QUANTUM VACUUM AND CLIFFORD ALGEBRAS

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Contents

Clifford space as an arena for physics

Distance in spacetime – line element

$$\boxed{\mathrm{d}s^2 = \mathrm{d}x^\mu \eta_{\mu\nu} \mathrm{d}x^\nu}$$

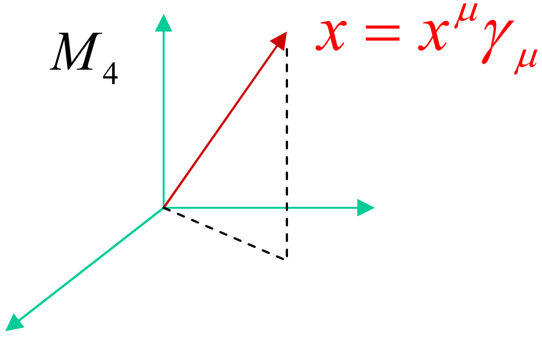
Square root of the distance – a vector

$$\boxed{\mathrm{d}s = \mathrm{d}x^\mu \gamma_\mu}$$

where the basis vectors γ_μ satisfy

$$\boxed{\gamma_\mu \cdot \gamma_\nu \equiv \frac{1}{2}(\gamma_\mu \gamma_\nu + \gamma_\mu \gamma_\nu) = \eta_{\mu\nu}}$$

Position in spacetime can be described as a vector:



This can be generalized to Clifford algebra:

$$X = \sigma \underline{1} + x^\mu \gamma_\mu + x^{\mu\nu} \gamma_\mu \wedge \gamma_\nu + x^{\mu\nu\alpha} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\alpha + x^{\mu\nu\alpha\beta} \gamma_\mu \wedge \gamma_\nu \wedge \gamma_\alpha \wedge \gamma_\beta$$

A Clifford number X describes position in Clifford space.

Clifford space is a manifold whose tangent space at any of its points is a Clifford algebra. In particular, a Clifford space can be flat, but in general it can be curved.

Instead of the usual relativity formulated in spacetime in which the interval is

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \eta_{\mu\nu} \mathrm{d}x^\nu = \mathrm{d}x^\mu \gamma_\mu \gamma_\nu \mathrm{d}x^\nu$$

we are studying the theory in which the interval is extended to the space of r-volumes (called Clifford space):

$$\mathrm{d}S^2 = \mathrm{d}x^M G_{MN} \mathrm{d}x^N = \langle \mathrm{d}x^M \gamma_M{}^\ddagger \gamma_N \mathrm{d}x^N \rangle_0 \qquad \mathrm{d}x^M \equiv \mathrm{d}x^{\mu_1 \dots \mu_r}, \qquad r = 0,1,2,3,4$$

Metric

$$G_{MN} = \gamma_M^\ddagger * \gamma_N \equiv \langle \gamma_M^\ddagger \gamma_N \rangle_0$$

Reversion

$$(\gamma_{\mu_1} \gamma_{\mu_2} \dots \gamma_{\mu_r})^\ddagger = \gamma_{\mu_r} \dots \gamma_{\mu_2} \gamma_{\mu_1}$$

Signature: $+ \; + + + + + + + - - - - - - - -$ (8,8)

In flat C-space: $\gamma_M \equiv \gamma_{\mu_1 \mu_2 \dots \mu_r} = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \dots \wedge \gamma_{\mu_r}$

at every point $\mathcal{E} \in C$

Instead of the usual relativity formulated in spacetime in which the interval is

$$\mathrm{d}s^2 = \mathrm{d}x^\mu \eta_{\mu\nu} \mathrm{d}x^\nu = \mathrm{d}x^\mu \gamma_\mu \gamma_\nu \mathrm{d}x^\nu$$

we are studying the theory in which the interval is extended to the space of r-volumes (called Clifford space):

$$\mathrm{d}S^2 = \mathrm{d}x^M G_{MN} \mathrm{d}x^N = \langle \mathrm{d}x^M \gamma_M^{\ddagger} \gamma_N \mathrm{d}x^N \rangle_0$$

$$\mathrm{d}x^M \equiv \mathrm{d}x^{\mu_1 \dots \mu_r}, \qquad r = 0,1,2,3,4$$

$$\gamma_M = \gamma_{\mu_1} \wedge \gamma_{\mu_2} \wedge \dots \wedge \gamma_{\mu_r}$$

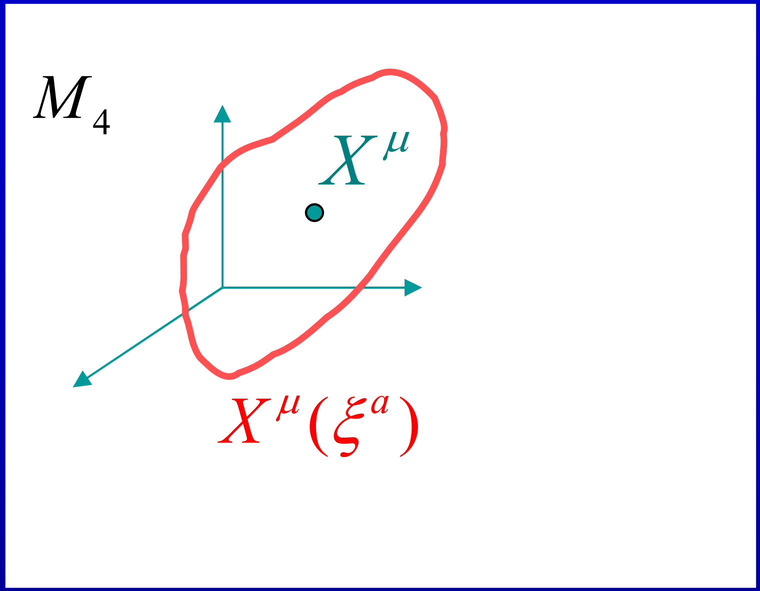
Coordinates of Clifford space can be used to model extended objects. They are a generalization of the concept of center of mass.

