approaches of Double Special Relativity (DSR). In addition one does not need to modify the energy-momentum addition (conservation) laws in the scattering of particles which break translational invariance. For a discussions on the open problems of Double Special Relativity theories based on kappa-deformed Poincare symmetries [63] and motivated by the anomalous Lorentz-violating dispersion relations in the ultra high energy cosmic rays [71, 72, 73], we refer to [70].

Related to the minimal Planck scale, an upper limit on the maximal acceleration principle in Nature was proposed by long ago Cainello [52]. This idea is a direct consequence of a suggestion made years earlier by Max Born on a Dual Relativity principle operating in phase spaces [49], [74] wherethere is an upper bound on the four-force (maximal string tension or tidal forces in the string case) acting on a particle as well as an upper bound in the particle velocity. One can combine the maximum speed of light with a minimum Planck scale into a maximal proper-accleration $a = c^2/L =$ within the framework of Finsler geometry [56]. For a recent status of the geometries behind maximal-acceleration see [73]; its relation to the Double Special Relativity programs was studied by [55] and the possibility that Moyal deformations of Poincare algebras could be related to the kappa-deformed Poincare algebras was raised in [68]. A thorough study of Finsler geometry and Clifford algebras has been undertaken by Vacaru [81].

Other several new physical implications of the maximal acceleration principle in Nature, like neutrino oscillations and other phenomena, have been studied by [54], [67], [42]. Recently, the variations of the fine structure constant α [64], with the cosmological accelerated expansion of the Universe, was recast as a renormalization group-like equation governing the cosmological reshift (Universe scale) variations of α based on this maximal acceleration principle in Nature [68]. The fine structure constant was smaller in the past. Pushing the cuttof scale to the minimum Planck scale led to the intriguing result that the fine structure constant could have been extremely small (zero) in the early Universe and that all matter in the Universe could have emerged via the Unruh-Rindler-Hawking effect (creation of radiation/matter) due to the acceleration w.r.t the vacuum frame of reference. For reviews on the alledged variations of the fundamental constants in Nature see [65] and for more astonishing variations of αdriven by quintessence see [66].

6.1 Clifford algebras in Phase space

We shall employ the procedure described in [15] to construct the Phase Space Clifford algebra that allowed [127] to reproduce the sub-maximally accelerated particle action of [53].

For simplicity we will focus on a two-dim phase space. Let e_p, e_q be the Clifford-algebra basis elements in a two-dim phase space obeying the following relations [15]:

$$e_p.e_q \equiv \frac{1}{2}(e_q e_p + e_p e_q) = 0. \tag{119}$$

and $e_p.e_p = e_q.e_q = 1$.

The Clifford product of e_p, e_q is by definition the sum of the scalar and the wedge product:

$$e_p e_q = e_p.e_q + e_p \wedge e_q = 0 + e_p \wedge e_q = i. \tag{120}$$

such that $i^2 = e_p e_q e_p e_q = -1$. Hence, the imaginary unit i, $i^2 = -1$ admits a very natural interpretation in terms of Clifford algebras, i.e., it is represented by the wedge product $i = e_p \wedge e_q$, a phase-space area element. Such imaginary unit allows us to express vectors in a C-phase space in the form:

$$\begin{aligned} Q &= Q = q e_q + p e_q \\ Q e_q &= q + p e_p e_q = q + ip = z \\ e_q Q &= q + p e_q e_p = q - ip = z^* \end{aligned} \tag{121}$$

which reminds us of the creation/anihilation operators used in the harmonic oscillator.

We shall now review the steps in [127] to reproduce the sub-maximally accelerated particle action [53]. The phase-space analog of the spacetime action is:

$$dQ.dQ = (dq)^2 + (dp)^2 \Rightarrow S = m \int \sqrt{(dq)^2 + (dp)^2}. \tag{122}$$

Introducing the appropriate length/mass scale parameters in order to have consistent units yields:

$$S = m \int \sqrt{(dq)^2 + (\frac{L}{m})^2 (dp)^2}. \tag{123}$$

where we have introduced the Planck scale L and have chosen the natural units $\hbar = c = 1$. A detailed physical discussion of the dilational invariant system of units $\hbar = c = G = 4\pi\epsilon_o = 1$ was presented in ref. [15]. G is the Newton constant and ϵ_o is the permitivity of the vacuum.

Extending this two-dim result to a $2n$-dim phase space result requires to have for Clifford basis the elements e_{p_μ}, e_{q_μ}, where $\mu = 1, 2, 3, ...n$. The action in the $2n$-dim phase space is:

$$S = m \int \sqrt{(dq^\mu dq_\mu) + (\frac{L}{m})^2 (dp^\mu dp_\mu)} = m \int d\tau \sqrt{1 + (\frac{L}{m})^2 (dp^\mu/d\tau)(dp_\mu/d\tau)}. \tag{124}$$

where we have factored-out of the square-root the infinitesimal proper-time displacement $(d\tau)^2 = dq^\mu dq_\mu$.

One can reccognize the action (124), up to a numerical factor of m/a, where a is the proper acceleration, as the same action for a sub-maximally accelerated particle given by Nesterenko [53] by rewriting $(dp^\mu/d\tau) = m(d^2x^\mu/d\tau^2)$:

$$S = m \int d\tau \sqrt{1 + L^2 (d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)}. \tag{125}$$

Postulating that the maximal proper-acceleration is given in terms of the speed of light and the minimal Planck scale by $a = c^2/L = 1/L$, the action above gives the Nesterenko action, up to a numerical m/a factor:

$$S = m \int d\tau \sqrt{1 + a^{-2}(d^2x^\mu/d\tau^2)(d^2x_\mu/d\tau^2)}. \tag{126}$$

The proper-acceleration is orthogonal to the proper-velocity and this can be easily verified by differentiating the timelike proper-velocity squared:

$$V^2 = \frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} = V^\mu V_\mu = 1 > 0 \Rightarrow \frac{dV^\mu}{d\tau}V_\mu = \frac{d^2x^\mu}{d\tau^2}V_\mu = 0. \tag{127}$$

which implies that the proper-acceleration is spacelike:

$$g^2(\tau) = -\frac{dx^\mu}{d\tau}\frac{dx_\mu}{d\tau} > 0 \Rightarrow S = m\int d\tau\sqrt{1 - \frac{g^2}{a^2}} = m\int d\omega. \tag{128}$$

where the analog of the Lorentz time-dilation factor for a sub-maximally accelerated particle is given by

$$d\omega = d\tau\sqrt{1 - \frac{g^2(\tau)}{a^2}}. \tag{129}$$

Therefore the dynamics of a sub-maximally accelerated particle can be reinterpreted as that of a particle moving in the spacetime tangent bundle whose Finsler-like metric is

$$(d\omega)^2 = g_{\mu\nu}(x^\mu, dx^\mu)dx^\mu dx^\nu = (d\tau)^2(1 - \frac{g^2(\tau)}{a^2}). \tag{130}$$

The invariant time now is no longer the standard proper-time τ but is given by the quantity $\omega(\tau)$. The deep connection between the physics of maximal acceleration and Finsler geometry has been analyzed by [56]. This sort of actions involving second derivatives have also been studied in the construction of actions associated with rigid particles (strings) [57], [58], [59], [60] among others.

The action is real-valued if, and only if, $g^2 < a^2$ in the same fashion that the action in Minkowski spacetime is real-valued if, and only if, $v^2 < c^2$. This is the physical reason why there is an upper bound in the proper-acceleration. In the special case of uniformly-accelerated motion $g(\tau) = g_o = constant$, the trajectory of the particle in Minkowski spacetime is a hyperbola.

6.2 Invariance under the $U(1,3)$ Group

In this section we will review in detail the principle of Maximal-acceleration Relativity [68] from the perspective of $8D$ Phase Spaces and the $U(1,3)$ Group. The $U(1,3) = SU(1,3) \otimes U(1)$ Group transformations, which leave invariant the phase-space intervals under rotations, velocity and acceleration boosts, were found by Low [74] and can be simplified drastically when the velocity/acceleration boosts are taken to lie in the z-direction, leaving the transverse directions x, y, p_x, p_y intact ; i.e., the $U(1,1) = SU(1,1)\otimes$

$U(1)$ subgroup transformations that leave invariant the phase-space interval are given by (in units of $\hbar = c = 1$)

$$(d\sigma)^2 = (dT)^2 - (dX)^2 + \frac{(dE)^2 - (dP)^2}{b^2} =$$

$$(d\tau)^2[1 + \frac{(dE/d\tau)^2 - (dP/d\tau)^2}{b^2}] = (d\tau)^2[1 - \frac{m^2 g^2(\tau)}{m_P^2 A_{max}^2}]. \tag{131}$$

where we have factored out the proper time infinitesimal $(d\tau)^2 = dT^2 - dX^2$ in eq.(131) and the maximal proper-force is set to be $b \equiv m_P A_{max}$. m_P is the Planck mass $1/L_P$ so that $b = (1/L_P)^2$, may also be interpreted as the maximal string tension when L_P is the Planck scale.

The quantity $g(\tau)$ is the proper four-acceleration of a particle of mass m in the z-direction which we take to be X. Notice that the invariant interval $(d\sigma)^2$ in eq.(131) is not strictly the *same* as the interval $(d\omega)^2$ of the Nesterenko action eq.(126), which was invariant under a pseudo-complexification of the Lorentz group [61]. Only when $m = m_P$, the two intervals agree. The interval $(d\sigma)^2$ described by Low [74] is $U(1,3)$-invariant for the most general transformations in the $8D$ phase-space. These transformatiosn are rather elaborate, so we refer to the references [74] for details. The analog of the Lorentz relativistic factor in eq.(131) involves the ratios of two proper *forces*. One variable force is given by ma and the maximal proper force sustained by an *elementary* particle of mass m_P (a *Planckton*) is assumed to be $F_{max} = m_{Planck} c^2/L_P$. When $m = m_P$, the ratio-squared of the forces appearing in the relativistic factor of eq.(131) becomes then g^2/A_{max}^2, and the phase space interval (131) coincides with the geometric interval of (126).

The transformations laws of the coordinates in that leave invariant the interval (131) are [74]:

$$T' = T cosh\xi + (\xi_v X + \frac{\xi_a P}{b^2})\frac{sinh\xi}{\xi}. \tag{132}$$

$$E' = E cosh\xi + (-\xi_a X + \xi_v P)\frac{sinh\xi}{\xi}. \tag{133}$$

$$X' = X cosh\xi + (\xi_v T - \frac{\xi_a E}{b^2})\frac{sinh\xi}{\xi}. \tag{134}$$

$$P' = P cosh\xi + (\xi_v E + \xi_a T)\frac{sinh\xi}{\xi}. \tag{135}$$

The ξ_v is velocity-boost rapidity parameter and the ξ_a is the force/acceleration-boost rapidity parameter of the primed-reference frame. They are defined respectively (in the special case when $m = m_P$):

$$tanh\frac{\xi_v}{c} = \pm\frac{v}{c}$$

$$tanh\frac{\xi_a}{b} = \pm\frac{ma}{m_P A_{max}} = \frac{a}{A_{max}} \tag{136}$$

The *effective* boost parameter ξ of the $U(1,1)$ subgroup transformations appearing in eqs.(132)–(135) is defined in terms of the velocity and acceleration boosts parameters ξ_v, ξ_a respectively as:

$$\xi \equiv \sqrt{\xi_v^2 + \frac{\xi_a^2}{b^2}}. \tag{137}$$

Our definition of the rapidity parameters are *different* than those in [74].