Instead of describing extended objects in ``full detail'', we can describe them in terms of the center of mass, area and volume coordinates.

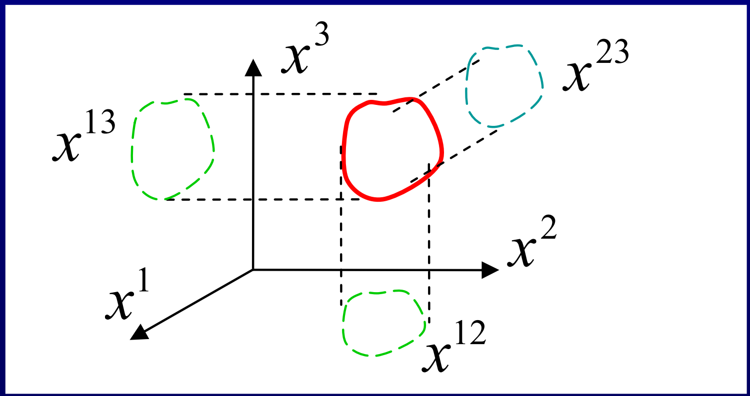
In particular, extended objects can be fundamental strings or branes.

Clifford space: a quenched configuration space of extended objects

Strings and branes have infinitely many degrees of freedom.
But at first approximation we can consider just the centre of mass.

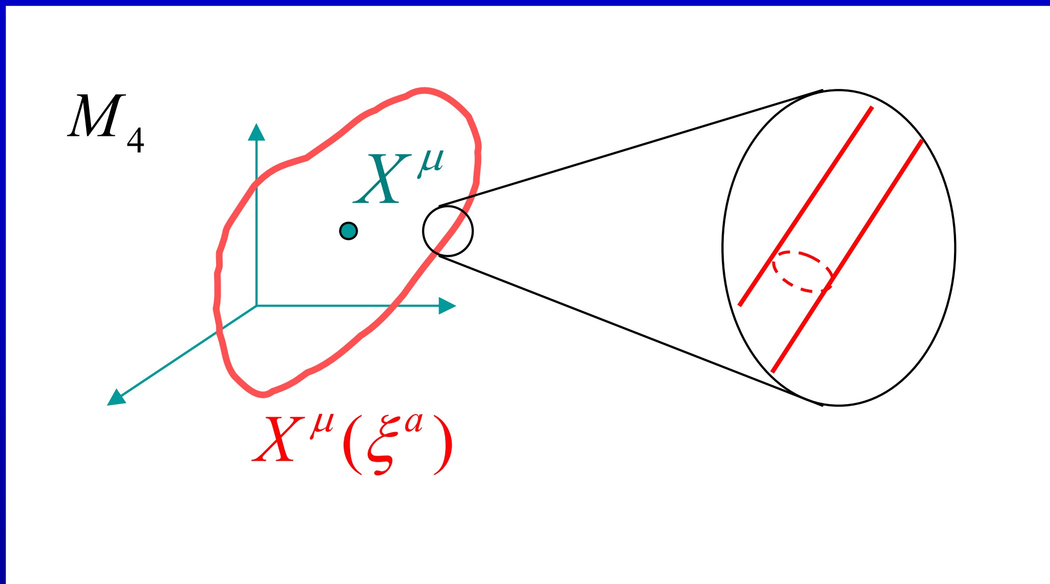


Next approximation is in considering the holographic coordinates of the oriented area enclosed by the string.



We may go further and search for eventual thickness of the object.

If the string has finite thickness, i.e., if actually it is not a string, but a 2-brane, then there exist the corresponding volume degrees of freedom.