Straightforward algebra allows us to verify that these transformations leave the interval of eq.(131) in classical phase space invariant. They are are fully consistent with Born's duality Relativity symmetry principle [49] $(Q, P) \rightarrow (P, -Q)$. By inspection we can see that under Born duality, the transformations in eqs.(132)–(135) are *rotated* into each other, up to numerical b factors in order to match units. When on sets $\xi_a = 0$ in (132)–(135) one recovers automatically the standard Lorentz transformations for the X, T and E, P variables *separately*, leaving invariant the intervals $dT^2 - dX^2 = (d\tau)^2$ and $(dE^2 - dP^2)/b^2$ separately.

When one sets $\xi_v = 0$ we obtain the transformations rules of the events in Phase space, from one reference-frame into another $uniformly$-accelerated frame of reference, $a = constant$, whose acceleration-rapidity parameter is in this particular case:

$$\xi \equiv \frac{\xi_a}{b}.\ tanh\xi = \frac{a}{A_{max}}. \tag{138}$$

The transformations for pure acceleration-boosts in are:

$$T' = Tcosh\xi + \frac{P}{b} sinh\xi. \tag{139}$$

$$E' = Ecosh\xi - bXsinh\xi. \tag{140}$$

$$X' = Xcosh\xi - \frac{E}{b} sinh\xi \tag{141}$$

$$P' = Pcosh\xi + bTsinh\xi. \tag{142}$$

It is straightforwad to verify that the transformations (139)–(141) leave invariant the fully phase space interval (131) but *does not* leave invariant the proper time interval $(d\tau)^2 = dT^2 - dX^2$. Only the *combination*:

$$(d\sigma)^2 = (d\tau)^2 (1 - \frac{m^2 g^2}{m_P^2 A_{max}^2}) \tag{143}$$

is truly left invariant under pure acceleration-boosts (139)–(141). One can verify as well that these transformations satisfy Born's duality symmetry principle:

$$(T, X) \rightarrow (E, P).\ (E, P) \rightarrow (-T, -X). \tag{144}$$

and $b \rightarrow \frac{1}{b}$. The latter Born duality transformation is nothing but a manifestation of the large/small tension duality principle reminiscent of the T-duality symmetry in string theory; i.e. namely, a small/large radius duality, a winding modes/ Kaluza-Klein modes duality symmetry in string compactifications and the Ultraviolet/Infrared entanglement in Noncommutative Field Theories. Hence, Born's duality principle in exchanging coordinates for momenta could be the underlying physical reason behind T-duality in string theory.

The composition of two succesive pure acceleration-boosts is another pure acceleration-boost with acceleration rapidity given by $\xi'' = \xi + \xi'$. The addition of *proper* accelerations follows the usual relativistic composition rule:

$$tanh\xi'' = tanh(\xi + \xi') = \frac{tanh\xi + \tanh\xi'}{1 + tanh\xi tanh\xi'} \Rightarrow \frac{a''}{A} = \frac{\frac{a}{A} + \frac{a'}{A}}{1 + \frac{aa'}{A^2}}. \qquad (145)$$

and in this fashion the upper limiting *proper* acceleration is never *surpassed* like it happens with the ordinary Special Relativistic addition of velocities.

The group properties of the full combination of velocity and acceleration boosts (132)–(135) requires much more algebra [68]. A careful study reveals that the composition *rule* of two succesive full transformations is given by $\xi'' = \xi + \xi'$ and the transformation laws are *preserved* if, and only if, the $\xi; \xi'; \xi''$...... parameters obeyed the suitable relations:

$$\frac{\xi_a}{\xi} = \frac{\xi'_a}{\xi'} = \frac{\xi''_a}{\xi''} = \frac{\xi''_a}{\xi + \xi'}. \qquad (146)$$

$$\frac{\xi_v}{\xi} = \frac{\xi'_v}{\xi'} = \frac{\xi''_v}{\xi''} = \frac{\xi''_v}{\xi + \xi'}. \qquad (147)$$

Finally we arrive at the compostion law for the effective, velocity and acceleration boosts parameters $\xi''; \xi''_v; \xi''_a$ respectively:

$$\xi''_v = \xi_v + \xi'_v. \qquad (148)$$

$$\xi''_a = \xi_a + \xi'_a. \qquad (149)$$

$$\xi'' = \xi + \xi'. \qquad (150)$$

The relations (146, 147, 148, 149, 150) are required in order to prove the *group* composition law of the transformations of (132)–(135) and, consequently, in order to have a truly Maximal-Acceleration Phase Space Relativity theory resulting from a phase-space change of coordinates in the cotangent bundle of spacetime.

6.3 Planck-Scale Areas are Invariant under Acceleration Boosts

Having displayed explicity the Group transformations rules of the coordinates in Phase space we will show why *infinite* acceleration-boosts (which is *not* the same as infinite proper acceleration) preserve Planck-Scale *Areas* [68] as a result of the fact that $b = (1/L_P^2)$ equals the *maximal* invariant force, or string tension, if the units of $\hbar = c = 1$ are used.

At Planck-scale L_P intervals/increments in one reference frame we have by definition (in units of $\hbar = c = 1$): $\Delta X = \Delta T = L_P$ and $\Delta E = \Delta P = \frac{1}{L_P}$ where $b \equiv \frac{1}{L_P^2}$ is the maximal tension. From eqs.(132)–(135) we get for the transformation rules of the finite intervals $\Delta X, \Delta T, \Delta E, \Delta P$, from one reference frame into another frame, in the $infinite$ acceleration-boost limit $\xi \to \infty$,

$$\Delta T' = L_P(cosh\xi + sinh\xi) \to \infty \tag{151}$$

$$\Delta E' = \frac{1}{L_P}(cosh\xi - sinh\xi) \to 0 \tag{152}$$

by a simple use of L'Hopital's rule or by noticing that both $cosh\xi; sinh\xi$ functions approach infinity at the same rate.

$$\Delta X' = L_P(cosh\xi - sinh\xi) \to 0. \tag{153}$$

$$\Delta P' = \frac{1}{L_P}(cosh\xi + sinh\xi) \to \infty \tag{154}$$

where the discrete displacements of two events in Phase Space are defined: $\Delta X = X_2 - X_1 = L_P$, $\Delta E = E_2 - E_1 = \frac{1}{L_P}$, $\Delta T = T_2 - T_1 = L_P$ and $\Delta P = P_2 - P_1 = \frac{1}{L_P}$.

Due to the identity:

$$\infty \times 0 = (cosh\xi + sinh\xi)(cosh\xi - sinh\xi) = cosh^2\xi - sinh^2\xi = 1 \tag{155}$$

one can see from eqs. (151)–(154) that the Planck-scale *Areas* are truly *invariant* under *infinite* acceleration-boosts $\xi = \infty$:

$$\Delta X'\Delta P' = 0 \times \infty = \Delta X \Delta P(cosh^2\xi - sinh^2\xi) = \Delta X \Delta P = \frac{L_P}{L_P} = 1. \tag{156}$$

$$\Delta T'\Delta E' = \infty \times 0 = \Delta T \Delta E(cosh^2\xi - sinh^2\xi) = \Delta T \Delta E = \frac{L_P}{L_P} = 1. \tag{157}$$

$$\Delta X'\Delta T' = 0 \times \infty = \Delta X \Delta T(cosh^2\xi - sinh^2\xi) = \Delta X \Delta T = (L_P)^2. \tag{158}$$

$$\Delta P'\Delta E' = \infty \times 0 = \Delta P \Delta E(cosh^2\xi - sinh^2\xi) = \Delta P \Delta E = \frac{1}{L_P^2}. \tag{159}$$

It is important to emphasize that the invariance property of the minimal Planck-scale *Areas* (maximal Tension) is *not* an exclusive property of $infinite$ acceleration boosts $\xi = \infty$, but, as a result of the identity $cosh^2\xi - sinh^2\xi = 1$, for all values of ξ, the minimal Planck-scale *Areas* are *always* invariant under *any* acceleration-boosts transformations. Meaning physically, in units of $\hbar = c = 1$, that the Maximal Tension (or maximal Force) $b = \frac{1}{L_P^2}$ is a true physical *invariant* universal quantity. Also we notice that the Phase-space areas, or cells, in units of $\hbar$, are also invariant ! The pure-acceleration boosts transformations are " symplectic ". It can be shown also that areas greater (smaller) than the Planck-area remain greater (smaller) than the invariant Planck-area under acceleration-boosts transformations.

The infinite acceleration-boosts are closely related to the infinite red-shift effects when light signals barely escape Black hole Horizons reaching an asymptotic observer with an infinite redshift factor. The important fact is that the Planck-scale *Areas* are truly maintained invariant under acceleration-boosts. This could reveal very important information about Black-holes Entropy and Holography. The logarithimic corrections to the Black-Hole Area-Entropy relation were obtained directly from Clifford-algebraic methods in C-spaces [21], in addition to the derivation of the maximal Planck temperature condition and the Schwarzchild radius in terms of the Thermodynamicsof a gas of p-loop-oscillatorsquanta represented by area-bits, volume-bits, ... hyper-volume-bits in Planck scale units. Minimal loop-areas, in Planck units, is also one of the most important consequences found in Loop Quantum Gravity long ago [111].

7 Some Further Important Physical Applications Related to the C-Space Physics

7.1 Relativity of signature

In previous sections we have seen how Clifford algebra can be used in the formulation of the point particle classical and quantum theory. The metric of spacetime was assumed, as usually, to have the Minkowski signature, and we have used the choice $(+---)$. There were arguments in the literature of why the spacetime signature is of the Minkowski type [113, 43]. But there are also studies in which signature changes are admitted [112]. It has been found out [16, 15, 30] that within Clifford algebra the signature of the underlying space is a matter of choice of basis vectors amongst available Clifford numbers. We are now going to review those important topics.

Suppose we have a 4-dimensional space V_4 with signature $(++++)$. Let e_μ, $\mu = 0, 1, 2, 3$, be basis vectors satisfying

$$e_\mu \cdot e_\nu \equiv \tfrac{1}{2}(e_\mu e_\nu + e_\nu e_\mu) = \delta_{\mu\nu} \ , \tag{160}$$

where $\delta_{\mu\nu}$ is the *Euclidean signature* of V_4. The vectors e_μ can be used as generators of Clifford algebra $\mathcal{C}_4$ over V_4 with a generic Clifford number (also called polyvector or Clifford aggregate) expanded in term of $e_J = (1, e_\mu, e_{\mu\nu}, e_{\mu\nu\alpha}, e_{\mu\nu\alpha\beta})$, $\mu < \nu < \alpha < \beta$,

$$A = a^J e_J = a + a^\mu e_\mu + a^{\mu\nu} e_\mu e_\nu + a^{\mu\nu\alpha} e_\mu e_\nu e_\alpha + a^{\mu\nu\alpha\beta} e_\mu e_\nu e_\alpha e_\beta. \tag{161}$$

Let us consider the set of four Clifford numbers $(e_0,\, e_i e_0)$, $i = 1, 2, 3$, and denote them as

$$\begin{aligned} e_0 &\equiv \gamma_0, \\ e_i e_0 &\equiv \gamma_i. \end{aligned} \tag{162}$$

The Clifford numbers γ_μ, $\mu = 0, 1, 2, 3$, satisfy

$$\tfrac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu) = \eta_{\mu\nu} \ , \tag{163}$$

where $\eta_{\mu\nu} = \mathrm{diag}(1,-1,-1,-1)$ is the *Minkowski tensor*. We see that the γ_μ behave as basis vectors in a 4-dimensional space $V_{1,3}$ with signature $(+---)$. We can form a Clifford aggregate

$$\alpha = \alpha^\mu \gamma_\mu \tag{164}$$

which has the properties of a *vector* in $V_{1,3}$. From the point of view of the space V_4 the same object α is a linear combination of a vector and bivector:

$$\alpha = \alpha^0 e_0 + \alpha^i e_i e_0. \tag{165}$$

We may use γ_μ as generators of the Clifford algebra $\mathcal{C}_{1,3}$ defined over the pseudo-Euclidean space $V_{1,3}$. The basis elements of $\mathcal{C}_{1,3}$ are $\gamma_J = (1, \gamma_\mu, \gamma_{\mu\nu}, \gamma_{\mu\nu\alpha}, \gamma_{\mu\nu\alpha\beta})$, with $\mu < \nu < \alpha < \beta$. A generic Clifford aggregate in $\mathcal{C}_{1,3}$ is given by

$$B = b^J \gamma_J = b + b^\mu \gamma_\mu + b^{\mu\nu} \gamma_\mu \gamma_\nu + b^{\mu\nu\alpha} \gamma_\mu \gamma_\nu \gamma_\alpha + b^{\mu\nu\alpha\beta} \gamma_\mu \gamma_\nu \gamma_\alpha \gamma_\beta. \tag{166}$$

With suitable choice of the coefficients $b^J = (b, b^\mu, b^{\mu\nu}, b^{\mu\nu\alpha}, b^{\mu\nu\alpha\beta})$ we have that B of eq. (166) is equal to A of eq.(161). Thus the same number A can be described either with e_μ which generate $\mathcal{C}_4$, or with γ_μ which generate $\mathcal{C}_{1,3}$. The expansions (166) and (161) exhaust all possible numbers of the Clifford algebras $\mathcal{C}_{1,3}$ and $\mathcal{C}_4$. Those expansions are just two different representations of the same set of Clifford numbers (also being called polyvectors or Clifford aggregates).

As an alternative to (162) we can choose

$$\begin{aligned} e_0 e_3 &\equiv \tilde{\gamma}_0, \\ e_i &\equiv \tilde{\gamma}_i, \end{aligned} \tag{167}$$

from which we have

$$\tfrac{1}{2}(\tilde{\gamma}_\mu \tilde{\gamma}_\nu + \tilde{\gamma}_\nu \tilde{\gamma}_\mu) = \tilde{\eta}_{\mu\nu} \tag{168}$$

with $\tilde{\eta}_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$. Obviously $\tilde{\gamma}_\mu$ are basis vectors of a pseudo-Euclidean space $\widetilde{V}_{1,3}$ and they generate the Clifford algebra over $\widetilde{V}_{1,3}$ which is yet another representation of the same set of objects (i.e., polyvectors). The spaces V_4, $V_{1,3}$ and $\widetilde{V}_{1,3}$ are different slices through C-space, and they span different subsets of polyvectors. In a similar way we can obtain spaces with signatures $(+-++)$, $(++-+)$, $(+++-)$, $(-+--)$, $(--+-)$, $(---+)$ and corresponding higher dimensional analogs. But we cannot obtain signatures of the type $(++--)$, $(+-+-)$, etc. In order to obtain such signatures we proceed as follows.

4-space. First we observe that the bivector $\bar{I} = e_3 e_4$ satisfies $\bar{I}^2 = -1$, commutes with e_1, e_2 and anticommutes with e_3, e_4. So we obtain that the set of Clifford numbers $\gamma_\mu = (e_1 \bar{I}, e_2 \bar{I}, e_3, e_3)$ satisfies

$$\gamma_\mu \cdot \gamma_\nu = \bar{\eta}_{\mu\nu} \ , \tag{169}$$

where $\bar{\eta} = \mathrm{diag}(-1,-1,1,1)$.

8-space. Let e_A be basis vectors of 8-dimensional vector space with signature $(+++++++ +)$. Let us decompose

$$e_A = (e_\mu, e_{\bar{\mu}}) \ , \qquad \begin{aligned} \mu &= 0,1,2,3, \\ \bar{\mu} &= \bar{0},\bar{1},\bar{2},\bar{3}. \end{aligned} \tag{170}$$

The inner product of two basis vectors

$$e_A \cdot e_B = \delta_{AB}, \tag{171}$$

then splits into the following set of equations:

$$\begin{aligned} e_\mu \cdot e_\nu &= \delta_{\mu\nu}\ , \\ e_{\bar\mu} \cdot e_{\bar\nu} &= \delta_{\bar\mu\bar\nu}\ , \\ e_\mu \cdot e_{\bar\nu} &= 0. \end{aligned} \tag{172}$$

The number $\bar I = e_{\bar 0} e_{\bar 1} e_{\bar 2} e_{\bar 3}$ has the properties

$$\begin{aligned} \bar I^2 &= 1, \\ \bar I e_\mu &= e_\mu \bar I, \\ \bar I e_{\bar\mu} &= -e_{\bar\mu} \bar I. \end{aligned} \tag{173}$$

The set of numbers

$$\begin{aligned} \gamma_\mu &= e_\mu\ , \\ \gamma_{\bar\mu} &= e_{\bar\mu} \bar I \end{aligned} \tag{174}$$

satisfies

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &= \delta_{\mu\nu}\ , \\ \gamma_{\bar\mu} \cdot \gamma_{\bar\nu} &= -\delta_{\mu\nu}\ , \\ \gamma_\mu \cdot \gamma_{\bar\mu} &= 0. \end{aligned} \tag{175}$$

The numbers $(\gamma_\mu,\, \gamma_{\bar\mu})$ thus form a set of basis vectors of a vector space $V_{4,4}$ with signature $(++++----)$.

10-space. Let $e_A = (e_\mu, e_{\bar\mu})$, $\mu = 1,2,3,4,5$; $\bar\mu = \bar 1, \bar 2, \bar 3, \bar 4, \bar 5$ be basis vectors of a 10-dimensional Euclidean space V_{10} with signature $(+++....)$. We introduce $\bar I = e_{\bar 1} e_{\bar 2} e_{\bar 3} e_{\bar 4} e_{\bar 5}$ which satisfies

$$\begin{aligned} \bar I^2 &= 1\ , \\ e_\mu \bar I &= -\bar I e_\mu\ , \\ e_{\bar\mu} \bar I &= \bar I e_{\bar\mu}. \end{aligned} \tag{176}$$

Then the Clifford numbers

$$\begin{aligned} \gamma_\mu &= e_\mu \bar I\ , \\ \gamma_{\bar\mu} &= e_\mu \end{aligned} \tag{177}$$

satisfy

$$\begin{aligned} \gamma_\mu \cdot \gamma_\nu &= -\delta_{\mu\nu}\ , \\ \gamma_{\bar\mu} \cdot \gamma_{\bar\nu} &= \delta_{\bar\mu\bar\nu}\ , \\ \gamma_\mu \cdot \gamma_{\bar\mu} &= 0. \end{aligned} \tag{178}$$

The set $\gamma_A = (\gamma_\mu, \gamma_{\bar{\mu}})$ therefore spans the vector space of signature $(-----+++++)$.

The examples above demonstrate how vector spaces of various signatures are obtained within a given set of polyvectors. Namely, vector spaces of different signature are different subsets of polyvectors within the same Clifford algebra. In other words, vector spaces of different signature are different subspaces of C-space, i.e., different sections through C-space[7].

This has important physical implications. We have argued that physical quantities are polyvectors (Clifford numbers or Clifford aggregates). Physical space is then not simply a vector space (e.g., Minkowski space), but a space of polyvectors, called C-space, a pandimensional continuum of points, lines, planes, volumes, etc., altogether. Minkowski space is then just a subspace with pseudo-Euclidean signature. Other subspaces with other signatures also exist within the pandimensional continuum C and they all have physical significance. If we describe a particle as moving in Minkowski spacetime $V_{1,3}$ we consider only certain physical aspects of the object considered. We have omitted its other physical properties like spin, charge, magnetic moment, etc.. We can as well describe the same object as moving in an Euclidean space V_4. Again such a description would reflect only a part of the underlying physical situation described by Clifford algebra.

7.2 Clifford space and the conformal group

7.2.1 Line element in C-space of Minkowski spacetime

In 4-dimensional spacetime a polyvector and its square (1) can be written as

$$\mathrm{d}X = \mathrm{d}\sigma + \mathrm{d}x^\mu \gamma_\mu + \frac{1}{2}\mathrm{d}x^{\mu\nu}\gamma_\mu \wedge \gamma_\nu + \mathrm{d}\tilde{x}^\mu \, I\gamma_\mu + \mathrm{d}\tilde{\sigma} I \tag{179}$$

$$|\mathrm{d}X|^2 = \mathrm{d}\sigma^2 + \mathrm{d}x^\mu \mathrm{d}x_\mu + \frac{1}{2}\mathrm{d}x^{\mu\nu}\mathrm{d}x_{\mu\nu} - \mathrm{d}\tilde{x}^\mu \mathrm{d}\tilde{x}_\mu - \mathrm{d}\tilde{\sigma}^2 \tag{180}$$

The minus sign in the last two terms of the above quadratic form occurs because in 4-dimensional spacetime with signature $(+---)$ we have $I^2 = (\gamma_0\gamma_1\gamma_2\gamma_3)(\gamma_0\gamma_1\gamma_2\gamma_3) = -1$, and $I^\dagger I = (\gamma_3\gamma_2\gamma_1\gamma_0)(\gamma_0\gamma_1\gamma_2\gamma_3) = -1$.

In eq.(180) the line element $\mathrm{d}x^\mu \mathrm{d}x_\mu$ of the ordinary special or general relativity is replaced by the line element in Clifford space. A "square root" of such a generalized line element is $\mathrm{d}X$ of eq.(179). The latter object is a *polyvector*, a differential of the coordinate polyvector field

$$X = \sigma + x^\mu \gamma_\mu + \frac{1}{2}x^{\mu\nu}\gamma_\mu \wedge \gamma_\nu + \tilde{x}^\mu I\gamma_\mu + \tilde{\sigma} I \tag{181}$$

whose square is

$$|X|^2 = \sigma^2 + x^\mu x_\mu + \frac{1}{2}x^{\mu\nu}x_{\mu\nu} - \tilde{x}^\mu \tilde{x}_\mu - \tilde{\sigma}^2 \tag{182}$$

The polyvector X contains not only the vector part $x^\mu\gamma_\mu$, but also a *scalar part* σ, *tensor part* $x^{\mu\nu}\gamma_\mu \wedge \gamma_\nu$, *pseudovector part* $\tilde{x}^\mu \, I\gamma_\mu$ and pseudoscalar part $\tilde{\sigma} I$. Similarly for the differential $\mathrm{d}X$.

[7]What we consider here should not be confused with the well known fact that Clifford algebras associated with vector spaces of different signatures (p, q), with $p + q = n$, are not all isomorphic.

When calculating the quadratic forms $|X|^2$ and $|\mathrm{d}X|^2$ one obtains in 4-dimensional spacetime with pseudo euclidean signature $(+ - --)$ the minus sign in front of the squares of the pseudovector and pseudoscalar terms. This is so, because in such a case the pseudoscalar unit square in flat spacetime is $I^2 = I^\dagger I = -1$. In 4-dimensions $I^\dagger = I$ regardless of the signature.

Instead of Lorentz transformations—pseudo rotations in spacetime—which preserve $x^\mu x_\mu$ and $\mathrm{d}x^\mu \mathrm{d}x_\mu$ we have now more general rotations—rotations in C-space—which preserve $|X|^2$ and $|\mathrm{d}X|^2$.

7.2.2 C-space and conformal transformations

From (180) and (182) we see [25] that a subgroup of the Clifford Group, or rotations in C-space is the group SO(4,2). The transformations of the latter group rotate x^μ, σ, $\tilde{\sigma}$, but leave $x^{\mu\nu}$ and $\tilde{x}^\mu$ unchanged. Although according to our assumption physics takes place in full C-space, it is very instructive to consider a subspace of C-space, that we shall call *conformal space* whose isometry group is SO(4,2).