X^{123}

In general, for an extended object in M_4, we have 16 coordinates

$$x^M \equiv x^{\mu_1 \dots \mu_r}, \qquad r = 0,1,2,3,4$$

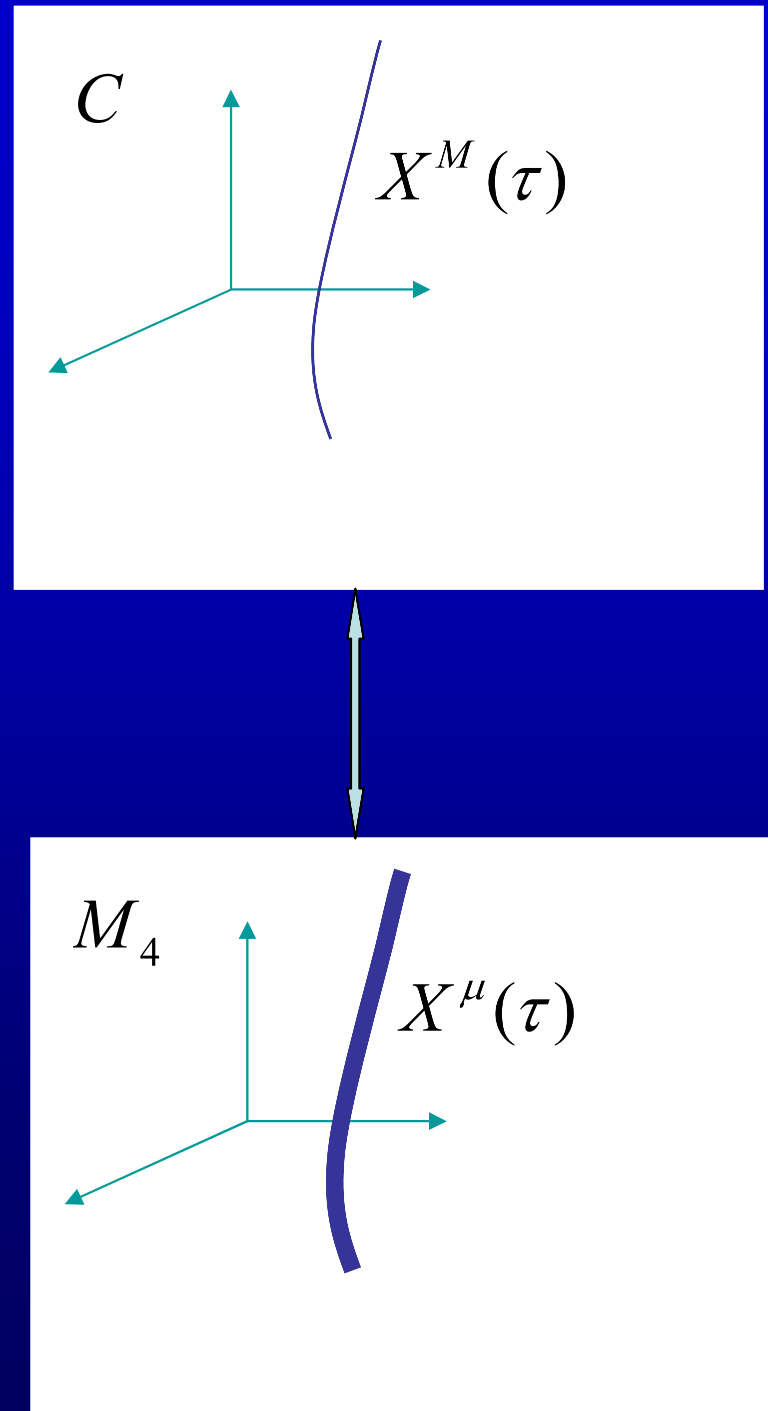
Polyvector coordinates

They are the projections of r-dimensional volumes (areas) onto the coordinate planes.

Oriented r-volumes can be elegantly described by Clifford algebra.

Thick point particles

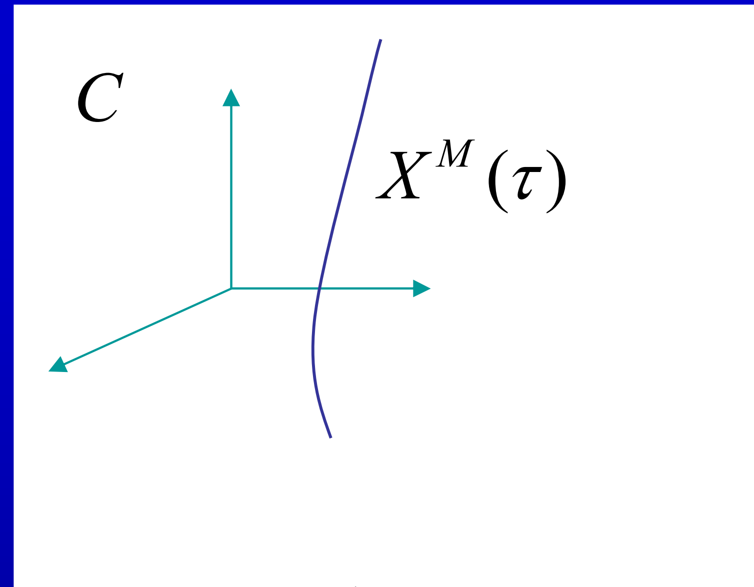
A world line in C represents the evolution of a `thick' particle in spacetime M_4

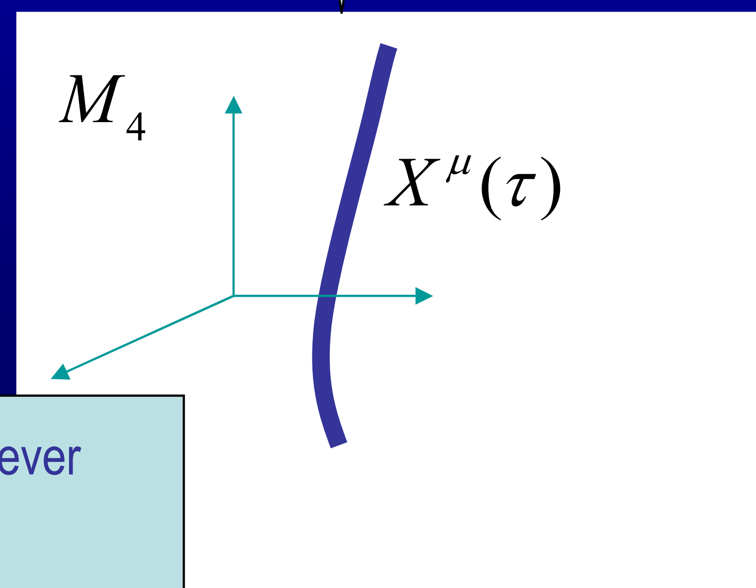


Thick particle can be an aggregate p-branes for various p=0,1,2,…

But such interpretation is not obligatory.

A world line in C represents the evolution of a `thick' particle in spacetime M_4





Thick particle can be an aggregate p-branes for various p=0,1,2,…

But such interpretation is not obligatory.

Thick particle may be a conglomerate of whatever extended objects that can be sampled by polyvector coordinates $X^M \equiv X^{\mu_1 \mu_2 \ldots \mu_r}$

A Toy Model: Harmonic Oscillator in Pseudo-Euclidean Space

Case $M_{1,1}$

$$L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - \frac{1}{2}\omega^2 (x^2 - y^2)$$

Equations of motion

$$\ddot{x} + \omega^2 x = 0\,, \qquad \ddot{y} + \omega^2 y = 0$$

The change of sign in front of the y-term has no influence on the equation of motion

Difference occurs when we calculate the canonical momenta

$$p_x = \frac{\partial L}{\partial \dot{x}} = \dot{x}\,, \qquad p_y = \frac{\partial L}{\partial \dot{y}} = -\dot{y}$$

and the Hamiltonian

$$H = p_x \dot{x} + p_y \dot{y} - L = \frac{1}{2}(p_x^2 - p_y^2) + \frac{\omega^2}{2}(x^2 - y^2)$$

The kinetic term for the y-component has negative sign, whilst that for the x-component has positive sign. Therefore, the equations of motion are

$$\ddot{x} = -\frac{\partial V}{\partial x}, \qquad \ddot{y} = \frac{\partial V}{\partial y}$$

$$\boxed{V = \frac{1}{2}\omega^2(x^2 - y^2)}$$

The criterion for the stability of motion for the y-degree of freedom is that the potential has to have a *maximum* in the (y,V)-plane.

Stability could be destroyed, if we include an extra interactive term into V.
I will demonstrate that even in the presence of an interaction,
stability can be preserved.

Presence of interactions

Classical Oscillator

$$L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - V\,, \qquad V = \frac{\omega}{2}(x^2 - y^2) + V_1$$

Interaction

Equation of motion:

$$\ddot{x} + \omega^2 x + \frac{\partial V_1}{\partial x} = 0$$

$$\ddot{y} + \omega^2 y - \frac{\partial V_1}{\partial x} = 0$$

$$V_1 = \frac{\lambda}{4}(x^2 - y^2)^2$$

As an example we will study this form of interaction

$$\ddot{x} + \omega^2 x + \lambda\, x(x^2 - y^2) = 0$$

$$\ddot{y} + \omega^2 y + \lambda\, y(x^2 - y^2) = 0$$

```
sol = NDSolve[{x''[t] + x[t] + 0.1 *x[t] * (x[t]^2 - y[t]^2) == 0,
   y''[t] + y[t] + 0.1*y[t] ... 0] == 0, x[0] == 0, y[0] == 1},
  {x, y}, {t, 1000}]
```

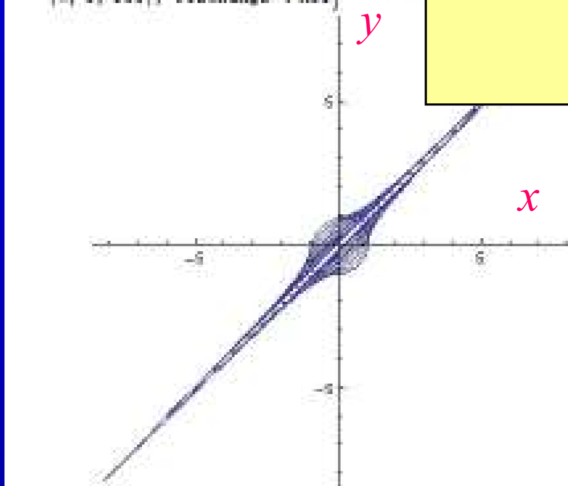
$$\ddot{x} + x + 0.1\,x(x^2 - y^2) = 0$$

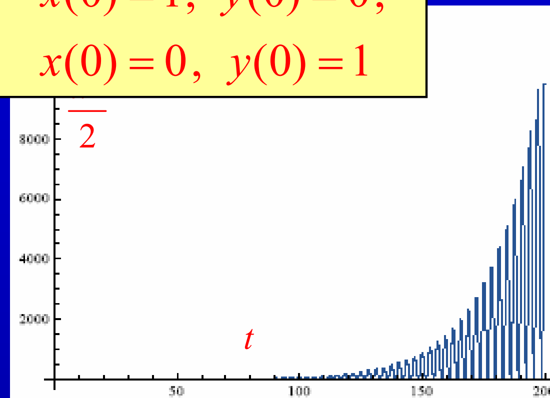
$$\ddot{y} + y + 0.1\,y(x^2 - y^2) = 0$$

$$\dot{x}(0) = 1,\ \ \dot{y}(0) = 0,$$

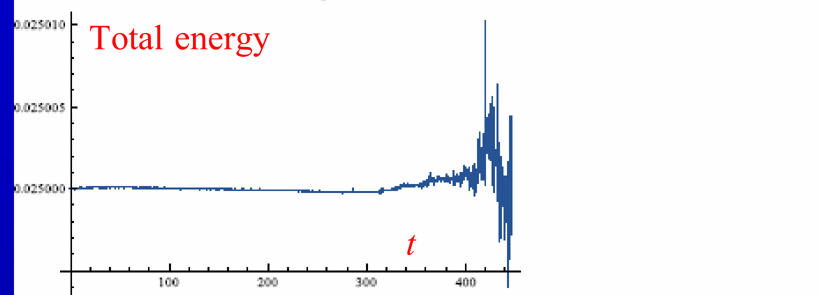
$$x(0) = 0,\ \ y(0) = 1$$

```
ParametricPlot[Evaluate[{x[t], y[t]} /. s
  {t, 0, 100}, PlotRange -> All]
```