Coordinates can be given arbitrary symbols. Let us now use the symbol η^μ instead of x^μ, and η^5,η^6 instead of $\tilde{\sigma}$, σ. In other words, instead of $(x^\mu, \tilde{\sigma}, \sigma)$ we write $(\eta^\mu, \eta^5, \eta^6) \equiv \eta^a$, $\mu = 0,1,2,3$, $a = 0,1,2,3,5,6$. The quadratic form reads

$$\eta^a \eta_a = g_{ab} \eta^a \eta^b \tag{183}$$

with

$$g_{ab} = \mathrm{diag}(1,-1,-1,-1,-1,1) \tag{184}$$

being the diagonal metric of the flat 6-dimensional space, a subspace of C-space, parametrized by coordinates η^a. The transformations which preserve the quadratic form (183) belong to the group SO(4,2). It is well known [38, 39] that the latter group, when taken on the cone

$$\eta^a \eta_a = 0 \tag{185}$$

is isomorphic to the 15-parameter group of conformal transformations in 4-dimensional spacetime [40].

Let us consider first the rotations of η^5 and η^6 which leave coordinates η^μ unchanged. The transformations that leave $-(\eta^5)^2 + (\eta^6)^2$ invariant are

$$\begin{aligned} \eta'^5 &= \eta^5 \cosh\alpha + \eta^6 \sinh\alpha \\ \eta'^6 &= \eta^5 \sinh\alpha + \eta^6 \cosh\alpha \end{aligned} \tag{186}$$

where α is a parameter of such pseudo rotations.

Instead of the coordinates η^5, η^6 we can introduce [38, 39] new coordinates κ, λ according to

$$\kappa = \eta^5 - \eta^6 \tag{187}$$

$$\lambda = \eta^5 + \eta^6 \tag{188}$$

In the new coordinates the quadratic form (183) reads

$$\eta^a \eta_a = \eta^\mu \eta_\mu - (\eta^5)^2 - (\eta^6)^2 = \eta^\mu \eta_\mu - \kappa\lambda \tag{189}$$

The transformation (186) becomes

$$\kappa' = \rho^{-1}\kappa \tag{190}$$

$$\lambda' = \rho\lambda \tag{191}$$

where $\rho = e^{\alpha}$. This is just a dilation of κ and the inverse dilation of λ.

Let us now introduce new coordinates x^{μ} according x^{μ} to[8]

$$\eta^{\mu} = \kappa x^{\mu} \tag{192}$$

Under the transformation (192) we have

$$\eta'^{\mu} = \eta^{\mu} \tag{193}$$

but

$$x'^{\mu} = \rho x^{\mu} \tag{194}$$

The latter transformation is *dilatation* of coordinates x^{μ}.

Considering now a line element

$$\mathrm{d}\eta^{a}\mathrm{d}\eta_{a} = \mathrm{d}\eta^{\mu}\mathrm{d}\eta_{\mu} - \mathrm{d}\kappa\mathrm{d}\lambda \tag{195}$$

we find that *on the cone* $\eta^{a}\eta_{a} = 0$ it is

$$\mathrm{d}\eta^{a}\mathrm{d}\eta_{a} = \kappa^{2}\,\mathrm{d}x^{\mu}\mathrm{d}x_{\mu} \tag{196}$$

even if κ is not constant. Under the transformation (190) we have

$$\mathrm{d}\eta'^{a}\mathrm{d}\eta'_{a} = \mathrm{d}\eta^{a}\mathrm{d}\eta_{a} \tag{197}$$

$$\mathrm{d}x'^{\mu}\mathrm{d}x'_{\mu} = \rho^{2}\,\mathrm{d}x^{\mu}\mathrm{d}x_{\mu} \tag{198}$$

The last relation is a *dilatation* of the 4-dimensional line element related to coordinates x^{μ}. In a similar way also other transformations of the group SO(4,2) that preserve (185) and (197) we can rewrite in terms of of the coordinates x^{μ}. So we obtain—besides dilations—translations, Lorentz transformations, and special conformal transformations; altogether they are called *conformal transformations*. This is a well known old observation [38, 39] and we shall not discuss it further. What we wanted to point out here is that conformal group SO(4,2) is a subgroup of the Clifford group.

7.2.3 On the physical interpretation of the conformal group SO(4,2)

In order to understand the physical meaning of the transformations (192) from the coordinates η^{μ} to the coordinates x^{μ} let us consider the following transformation in 6-dimensional space V_6 :

$$\begin{aligned} x^{\mu} &= \kappa^{-1}\eta^{\mu} \\ \alpha &= -\kappa^{-1} \\ \Lambda &= \lambda - \kappa^{-1}\eta^{\mu}\eta_{\mu} \end{aligned} \tag{199}$$

[8] These new coordinates x^{μ} should not be confused with coordinate x^{μ} used in Sec.2.

This is a transformation from the coordinates $\eta^a = (\eta^\mu, \kappa, \lambda)$ to the new coordinates $x^a = (x^\mu, \alpha, \Lambda)$. No extra condition on coordinates, such as (185), is assumed now. If we calculate the line element in the coordinates η^a and x^a, respectively, we find the the following relation [27]

$$\mathrm{d}\eta^\mu \mathrm{d}\eta^\nu \, g_{\mu\nu} - \mathrm{d}\kappa \, \mathrm{d}\lambda = \alpha^{-2} (\mathrm{d}x^\mu \mathrm{d}x^\nu \, g_{\mu\nu} - \mathrm{d}\alpha \mathrm{d}\Lambda) \tag{200}$$

We can interpret a transformation of coordinates *passively* or *actively.* Geometric calculus clarifies significantly the meaning of passive and active transformations. Under a *passive transformation* a vector remains the same, but its components and basis vector change. For a vector $\mathrm{d}\eta = \mathrm{d}\eta^a \gamma_a$ we have

$$\mathrm{d}\eta' = \mathrm{d}\eta'^a \gamma'_a = \mathrm{d}\eta^a \gamma_a = \mathrm{d}\eta \tag{201}$$

with

$$\mathrm{d}\eta'^a = \frac{\partial \eta'^a}{\partial \eta^b} \, \mathrm{d}\eta^b \tag{202}$$

and

$$\gamma'_a = \frac{\partial \eta^b}{\partial \eta'^a} \, \gamma_b \tag{203}$$

Since the vector is invariant, so it is its square:

$$\mathrm{d}\eta'^2 = \mathrm{d}\eta'^a \gamma'_a \, \mathrm{d}\eta'^b \gamma'_b = \mathrm{d}\eta'^a \mathrm{d}\eta'^b g'_{ab} = \mathrm{d}\eta^a \mathrm{d}\eta^b g_{ab} \tag{204}$$

From (203) we read that the well known relation between new and old coordinates:

$$g'_{ab} = \frac{\partial \eta^c}{\partial \eta'^a} \frac{\partial \eta^d}{\partial \eta'^b} \, g_{cd} \tag{205}$$

Under an *active transformation* a vector changes. This means that in a fixed basis the components of a vector change:

$$\mathrm{d}\eta' = \mathrm{d}\eta'^a \gamma_a \tag{206}$$

with

$$\mathrm{d}\eta'^a = \frac{\partial \eta'^a}{\partial \eta^b} \, \mathrm{d}\eta^b \tag{207}$$

The transformed vector $\mathrm{d}\eta'$ is different from the original vector $\mathrm{d}\eta = \mathrm{d}\eta^a \gamma_a$. For the square we find

$$\mathrm{d}\eta'^2 = \mathrm{d}\eta'^a \mathrm{d}\eta'^b g_{ab} = \frac{\partial \eta'^a}{\partial \eta^c} \frac{\partial \eta'^b}{\partial \eta^d} \, \mathrm{d}\eta^c \mathrm{d}\eta^d g_{ab} \tag{208}$$

i.e., the transformed line element $\mathrm{d}\eta'^2$ is different from the original line element.

Returning now to the coordinate transformation (199) with the identification $\eta'^a = x^a$, we can interpret eq. (200) passively or actively.

In the *passive interpretation* the metric tensor and the components $\mathrm{d}\eta^a$ change under a transformation, so that in our particular case the relation (204) becomes

$$\mathrm{d}x^a \, \mathrm{d}x^b \, g'_{ab} = \alpha^{-2} (\mathrm{d}x^\mu \mathrm{d}x^\nu \, g_{\mu\nu} - \mathrm{d}\alpha \, \mathrm{d}\Lambda) = \mathrm{d}\eta^a \mathrm{d}\eta^b g_{ab} = \mathrm{d}\eta^\mu \mathrm{d}\eta^\nu g_{\mu\nu} - \mathrm{d}\kappa \, \mathrm{d}\lambda \tag{209}$$

with

$$g'_{ab} = \alpha^{-2} \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \quad , \quad g_{ab} = \begin{pmatrix} g_{\mu\nu} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 0 \end{pmatrix} \tag{210}$$

In the above equation the same infinitesimal distance squared is expressed in two different coordinates η^a or x^a.

In *active interpretation*, only $\mathrm{d}\eta^a$ change, whilst the metric remains the same, so that the transformed element is

$$\mathrm{d}x^a\,\mathrm{d}x^b\,g_{ab} = \mathrm{d}x^\mu \mathrm{d}x^\nu\,g_{\mu\nu} - \mathrm{d}\alpha\,\mathrm{d}\Lambda = \kappa^{-2}\,\mathrm{d}\eta^a \mathrm{d}\eta^b g_{ab} = \kappa^{-2}(\mathrm{d}\eta^\mu \mathrm{d}\eta^\nu g_{\mu\nu} - \mathrm{d}\kappa\,\mathrm{d}\lambda) \tag{211}$$

The transformed line lelement $\mathrm{d}x^a \mathrm{d}x_a$ is physically different from the original line element $\mathrm{d}\eta^a \mathrm{d}\eta_a$ by a factor $\alpha^2 = \kappa^{-2}$

A rotation (186) in the plane (η^5, η^6) (i.e., the transformation (190),(191) of (κ, λ)) manifests in the new coordinates x^a as a *dilatation* of the line element $\mathrm{d}x^a \mathrm{d}x_a = \kappa^{-2}\,\mathrm{d}\eta^a \eta_a$:

$$\mathrm{d}x'^a \mathrm{d}x'_a = \rho^2 \mathrm{d}x^a \mathrm{d}x_a \tag{212}$$

All this is true in the full space V_6. On the cone $\eta^a \eta_a = 0$ we have $\Lambda = \lambda - \kappa \eta^\mu \eta_\mu = 0$, $\mathrm{d}\Lambda = 0$ so that $\mathrm{d}x^a \mathrm{d}x_a = \mathrm{d}x^\mu \mathrm{d}x_\mu$ and we reproduce the relations (198) which is a dilatation of the 4-dimensional line element. It can be interpreted either passively or actively. In general, the pseudo rotations in V_6, that is, the transformations of the 15-parameter group SO(4,2) when expressed in terms of coordinates x^a, assume on the cone $\eta^a \eta_a = 0$ the form of the ordinary conformal transformations. They all can be given the active interpretation [27, 28].