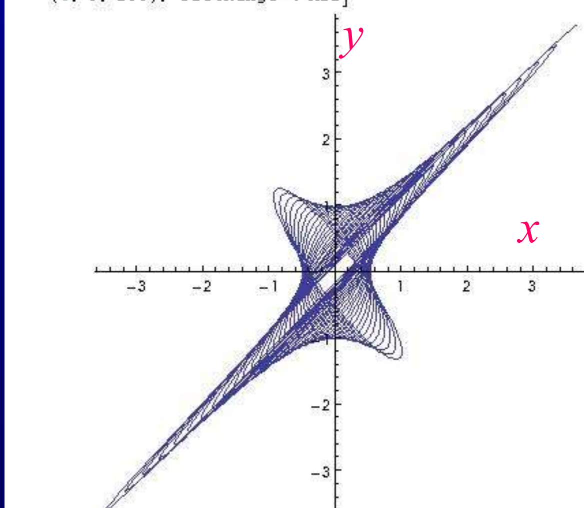



```
Plot[Evaluate[
  x'[t]^2 / 2 - y'[t]^2 / 2 + x[t]^2 / 2 - y[t]^2 / 2 + (0.1 / 4) (x[t]^2 - y[t]^2)^2 /. sol],
  {t, 0, 447}, PlotRange -> All]
```

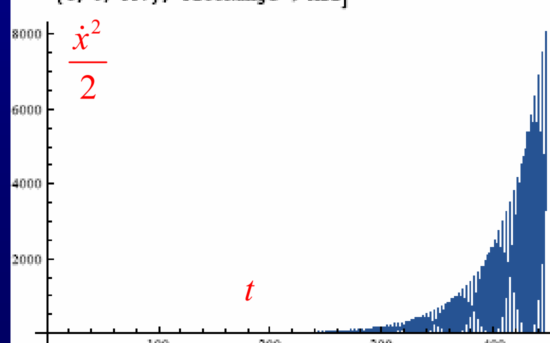



```
sol = NDSolve[{x''[t] + x[t] + 0.1 *x[t] * (x[t]^2 - y[t]^2) == 0,
   y''[t] + y[t] + 0.1*y[t] * (x[t]^2 - y[t]^2) == 0, x'[0] == 1, y'[0] == -1.2, x[0] == 0, y[0] == 0.5},
  {x, y}, {t, 3000}]
```

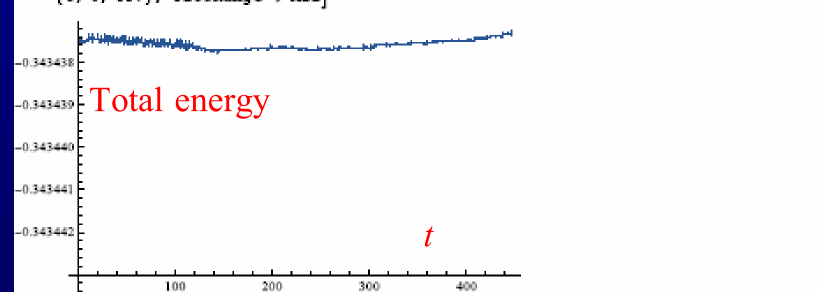
```
ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
  {t, 0, 200}, PlotRange -> All]
```




```
Plot[Evaluate[x'[t]^2 /. sol],
  {t, 0, 447}, PlotRange -> All]
```



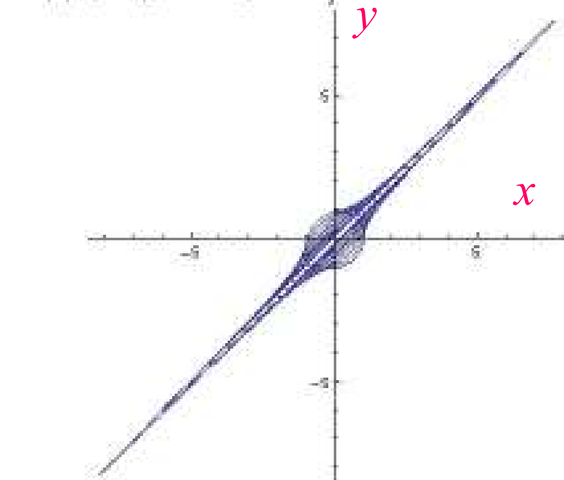

```
Plot[Evaluate[
  x'[t]^2 / 2 - y'[t]^2 / 2 + x[t]^2 / 2 - y[t]^2 / 2 + (0.1 / 4) (x[t]^2 - y[t]^2)^2 /. sol],
  {t, 0, 447}, PlotRange -> All]
```



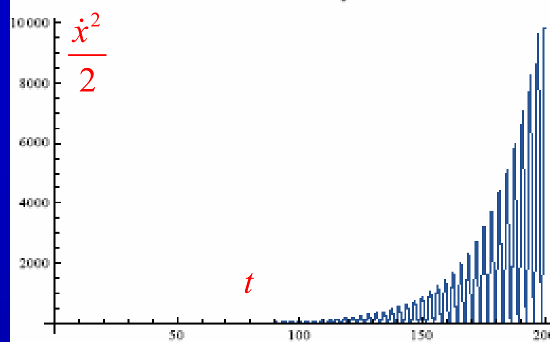
Calculations executed with Mathematica, by using NDSolve and ParametricPlot

```
sol = NDSolve[{x''[t] + x[t] + 0.1 *x[t] * (x[t] ^2 - y[t] ^2) == 0,
   y''[t] + y[t] + 0.1 *y[t] * (x[t] ^2 - y[t] ^2) == 0, x'[0] == 1, y'[0] == 0, x[0] == 0, y[0] == 1},
  {x, y}, {t, 1000}]
```