We started from the new paradigm that physical phenomena actually occur not in spacetime, but in a larger space, the so called *Clifford space* or C-space which is a manifold associated with the Clifford algebra generated by the basis vectors γ_μ of spacetime. An arbitrary element of Cliffod algebra can be expanded in terms of the objects E_A, $A = 1, 2, ..., 2^D$, which include, when $D = 4$, the scalar unit $\mathbf{1}$, vectors γ_μ, bivectors $\gamma_\mu \wedge \gamma_\nu$, pseudovectors $I\gamma_\mu$ and the pseudoscalar unit $I \equiv \gamma_5$. C-space contains 6-dimensional subspace V_6 spanned[9] by $\mathbf{1}$, γ_μ, and γ_5. The metric of V_6 has the signature $(+----+)$. It is well known that the rotations in V_6, when taken on the conformal cone $\eta^a \eta_a = 0$, are isomorphic to the non linear transformations of the conformal group in spacetime. Thus we have found out that C-space contains —as a subspace— the 6-dimensional space V_6 in which the conformal group acts linearly. From the physical point of view this is an important and, as far as we know, a novel finding, although it might look mathematically trivial. *So far it has not been clear what could be a physical interpretation of the 6 dimensional conformal space.* Now we see that it is just a subspace of Clifford space. The two extra dimensions, parametrized by κ and λ, are not the ordinary extra dimensions; they are coordinates of Clifford space C_4 of the 4-dimensional Minkowski spacetime V_4.

[9]It is a well known observation that the generators L_{ab} of SO(4,2) can be realized in terms of $\mathbf{1}$, γ_μ, and γ_5. Lorentz generators are $M_{\mu\nu} = -\frac{i}{4}[\gamma_\mu, \gamma_\nu]$, dilatations are generated by $D = L_{65} = -\frac{1}{2}\gamma_5$, translations by $P_\mu = L_{5\mu} + L_{6\mu} = \frac{1}{2}\gamma_\mu(1 - i\gamma_5)$ and the special conformal transformations by $L_{5\mu} - L_{6\mu} = \frac{1}{2}\gamma_\mu(1 + i\gamma_5)$. This essentially means that the generators are $L_{ab} = -\frac{i}{4}[e_a, e_b]$ with $e_a = (\gamma_\mu, \gamma_5, \mathbf{1})$, where care must be taken to replace commutators $[\mathbf{1}, \gamma_5]$ and $[\mathbf{1}, \gamma_\mu]$ with $2\gamma_5$ and $2\gamma_\mu$

We take C-space seriously as an arena in which physics takes place. The theory is a very natural, although not trivial, extension of the special relativity in spacetime. In special relativity the transformations that preserve the quadratic form are given an *active interpretation*: they relate the objects or the systems of reference in *relative translational motion*. Analogously also the transformations that preserve the quadratic form (180) or (182) in C-space should be given an active interpretation. We have found that among such transformations (rotations in C-space) there exist the transformations of the group SO(4,2). Those transformations also should be given an active interpretation as the transformations that relate different physical objects or reference frames. Since in the ordinary relativity we do not impose any constraint on the coordinates of a freely moving object so we should not impose any constraint in C-space, or in the subspace V_6. However, by using the projective coordinate transformation (199), without any constraint such as $\eta^a \eta_a = 0$, we arrived at the relation (211) for the line elements. If in the coordinates η^a the line element is constant, then in the coordinates x^a the line element is changing by a scale factor κ which, in general, depends on the evolution parameter τ. The line element need not be one associated between two events along a point particle's worldline: it can be between two arbitrary (space-like or time-like) events within an extended object. We may consider the line element ($\equiv$ distance squared) between two infinitesimally separated events within an extended object such that both events have the same coordinate label Λ so that $\mathrm{d}\Lambda = 0$. Then the 6-dimensional line element $\mathrm{d}x^\mu \mathrm{d}x^\nu\, g_{\mu\nu} - \mathrm{d}\alpha\, \mathrm{d}\Lambda$ becomes the 4-dimensional line element $\mathrm{d}x^\mu \mathrm{d}x^\nu\, g_{\mu\nu}$ and, because of (211) it changes with τ when κ does change. This means that the object changes its *size*, it is moving dilatationally [27, 28]. We have thus arrived at a very far reaching observation that the relativity in C-space implies *scale changes of physical objects as a result of free motion, without presence of any forces or such fields as assumed in Weyl theory.* This was advocated long time ago [27, 28], but without recurse to C-space. However, if we consider the full Clifford space C and not only the Minkowski spacetime section through C, then we arrive at a more general dilatational motion [17] related to the polyvector coordinates $x^{\mu\nu}, x^{\mu\nu\alpha}$ and $x^{0123} \equiv \tilde{\sigma}$ (also denoted s) as reviewed in section 3.

7.3 C-space Maxwell Electrodynamics

Finally, in this section we will review and complement the proposal of ref.[75] to generalize Maxwell Electrodynamics to C-spaces, namely, construct the Clifford algebra-valued extension of the Abelian field strength $F = dA$ associated with ordinary vectors A_μ. Using Clifford algebraic methods we shall describe how to generalize Maxwell's theory of Electrodynamics asociated with ordinary point-charges to a generalized Maxwell theory in Clifford spaces involving *extended* charges and p-forms of arbitrary rank, not unlike the couplings of p-branes to antisymmetric tensor fields.

Based on the standard definition of the Abelian field strength $F = dA$ we shall use the same definition in terms of polyvector-valued quantities and differential operators in C-space

$$A = A_N E^N = \phi \underline{1} + A_\mu \gamma^\mu + A_{\mu\nu} \gamma^\mu \wedge \gamma^\nu + ... \tag{213}$$

The first component in the expansion ϕ is a scalar field; A_μ is the standard Maxwell field A_μ, the third component $A_{\mu\nu}$ is a rank two antisymmetric tensor field....and the last component of the expansion is a pseudo-scalar. The fact that a scalar and pseudo-scalar field appear very naturally in the expansion of the C-space polyvector valued field A_N suggests that one could attempt to identify the latter fields with a dilaton-like and axion-like field, respectively.Once again= , in order to match units in the expansion (213), it requires the introduction of suitable powers of a length scale parameter, the Planck scalewhich is conveniently set to unity.

The differential operator is the generalized Dirac operator

$$d = E^M \partial_M = \underline{1}\partial_\sigma + \gamma^\mu \partial_{x_\mu} + \gamma^\mu \wedge \gamma^\nu \partial_{x_{\mu\nu}} + ...) \tag{214}$$

the polyvector-valued indices $M, N....$ range from $1, 2.....2^D$ since a Clifford algebra in D-dim has 2^D basis elements. The generalized Maxwell field strength in C-space is

$$F = dA = E^M \partial_M (E^N A_N) = E^M E^N \partial_M A_N = \frac{1}{2}\{E^M, E^N\}\partial_M A_N +$$

$$\frac{1}{2}[E^M, E^N]\partial_M A_N = \frac{1}{2}F_{(MN)}\{E^M, E^N\} + \frac{1}{2}F_{[MN]}[E^M, E^N]. \tag{215}$$

where one has *decomposed* the Field strength components into a symmetric plus antisymmetric piece by simply writing the Clifford geometric product of two polyvectors $E^M E^N$ as the sum of an anticommutator plus a commutator piece respectively,

$$F_{(MN)} = \frac{1}{2}(\partial_M A_N + \partial_N A_M). \tag{216}$$

$$F_{[MN]} = \frac{1}{2}(\partial_M A_N - \partial_N A_M). \tag{217}$$

Let the C-space Maxwell action (up to a numerical factor) be given in terms of the antisymmetric part of the field strength:

$$I[A] = \int [\mathrm{D}X] F_{[MN]} F^{[MN]}. \tag{218}$$

where $[\mathrm{D}X]$ is a C-space measure comprised of all the (holographic) coordinates degrees of freedom

$$[\mathrm{D}X] \equiv (d\sigma)(dx^0 dx^1 ...)(dx^{01} dx^{02} ...)....(dx^{012...D}). \tag{219}$$

Action (218) is invariant under the gauge transformations

$$A'_M = A_M + \partial_M \Lambda \tag{220}$$

The matter-field minimal coupling (interaction term) is:

$$\int A_M dX^M = \int [\mathrm{D}X] J_M A^M, \tag{221}$$

where one has reabsorbed the coupling constant, the C-space analog of the electric charge, within the expression for the A field itself. Notice that this term (221) has the same form as the coupling of p-branes (whose world volume is $p+1$-dimensional) to antisymmetric tensor fields of rank $p+1$.

The open line integral in C-space of the matter-field interaction term in the action is taken from the polyparticle's proper time interval S ranging from $-\infty$ to $+\infty$ and can be recast via the Stokes law solely in terms of the antisymmetric part of the field strength. This requires closing off the integration countour by a semi-circle that starts at $S=+\infty$, goes all the way to C-space infinity, and comes back to the point $S=-\infty$. The field strength vanishes along the points of the semi-circle at infinity, and for this reason the net contribution to the contour integral is given by the open-line integral. Therefore, by rewriting the $\int A_M dX^M$ via the Stokes law relation, it yields

$$\int A_M dX^M = \int F_{[MN]} dS^{[MN]} = \int F_{[MN]} X^M dX^N = \int dS F_{[MN]} X^M (dX^N/dS). \tag{222}$$

where in order to go from the second term to the third term in the above equation we have integrated by parts and then used the Bianchi identity for the antisymmetric component $F_{[MN]}$.

The integration by parts permits us to go from a C-space domain integral, represented by the Clifford-value hypersurface S^{MN}, to = a C-space boundary-line integral

$$\int dS^{MN} = \frac{1}{2}\int (X^M dX^N - X^N dX^M). \tag{223}$$

The pure matter terms in the action are given by the analog of the proper time integral spanned by the motion of a particle in spacetime:

$$\kappa \int dS = \kappa \int dS \sqrt{\frac{dX^M}{dS}\frac{dX_M}{dS}}. \tag{224}$$

where κ is a parameter whose dimensions are $(mass)^{p+1}$ and S is the polyparticle proper time in C-space.

The Lorentz force relation in C-space is directly obtained from a variation of

$$\int dS F_{[MN]} X^M (dX^N/dS). \tag{225}$$

and

$$\kappa \int dS = \kappa \int \sqrt{dX^M dX_M}. \tag{226}$$

with respect tothe X^M variables:

$$\kappa \frac{d^2 X_M}{dS^2} = e F_{[MN]} \frac{dX^N}{dS}. \tag{227}$$

where we have re-introduced the C-space charge e back into the Lorentz force equation in C-space. A variation of the terms in the action w.r.t the A_M field furnishes the following equation of motion for the A fields:

$$\partial_M F^{[MN]} = J^N. \tag{228}$$

By taking derivatives on both sides of the last equation with respect to the X^N coordinate, one obtains due to the symmmetry condition of $\partial_M \partial_N$ versus the antisymmetry of $F^{[MN]}$ that

$$\partial_N \partial_M F^{[MN]} = 0 = \partial_N J^N = 0. \tag{229}$$

which is precisely the continuity equation for the current.

The continuity equation is essential to ensure that the matter-field coupling term of the action $\int A_M dX^M = \int [\mathrm{D}X] J^M A_M$ is also gauge invariant, which can be readily verified after an integration by parts and setting the boundary terms to zero:

$$\delta \int [\mathrm{D}X] J^M A_M = \int [\mathrm{D}X] J^M \partial_M \Lambda = - \int [\mathrm{D}X] (\partial_M J^M) \Lambda = 0. \tag{230}$$

Gauge invariance also ensures the conservation of the energy-momentum (via Noether's theorem) defined in tems of the Lagrangian density variation. We refer to [75] for further details.

The gauge invariant C-space Maxwell action as given in eq. (218) is in fact only a part of a more general action given by the expression

$$I[A] = \int [\mathrm{D}X]\, F^\dagger * F = \int [\mathcal{D}X] \;\; < F^\dagger F >_{scalar} . \tag{231}$$

The latter action is not gauge invariant, since it contains not only the antisymmetric but also the symmetric part of F. Therefore the action if the form (231) contains a gauge fixing term.