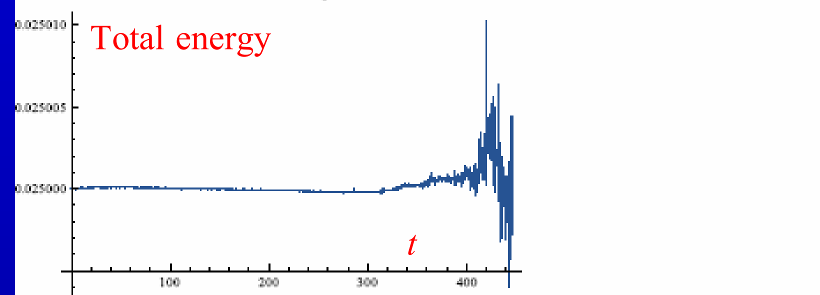
```
ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
  {t, 0, 100}, PlotRange -> All]
```




```
Plot[Evaluate[x'[t]^2 /. sol],
  {t, 0, 200}, PlotRange -> All]
```

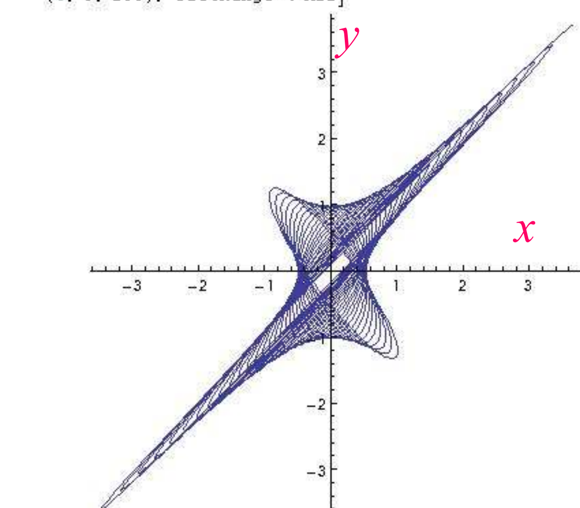



```
Plot[Evaluate[
  x'[t]^2 / 2 - y'[t]^2 / 2 + x[t]^2 / 2 - y[t]^2 / 2 + (0.1 / 4) (x[t]^2 - y[t]^2)^2 /. sol],
  {t, 0, 447}, PlotRange -> All]
```

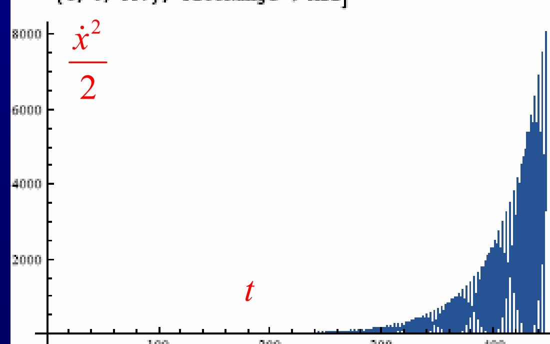



```
sol = NDSolve[{x''[t] + x[t] + 0.1 *x[t] * (x[t] ^2 - y[t] ^2) == 0,
   y''[t] + y[t] + 0.1 *y[t] * (x[t] ^2 - y[t] ^2) == 0, x'[0] == 1, y'[0] == -1.2, x[0] == 0, y[0] == 0.5},
  {x, y}, {t, 3000}]
```

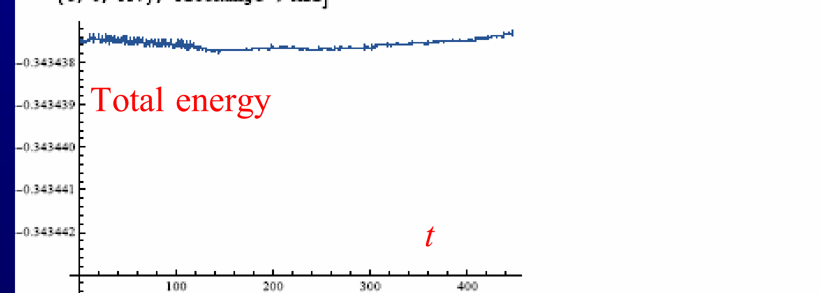
```
ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
  {t, 0, 200}, PlotRange -> All]
```




```
Plot[Evaluate[x'[t]^2 /. sol],
  {t, 0, 447}, PlotRange -> All]
```




```
Plot[Evaluate[
  x'[t]^2 / 2 - y'[t]^2 / 2 + x[t]^2 / 2 - y[t]^2 / 2 + (0.1 / 4) (x[t]^2 - y[t]^2)^2 /. sol],
  {t, 0, 447}, PlotRange -> All]
```



Calculations executed with Mathematica, by using NDSolve and ParametricPlot

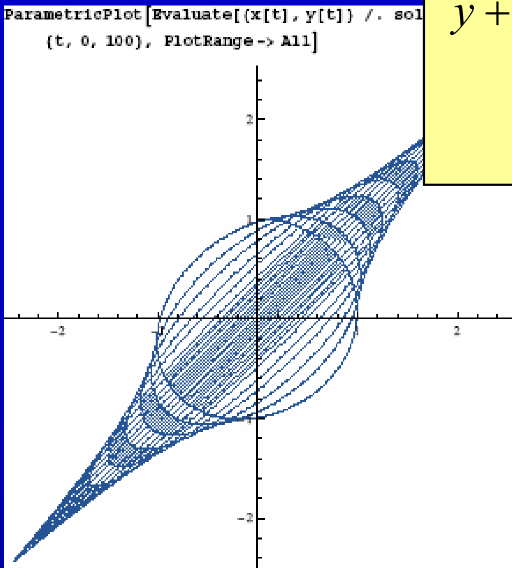
```
sol = NDSolve[{x''[t] + 1.01 x[t] + 1.01 * 0.1 * x[t] * (x[t]^2 - y[t]^2) == 0,
   y''[t] + y[t] + 0.1 * y[t
   y'[0] == 0, x[0] == 0, y
```