A lesson that we have from these considerations is that the C-space Maxwell action written in the form (231) automatically contains a gauge fixing term. Analogous result for *ordinary* Maxwell field is known from Hestenes work [1], although formulated in a slightly different way, namely by direclty considering the field equations without emplying the action.

The action considered in [75] was :

$$S[A] = \int [\mathcal{D}X] \;\; (F_{(MN)} F^{(MN)} + F_{[MN]} F^{[MN]}) \tag{232}$$

The latter action is strictly speaking not gauge invariant, since it contains not only the antisymmetric but also the symmetric part of F. It is invariant under a *restricted* gauge symmetry transformations. It is invariant (up to total derivatives) under $infinitesimal$ gauge transformations provided the symmetric part of F is divergence-free $\partial_M F^{(MN)} = 0$ [75] . This divergence-free condition has the same effects as if one were fixing a gauge leaving a *residual* symmetry of *restricted* gauge transformations such that the gauge symmetry parameter obeys the Laplace-like equation $\partial_M \partial^M \Lambda = 0$. Such residual (restricted) symmetries are precisely those that leave invariant the divergence-free condition on the symmetric part of F. Residual, restricted symmetries occur, for example, in the light-cone gauge of p-brane actions leaving a residual symmetry of *volume*-preserving diffs. They also occur in string theory when the conformal gauge is chosen leaving a residual symmetry under conformal reparametrizations; i.e. the so-called Virasoro algebras whose symmetry transformations are given by holomorphic and anti-holomorphic reparametrizations of the string world-sheet.

This Laplace-like condition on the gauge parameter is also the one required such that the action in [75] is invariant under *finite* (restricted) gauge transformations since under such (restricted) finite transformations the Lagrangian changes by second-order terms of the form $(\partial_M \partial_N \Lambda)^2$, which are total derivatives if, and only if, the gauge parameter is restricted to obey the analog of Laplace equation $\partial_M \partial^M \Lambda = 0$ Therefore the action of eq-(233) is invariant under a *restricted* gauge transformation which bears a resemblance to *volume*-preserving diffeomorphisms of the p-branes action in the light-cone gauge.

It remains to be seen if this construction of C-space generalized Maxwell Electrodynamics of p-forms can be generalized to the Nonabelian case when we replace ordinary derivatives by gauge-covariant ones:

$$F = dA \to F = DA = (dA + A \bullet A). \tag{233}$$

For example, one could define the graded-symmetric product $E_M \bullet E_N$ based on the graded commutator of Superalgebras:

$$[A, B] = AB - (-1)^{s_A s_B} BA. \tag{234}$$

s_A, s_B is the grade of A and B respectively. For bosons the grade is even and for fermions is odd. In this fashion the graded commutator captures both the anti-commutator of two fermions and the commutator of two bosons in one stroke. One may extend this graded bracket definition to the graded structure present in Clifford algebras, and define

$$E_M \bullet E_N = E_M E_N - (-1)^{s_M s_N} E_N E_M. \tag{235}$$

s_M, s_N is the grade of E_M and E_N respectively. Even or odd depending on the grade of the basis elements.

One may generalize Maxwell's theory to Born-Infeld nonlinear Electrodynamics in C-spacesbased on this extension of Maxwell Electrodynamics in C-spaces and to couple a C-space version of a Yang-Mills theory to C-space gravity, a higher derivative gravity with torsion, this will be left for a future publication. Clifford algebras have been used in the past [62] to study the Born-Infeld model in ordinary spacetime and to write a nonlinear version of the Dirac equation. The natural incorporation of monopoles in Maxwell's theory was investigated by [89] and a recent critical analysis of " unified " theories of gravity with electromagnetism has been presented by [90]. Most recently [22] has studied the covariance of Maxwell's theory from a Clifford algebraic point of view.

8 Concluding Remarks

We have presented a brief review of some of the most important features of the Extended Relativity theory in Clifford-spaces (C-spaces). The "coordinates" X are noncommuting Clifford-valued quantities which incoporate the lines, areas, volumes,....degrees of freedom associated with the collective particle, string, membrane,... dynamics underlying the center-of-mass motion and holographic projections of the p-loops onto the embedding target spacetime backgrounds. C-space Relativity incoporates the idea of an invariant length, which upon quantization, should lead to the notion of minimal Planck scale [23].

Other relevant features are those of maximal acceleration [52], [49] ; the invariance of Planck-areas under acceleration boosts; the resolution of ordering ambiguities in QFT; supersymmetry ; holography [119]; the emergence of higher derivative gravity with torsion ;and the inclusion of variable dimensions/signatures that allows to study the dynamics of all (closed) p-branes, for all values of p, in one single unified footing, by starting with the C-space brane action constructed in this work.

The Conformal group construction presented in **7** , as a natural subgroup of the Clifford group in four-dimenions, needs to be generalized to other dimensions, in particular to two dimensions where the Conformal group is infinite-dimensional. Kinani [130] has shown that the Virasoro algebra can be obtained from generalized Clifford algebras. The construction of area-preserving diffs algebras, like w_∞ and $su(\infty)$, from Clifford algebras remains an open problem. Area-preserving diffs algebras are very important in the study of membranes and gravity since Higher-dim Gravity in $m + n$-dim has been shown a while ago to be equivalent to a lower m-dim Yang-Mills-like gauge theory of diffs of an internal n-dim space [120] and that amounts to another explanation of the holographic principle behind the AdS/CFT duality conjecture [121]. We have shown how C-space Relativity involves scale changes in the sizes of physical objects, in the absence of forces and Weyl'gauge field of dilations. The introduction of scale-motion degrees of freedom has recently been implemented in the wavelet-based regularization procedure of QFT by [87]. The connection to Penrose's Twistors program is another interesting project worthy of investigation.

The quantization and construction of QFTs in C-spaces remains a very daunting task since it may involve the construction of QM in Noncommutative spacetimes [102], braided Hopf quantum Clifford algebras [86], hypercomplex extensions of QM like quaternionic and octonionic QM [99], [97], [98], exceptional group extensions of the Standard Model [85],hyper-matrices and hyper-determinants [88], multi-symplectic mechanics, the de Donde-Weyl formulations of QFT [82], to cite a few, for example. The quantization program in C-spaces should share similar results as those in Loop Quantum Gravity [111], in particular the minimal Planck areas of the expectation values of the area-operator.

Spacetime at the Planck scale may be discrete, fractal, fuzzy, noncommutative... The original Scale Relativity theory in fractal spacetime [23] needs to be extended futher to incoporate the notion of fractal "manifolds". A scale-fractal calculus and a fractal-analysis construction that are esential in building the notion of a fractal "manifold" has been initiated in the past years by [129]. It remains yet to be proven that a scale-fractal calculus in fractal spacetimes is another realization of a Connes Noncommutative Geometry. Fractal strings/branes and their spectrum have been studied by [104] that may require generalized Statistics beyond the Boltzmann-Gibbs, Bose-Einstein and Fermi-Dirac, investigated by [105], [103], among others.

Non-Archimedean geometry has been recognized long ago as the natural one operating at the minimal Planck scale and requires the use p-adic numbers instead of ordinary numbers [101]. By implementing the small/large scale, ultraviolet/infrared duality principle associated with QFTs in Noncommutative spaces, see [125] for a review, one would expect an upper maximum scale [23] and a maximum temperature [21] to be operating in Nature. Non-Archimedean Cosmologies based on an upper scale has been investigated by [94].

An upper/lower scale can be accomodated simultaneously and very naturally in the q-Gravity theory of [114], [69] based on bicovariant quantum group extensions of the Poincare, Conformal group, where the q deformation parameter could be equated to the quantity $e^{\Lambda/L}$, such that both $\Lambda = 0$ and $L = \infty$, yield the same classical $q = 1$ limit. For a review of q-deformations of Clifford algebras and their generalizations see [86], [128].

It was advocated long ago by Wheeler and others, that information theory [106], set theory and number theory, may be the ultimate physical theory. The important role of Clifford algebras in information theory have been known for some time [95]. Wheeler's spacetime foam at the Planck scale may be the background source generation of Noise in the Parisi-Wu stochastic qunatization [47] that is very relevant in Number theory [100]. The pre-geometry cellular-networks approach of [107] and the quantum-topos views based on gravitational quantum causal sets, noncommutative topology and category theory [109], [110], [124] deserves a futher study within the C-space Relativity framework, since the latter theory also invokes a Category point of view to the notion of dimensions. C-space is a pandimensional continuum [14], [8]. Dimensions are topological invariants and, since the dimensions of the extended objects change in C-space, topology-change is another ingredient that needs to be addressed in C-space Relativity and which may shed some light into the physical foundations of string/M theory [118]. It has been speculated that the universal symmetries of string theory [108] may be linked to Borcherds Vertex operator algebras (the Monstruous moonshine) that underline the deep interplay between Conformal Field Theories and Number theory. A lot remains to be done to bridge together these numerous branches of physics and mathematics. Many surprises may lie ahead of us.

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References

[1] D. Hestenes, "Spacetime Algebra" Gordon and Breach, New York, 1996.

D. Hestenes and G. Sobcyk, "Clifford Algebra to Geometric Calculus" D. Reidel Publishing Company, Dordrecht, 1984.

[2] I. R. Porteous, Clifford Algebras and the Classical Groups, Cambridge University Press, 1995.

[3] W. Baylis, Electrodynamics, A Modern Geometric Approach, Boston, Birkhauser, 1999.

[4] G. Trayling and W. Baylis, J.Phys. A34 (2001) 3309-3324; Int.J.Mod.Phys. A16S1C (2001) 909-912.

[5] B. Jancewicz, Multivectors and Clifford Algebra in Electrodynamics World Scientific, Singapore 1989.

[6] "Clifford Algebras and their applications in Mathematical PhysicsVol 1: Algebras and Physics. eds by R. Ablamowicz, B. Fauser. Vol 2: Clifford analysis. eds by J. Ryan, W. Sprosig Birkhauser, Boston 2000.

[7] P. Lounesto, "Clifford Algebras and Spinors". Cambridge University Press. 1997.

[8] C. Castro, Chaos, Solitons and Fractals **10** (1999) 295. Chaos, Solitons and Fractals **12** (2001) 1585. " The Search for the Origins of M Theory: Loop Quantum Mechanics, Loops/Strings and Bulk/Boundary Dualities " [arXiv: hep-th/9809102].

[9] C. Castro *The Programs of the Extended Relavity in C-spaces, towards physical foundations of String Theory*, Advance NATO Workshop on the Nature of Time, Geometry and the Physics of Perception, Tatranksa Lomnica, Slovakia, May 2001. Kluwer Publishers 2003.

[10] C. Castro, Chaos, Solitons and Fractals **11** (2000) 1663. Foundations of Physics **30** (2000) 1301.

[11] D. Amati, M. Ciafaloni and G. Veneziano, Phys. Letts **B 197** (1987) 81. D. Gross, P. Mende, Phys. Letts **B 197** (1987) 129. M.Maggiore, Phys. Lett **B 304** (1993) 65.

[12] S. Ansoldi, C.Castro and E. Spalluci, Class. Quant. Grav **18** (1999) 1833. C. Castro, Chaos, Solitons and Fractals **11** (2000) 1721

[13] Y.Hosotani, Phys. Rev. Lett **55** (1985) 1719. L.Carson, C.H. Ho and Y. Hosotani, Phys. Rev. **D 37** (1988) 1519. C.H. Ho, Jour. Math. Phys **30** (1989) 2168.