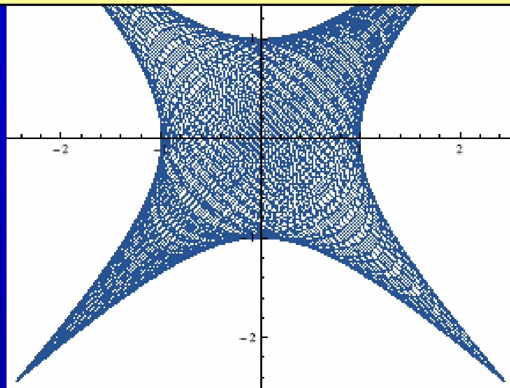
$$\ddot{x} + 1.01\,x + 1.01 \times 0.1\,x(x^2 - y^2) = 0$$

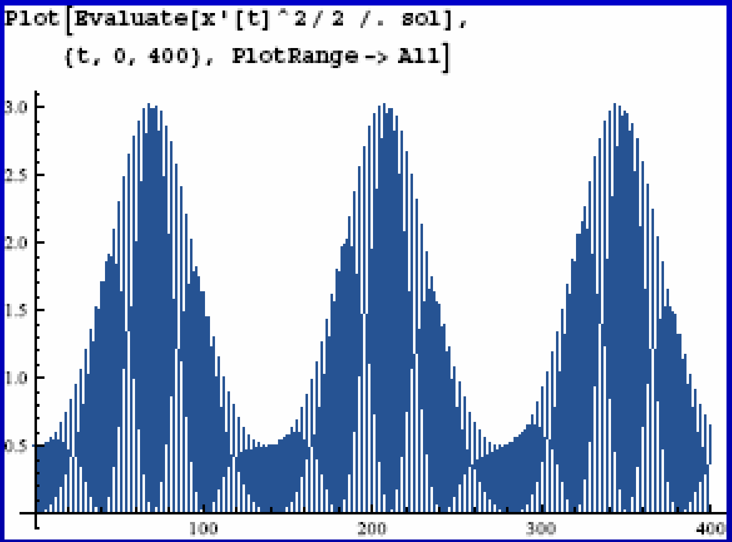
$$\ddot{y} + y + 0.1\,y(x^2 - y^2) = 0$$