[14] W. Pezzaglia, "Physical Applications of a Generalized Geometric Calculus" [arXiv: gr-qc/9710027]. Dimensionally Democratic calculus and Principles of Polydimensional Physics [arXiv: gr-qc/9912025]. Classification of Multivector Theories and Modifications of the Postulates of Physics [arXiv: gr-qc/9306006] ; Physical Applications of Generalized Clifford Calculus: Papatetrou equations and Metamorphic Curvature [arXiv: gr-qc/9710027]. Classification of Multivector theories and modification of the postulates of Physics [arXiv: gr-qc/9306006].

[15] M.Pavšič: "The Landscape of Theoretical Physics: A Global View; From Point Particle to the Brane World and Beyond, in Search of Unifying Principle", Kluwer Academic, Dordrecht 2001.

[16] M.Pavšič, Foundations of Physics **31** (2001) 1185.

[17] M.Pavšič, Foundations of Physics **33** (2003) 1277 gr-qc/0211085.

[18] S. Ansoldi, A. Aurilia, C. Castro, E. Spallucci, Phys. Rev. **D 64** 026003 (2001)

[19] A. Aurilia, S. Ansoldi and E. Spallucci, Class. Quant. Grav. **19** (2002) 3207 [arXiv:hep-th/0205028].

[20] C. Castro, M. Pavšič, Phys. Lets. **B 539** (2002) 133.

[21] C. Castro, Jour. of Entropy **3** (2001) 12-26. C. Castro and Alex Granik, Foundations of Physics **33** (3) (2003) 445.

[22] T. Ivezic, Foundations of Physics Lett **15** (2002) 27. Invariant Relativistic Electrodynamics, Clifford Algebra Approach arXiv: hep-th/0207250].

[23] L. Nottale: *Fractal Spacetime and Microphysics, towards the Theory of Scale Relativity*, World Scientific 1992. La Relativité dans tous ses états, Hachette Literature, Paris,

[24] S. Ansoldi, A. Aurilia and E. Spallucci, Phys. Rev **D 56** no. 4 (1997) 2352. Chaos, Solitons and Fractals **10** (1999) 197.

[25] C. Castro and M. Pavšič, Int. J. Theor. Phys. **42** (2003) 1693.

[26] M. Pavšič, Class. Quant. Grav. **20** (2003) 2697 [arXiv:gr-qc/0111092].

[27] M. Pavšič, Nuovo Cim. B **41** (1977) 397; *International Journal of Theoretical Physics* **14**, 299 (1975)

[28] M. Pavšič, J. Phys. A **13** (1980) 1367.

[29] M. Pavšič, Found. Phys. **26** (1996) 159 [arXiv:gr-qc/9506057].

[30] M. Pavšič, arXiv:gr-qc/0210060.

[31] J. P. Terletsky, *Doklady Akadm. Nauk SSSR*, **133**, 329 (1960); M. P. Bilaniuk, V. K. Deshpande and E.C. G. Sudarshan, *American Journal of Physics* **30**, 718 (1962); E. Recami and R. Mignani, *Rivista del Nuovo Cimento* **4**, 209 (1974); E. Recami, *Rivista del Nuovo Cimento* **9**, 1 (1986)

[32] M. Pavšič, J. Phys. A **14** (1981) 3217.

[33] E.C.G. Stueckelberg, *Helv. Phys. Acta*, **14**, 322 (1941); **14**, 588 (1941); **15**, 23 (1942)

[34] L.P. Horwitz and C. Piron, *Helv. Phys. Acta*, **46**, 316 (1973); L.P. Horwitz and F. Rohrlich, *Physical Review D* **24**, 1528 (1981); **26**, 3452 (1982); L.P. Horwitz, R.I. Arshansky and A.C. Elitzur *Found. Phys* **18**, 1159 (1988); R.I. Arshansky, L.P. Horwitz andY. Lavie, *Foundations of Physics* **13**, 1167 (1983)

[35] M. Pavšič, *Found. Phys.* **21**, 1005 (1991); *Nuovo Cim.* **A104**, 1337 (1991); *Doga, Turkish Journ. Phys.* **17**, 768 (1993) *Found. Phys.* **25**, 819 (1995); *Nuovo Cimento A* **108**, 221 (1995); *Foundations of Physics* **26**, 159 (1996); M. Pavšič, *Nuovo Cimento A* **110**, 369 (1997); Found. Phys. **28** (1998); 1443.Found. Phys. **28** (1998) 1453.

[36] C.W. Misner, K. S. Thorne and J. A. Wheeler, *Gravitation*, Freeman, San Francisco, 1973), p. 1215

[37] A. Schild, *Phys.Rev.* **D16**, 1722 (1977)

[38] H.A. Kastrup, *Annaled der Physik (Lpz.)* **7**, 388 (1962)

[39] A.O. Barut and R. B. Haugen, *Annals of Physics* **71**, 519 (1972)

[40] E. Cunningham, Proc. London Math. Soc. **8**, 77 (1909); H. Bateman, Proc. London Math. Soc. **8**, 223 (1910); T. Fulton, F. Rohrlich and L. Witten, Rev. Mod. Phys. **34**, 442 (1962); J. Wess, Nuovo Cim., **18**, 1086 (1960); G. Mack, Nucl. Phys. **B5**, 499 (1968); A.F. Grillo, Riv. Nuovo Cim. **3**, 146 (1973); J. Niederle and J. Tolar, Czech. J. Phys. **B23**, 871 (1973)

[41] D. Hestenes and R. Ziegler, Projective Geometry with Clifford Algebra, Acta Applicandae Mathematicae **23** (1991) 25-63.

[42] D. V Ahluwalia and C. Burgard, General Relativity and Gravitation **28** (10) (1996) 1163. D.V. Ahluwalia: General Relativity and Gravitation **29** (12) (1997) 1491. D.V. Ahluwalia, Phys. Letts **A 275** (2000) 31. G.Adunas, E. Rodriguez-Milla and D.V. Ahluwalia, Phys. Letts **B 485** (2000) 215.

[43] F.D. Smith, Int. Jour. of Theoretical Physics **24** (1985) 155. Int. Jour. of Theoretical Physics **25** (1985) 355. From Sets to Quarks [arXiv: hep-ph/9708379]. http://www.innerx.net/personal/tsmith/clfpq.html.

[44] F. I. Cooperstock, V. Faraoini, " Extended Planck Scale " [arXiv: hep-th/0302080].

[45] H. I. Arcos and J. G. Pereira, " Kerr-Newman solutions as a Dirac particle" [arXiv: hep-th/0210103].

[46] W. Smilga: Higher order terms in the contraction of $S0(3,2)$ [arXiv: hep-th/0304137]. A. Wyler, C. R. Acad. Sc. Paris **269** (1969) 743-745.

[47] C.Beck, *Spatio-Temporal Vacuum fluctuations of Quantized Fields* . World Scientific series in Advances in Nonlinear Dynamics, vol. 21 (2002).

[48] G. Trayling and W. Baylis, J. Phys. **A 34** (2001) 3309. J. Chisholm and R. Farwell: J. Phys. **A 32** (1999) 2805. J. Chisholm and R. Farwell: Foundations of Physics **25** (1995) 1511.

[49] M. Born: Proc. Royal Society **A 165** (1938) 291. Rev. Mod. Physics **21** (1949) 463.

[50] *Clifford (Geometric) Algebras, with applications to Physics, Mathematics and Engineering* W. E. Baylis, editor; Birkhauser, Boston, 1997.

[51] S. Somaro: Higher Spin and the Spacetime Algebra V. Dietrich et al. (eds), *Clifford Algebras and their Applications in Mathematical Physics* (1998) 347-368; Kluwer Academic Publishers, the Netherlands.

[52] E. Caianiello, “Is there a maximal acceleration?”, Lett. Nuovo Cimento **32** (1981) 65.

[53] V. Nesterenko, Class. Quant. Grav. **9** (1992) 1101; Phys. Lett. **B 327** (1994) 50;

[54] V. Bozza, A. Feoli, G. Lambiase, G. Papini and G. Scarpetta, Phys. Let **A 283** (2001) 53. V. Nesterenko, A. Feoli, G. Lambiase and G. Scarpetta, Phys. Rev **D 60**, 065001 (1999).

[55] K. Rama, ”Classical velocity in kappa-deformed Poincare algebra and amaximal acceleration” [arXiv: hep-th/0209129].

[56] H. Brandt: Contemporary Mathematics **196** (1996) 273. Chaos, Solitons and Fractals **10** (2-3) (1999) 267.

[57] M.Pavšič: Phys. Lett **B 205** (1988) 231 ; Phys. Lett **B 221** (1989) 264. H. Arodz, A. Sitarz, P. Wegrzyn: Acta Physica Polonica **B 20** (1989) 921

[58] M. Plyushchay, ” Comment on the relativistic particle with curvature and torsion of world trajectory” [arXiv: hep-th/9810101]. Mod.Phys.Lett. A10 (1995) 1463-1469

[59] E. Ramos, J. Roca, Nucl.Phys. B477 (1996) 606-622

[60] H. Kleinert, A.M Chervyakov, ” Evidence for Negative Stiffness of QCD Strings ” [arXiv: hep-th/9601030].

[61] F. Schuller, Annals of Phys. **299** (2002) 174.

[62] A. Chernitskii: Born-Infeld electrodynamics, Clifford numbers and spinor representations [arXiv: hep-th/0009121].

[63] J. Lukierski, A. Nowicki, H. Ruegg, V. Tolstoy, Phys. Lett **264** (1991) 331. J. Lukierski, H. Ruegg, W. Zakrzewski: Ann. Phys **243** (1995) 90. J. Lukierski, A. Nowicki: Double Special Relativity verus kappa-deformed dynamics. [arXiv: hep-th/0203065].

[64] J. Webb, M. Murphy, V. Flambaum, V. Dzuba, J. Barrow, C. Churchill, J. Prochaska, and A. Wolfe, ” Further evidence for Cosmological Evolution of the Fine Structure Constant ” Monthly Notices of the Royal Astronomical Society **327** (2001) 1208.

[65] J.P. Uzan, ” The fundamental constants and their variations: observational status and theoretical motivations” [arXiv: hep-ph/0205340].

[66] L. Anchordogui, H. Goldberg, Phys. Rev. **D 68** (2003) 083513.

[67] G. Lambiase, G.Papini. G. Scarpetta: Maximal Acceleration Corrections to the Lamb Shift of one Electron Atoms [arXiv: hep-th/9702130]. G. Lambiase, G.Papini. G. Scarpetta, Phys. Let **A 224** (1998) 349. G. Papini, ” Shadows of a maximal acceleration” [arXiv: gr-qc/0211011].

[68] C. Castro, Int. J. Mod. Phys. A **18** (2003) 5445 [arXiv: hep-th/0210061]

[69] L. Castellani: Phys. Lett **B 327** (1994) 22. Comm. Math. Phys **171** (1995) 383.

[70] G.Amelino-Camelia, Phys. Lett **B 510** (2001) 255. Int. J. Mod. Phys **D 11** (2002) 35. Int. J. Mod. Phys **D 11** (2002) 1643.

[71] K. Greisen, Phys. Rev. Lett **16** (1966) 748. G.T. Zatsepin, V. Kurmin, Sov. Phys. JETP Lett **4** (1966) 78.

[72] J. Ellis, N. Mavromatos and D. V. Nanopolous, Chaos, Solitons and Fractals, **10** (1999) 345.

[73] M. Toller, "The Geometry of Maximal Acceleration" [ArXiv: hep-th/0312016].

[74] S. Low: Jour. Phys **A** Math. Gen **35** (2002) 5711.

[75] C.Castro, Mod. Phys. Letts. **A 19** (2004) 19

[76] Y. Choquet-Bruhat, C. DeWitt-Morette and M. Dillard-Bleick, *Analysis, Manifolds and Physics* (revised edition), North Holland Publ. Co., Amsterdam, 1982.

[77] A. Crumeyrole, *Orthogonal and Sympletic Clifford Algebras*, Kluwer Acad, Publ., Dordrecht, 1990.

[78] H. Blaine Lawson and M.L. Michelson, *Spin Geometry*, Princeton University Press, Princeton, 1980.

[79] A.M.Moya, V.V Fernandez and W.A. Rodrigues, Int.J.Theor.Phys. **40** (2001) 2347-2378 [arXiv: math-ph/0302007]; *Multivector Functions of a Multivector Variable* [arXiv: math.GM/0212223]; *Multivector Functionals* [arXiv: math.GM/0212224]

[80] C. Castro, "The charge-mass-spin relationship of a Clifford polyparticle, Kerr-Newman Black holes and the Fine structure Constant", to appear in Foundations of Physics.

[81] S. Vacaru and N Vicol: Nonlinear Connections and Clifford Structures math.DG/0205190. S. Vacaru: " (Non) Commutative Finsler Geometry from String/M Theory " [arXiv: hep-th/0211068].

[82] I.Kanatchikov, Rep. Math. Phys **43** (1999) 157. V. Kisil, [arXiv: quant-ph/0306101].

[83] N. Bjerrus-Bohr, " Quantum Gravity at large number of dimensions" [arXiv: hep-th/0310263]. J.F. Donoghue, Phys. Rev. **D 50** (1994) 3874. A. Strominger, Phys. Rev. **D 24** (1981) 3082.

[84] W.A. Rodrigues, Jr, J. Vaz, Jr, Adv. Appl. Clifford Algebras **7** (1997) 457-466. E.C de Oliveira and W.A. Rodrigues, Jr, Ann. der Physik **7** (1998) 654-659. Phys. Lett bf A 291 (2001) 367-370. W.A. Rodrigues, Jr, J.Y.Lu, Foundations of Physics **27** (1997) 435-508.

[85] P. Ramond, " Exceptional Groups and Physics " [arXiv: hep-th/0301050]

[86] . B. Fauser, " A treatise on Quantum Clifford Algebras" [arXiv: math.QA/0202059.]

[87] M. Altaisky, " Wavelet based regularization for Euclidean field theory and stochastic quantization " [arXiv: hep-th/0311048].

[88] V. Tapia, " Polynomial identities for Hypermatrices" [arXiv: math-ph/0208010].

[89] M. Defaria-Rosa, E. Recami and W. Rodrigues, Phys. Letts **B 173** (1986) 233.

[90] E. Capelas de Oliveira and W.A Rodrigues, " Clifford valued Differential Forms, Algebraic spinor fields, Gravitation, Electromagnetism and " unified " theories " [arXiv: math-ph/0311001].

[91] C. Lanczos, Z. Physik **57** (1929) 447–473; ibid 474-483; ibid 484–493; C. Lanczos, Physik. Zeitschr. **31** (1930) 120–130.

[92] F. Gursey, *Applications of Quaternions to Field Equations*, Ph.D thesis, University of London (1950) 204 pp.

[93] R.A. Mosna and W.A. Rodrigues, Jr., J. Math. Phys. **45** (2004) 7 [arXiv: math-ph/0212033]; W.A. Rodrigues, J. Math. Phys. **45** (2004) (to appear) [arXiv: math-ph/0212030]

[94] K. Avinash and V. I. Rvachev, Foundations of Physics **30**(2000) 139.

[95] D. Gottesman, "The Heisenberg Representation of Quantum Computers" [ArXiv: quant-ph9807006].

[96] J. Baugh, D, Finkelstein, A. Galiautdinov and M. Shiri-Garakaui, Found.Phys. 33 (2003) 1267-1275

[97] S.L. Adler, *Quaternionic Quantum Mechanics and Quantum Fields* Oxford Univ. Press, New York, 1995.

[98] S. de Leo and K. Abdel-Khalek, " Towards an Octonionic World " [arXiv:hep-th/9905123]. S. de Leo, " Hypercomplex Group Theory " [arXiv: physics/9703033].

[99] C.Castro, " Noncommutative QM and Geometry from quantization in C-spaces " [arXiv: hep-th/0206181].

[100] M. Watkins, " Number theory and Physics " website, http://www.maths.ex.ac.uk/mwatkins

[101] V. Vladimorov, I. Volovich and I. Zelenov, *p-adic Numbers in Mathematical Physics* World Scientific, 1994, Singapore. L. Brekke and P. Freund, Phys. Reports **231** (1991) 1. M. Pitkanen, *Topological Geometrodynamics* http://www.physics.helsinki.fi/matpitka/tgd.html

[102] C.N. Yang, Phys. Rev. **72** (1947) 874; H.S. Snyder, Phys. Rev. **71** (1947) 38. H.S. Snyder, Phys. Rev **72** (1947) 68. S. Tanaka, Yang's Quantized Spacetime Algebra and Holographic Hypothesis [arXiv: hep-th/0303105].

[103] W. da Cruz: "Fractal von Neumann Entropy" [arXiv: cond-mat/0201489]. " Fractons and Fractal Statistics " arXiv: hep-th/9905229.

[104] C.Castro, Chaos, Solitons and Fractals **14** (2002) 1341. **ibid 15** (2003) 797. M. Lapidus and M. Frankenhuysen, *Fractal strings, complex dimensions and the zeros of the zeta function* Birkhauser, New York (2000).

[105] C. Castro, " The Nonextensive Statistics of Fractal Strings and Branes " submitted to Physica A. J. Havrda and F. Charvat: Kybernatica **3** (1967) 30. C-Tsallis, Jour. of Statistical Physics **52** (1988) 479. C.Tsallis, " Non-extensive Statistical mechanics: A brief review of its present status " [arXiv: cond-mat/0205571].

[106] D. Frieden, *Physics from Fisher Information Theory*, Cambridge University Press 1998.

[107] T. Nowotny, M. Requardt, Chaos, Solitons and Fractals **10** (1999) 469.

[108] , F. Lizzi, R.J Szabo, Chaos, Solitons and Fractals **10** (1999) 445.

[109] I. Raptis, " Quantum Space-Time as a Quantum Causal Set" [arXiv gr-qc/0201004]. " Presheaves, Sheaves and their Topoi in Quantum Gravity and Quantum Logic" [arXiv: gr-qc/0110064]. " Non-Commutative Topology for Curved Quantum Causality " [arXiv: gr-qc/0101082].

[110] C. Isham and J. Butterfield, Found.Phys. 30 (2000) 1707-1735. A. K. Guts, E. B. Grinkevich, " Toposes in General Theory of Relativity " [arXiv: gr-qc/9610073].

[111] A. Ashtekar, C. Rovelli and L. Smolin, Phys. Rev. Lett **69** (1992) 237. C. Rovelli, " A dialog on quantum gravity " [arXiv: hep-th/0310077] L. Freidel, E. Livine and Carlo Rovelli, Class.Quant.Grav. 20 (2003) 1463-1478. L. Smolin, " How far are we from the quantum theory of gravity? " [arXiv:hep-th/0303185]

[112] M.Saniga, Chaos, Solitons and Fractals 19 (2004) 739-741

[113] Norma Mankoc Borstnik and H.B. Nielsen, J. Math. Phys. 44 (2003) 4817-4827

[114] L. Castellani, Class. Quant. Grav. 17 (2000) 3377-3402 Phys.Lett. B327 (1994) 22-28

[115] E. Guendelman, E. Nissimov and S. Pacheva, " Strings, p-Branes and Dp-Branes With Dynamical Tension ". hep-th/0304269

[116] P. Bouwknegt, K. Schouetens, Phys. Reports **223** (1993) 183-276. E. Sezgin, " Aspects of W_∞ Symmetry " hep-th/9112025. X. Shen, Int.J.Mod.Phys. A7 (1992) 6953-6994.

[117] M. Vasiliev: " Higher Spin Gauge Theories, Star Product and AdS spaces " [arXiv: hep-th/9910096]. M. Vasiliev, S. Prokushkin: "Higher-Spin Gauge Theories with Matter". [arXiv: hep-th/9812242, hep-th/9806236].

[118] Y. Ne'eman, E. Eizenberg, "Membranes and Other Extendons (p-branes) ", World Scientific Lecture Notes in Physics vol. **39** 1995. J. Polchinski, *Superstrings*. Cambridge University Press (2000). M. Green, J. Schwarz and E. Witten, *Superstring Theory*. Cambridge University Press (1986).

[119] J. Maldacena, Adv. Theor. Math. Phys **2** (1998) 231.

[120] Y.Cho, K, Soh, Q. Park, J. Yoon: Phys. Lets **B 286** (1992) 251. J. Yoon, Phys. Letts **B 308** (1993) 240; J. Yoon, Phys. Lett **A 292** (2001) 166. J. Yoon, Class. Quan. Grav **16** (1999) 1863.

[121] C. Castro, Europhysics Letters **61** (4) (2003) 480. Class. Quant. Gravity **20** no. 16 (2003) 3577.

[122] A.B. Zamoldchikov, Teor. Fiz **65** (1985) 347 Pope et al C. Pope, Nucl.Phys. B413 (1994) 413-432 C. Pope, L. Romans, X. Shen, Nuc. Phys, **B 339** (1990) 191. C. Pope, L. Romans, X. Shen, Phys. Letts **B 236** (1990) 173. C. Pope, L. Romans, X. Shen, Phys. Letts **B 242** (1990) 401.

[123] B. Dolan, Tchrakian: Phys. Letts **B 202** (2) (1988) 211.

[124] E. Hawkins, F. Markopoulou and H. Sahlmann " Evolution in Quantum Causal Histories " [arXIv ; hep-the/0302111]. F. Markoupolu, L. Smolin, " Quantum Theory from Quantum Gravity " [arXiv: gr-qc/0311059].

[125] M.Douglas, N. Nekrasov, Rev.Mod.Phys. 73 (2001) 977-1029

[126] T. Eguchi, *Phys. Rev. Lett.* **44**, 126 (1980)

[127] C. Castro, " Maximal-acceleration phase space relativity from Clifford algebras " [arXiv: hep-th/0208138].

[128] V. Abramov, R. Kerner and B. Le Roy, J.Math.Phys. 38 (1997) 1650-1669

[129] J. Cresson, " Scale calculus and the Schrodinger equation " [arXiv: math.GM/0211071].

[130] E.H. Kinani, " Between Quantum Virasoro Algebras and Generalized Clifford Algebras [arXiv: math-ph/0310044].

[131] S. Capozziello, S. Carloni and A. Troisi, " Quintessence without scalar fields " [arXiv: astro-ph/0303041]. S. Carroll, V. Duvvuri, M. Trodden and M. Turner, " Is Cosmic Speed-Up Due to New Gravitational Physics? " [arXiv: astro-ph/0306438]. A. Lue, R. Scoccimarro and G. Strakman, " Differentiating between Modified Gravity and Dark Energy " [arXiv: astro-ph/0307034].

[132] A. Aurilia, A. Smailagic and E. Spallucci, *Physical Review D* **47**,2536 (1993); A. Aurilia and E. Spallucci, *Classical and Quantum Gravity* **10**, 1217 (1993); A. Aurilia, E. Spallucci and I. Vanzetta, *Physical Review D* **50**, 6490 (1994); S. Ansoldi, A. Aurilia and E. Spallucci, *Physical Review D* **53**, 870 (1996); S. Ansoldi, A. Aurilia and E. Spallucci, *Physical Review D* **56**, 2352 (1997)