$$\dot{x}(0) = 1, \ \dot{y}(0) = 0,$$

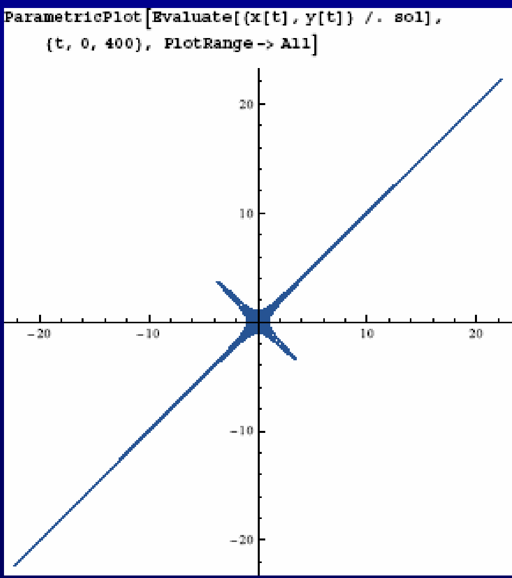
$$x(0) = 0, \ y(0) = 1$$

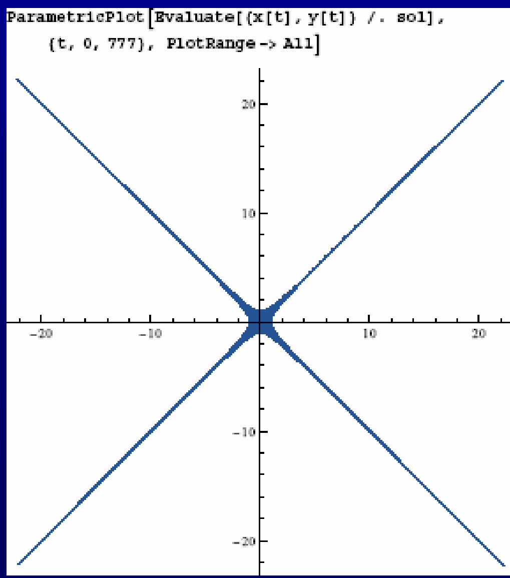


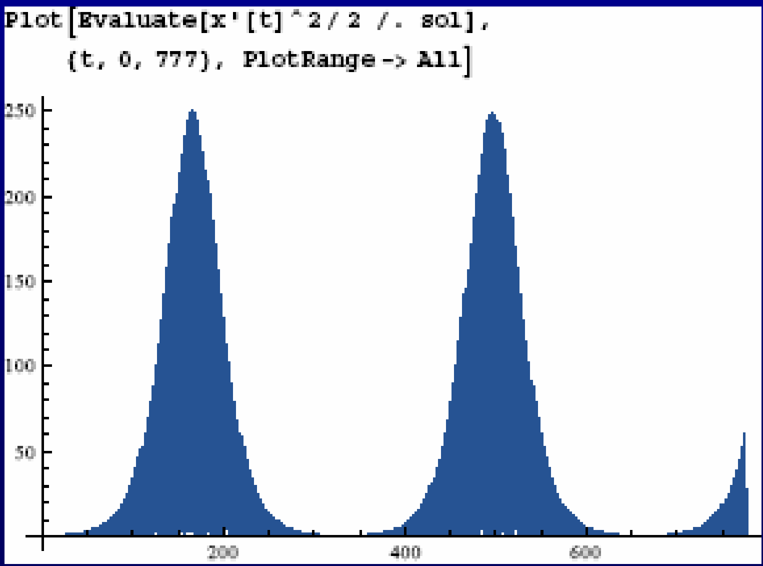





```
sol = NDSolve[{x''[t] + 1.0001 x[t] + 1.0001 * 0.1 * x[t] * (x[t]^2 - y[t]^2) == 0,
   y''[t] + y[t] + 0.1 * y[t] * (x[t]^2 - y[t]^2) == 0, x'[0] == 1,
   y'[0] == 0, x[0] == 0, y[0] == 1}, {x, y}, {t, 1000}]
```

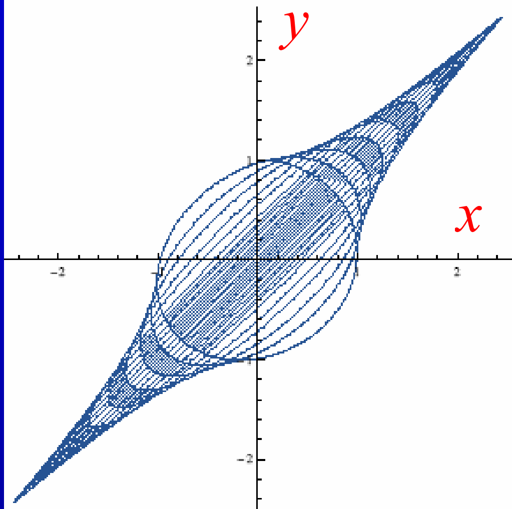




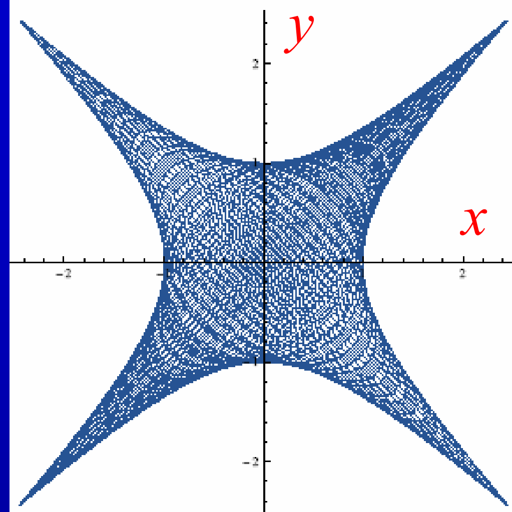


```
sol = NDSolve[{x''[t] + 1.01 x[t] + 1.01 * 0.1 * x[t] * (x[t]^2 - y[t]^2) == 0,
   y''[t] + y[t] + 0.1 * y[t] * (x[t]^2 - y[t]^2) == 0, x'[0] == 1,
   y'[0] == 0, x[0] == 0, y[0] == 1}, {x, y}, {t, 1000}]
```

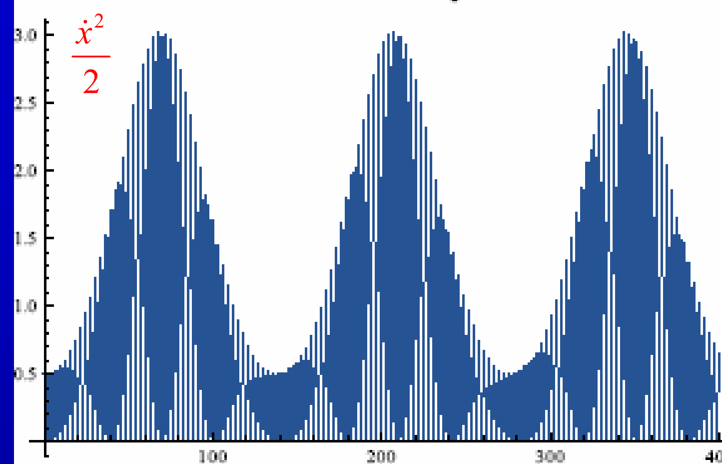
```
ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
   {t, 0, 100}, PlotRange -> All]
```




```
ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
   {t, 0, 400}, PlotRange -> All]
```

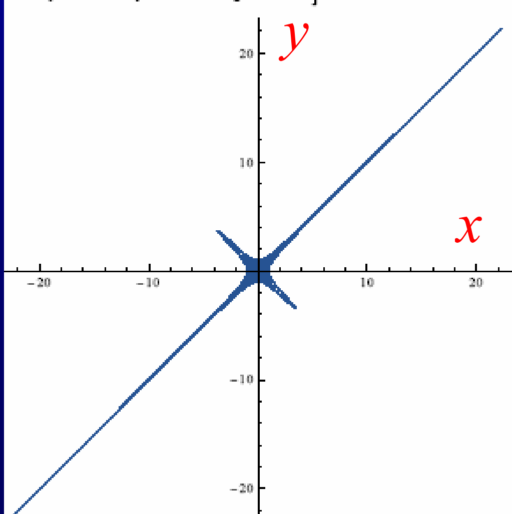



```
Plot[Evaluate[x'[t]^2/2 /. sol],
   {t, 0, 400}, PlotRange -> All]
```

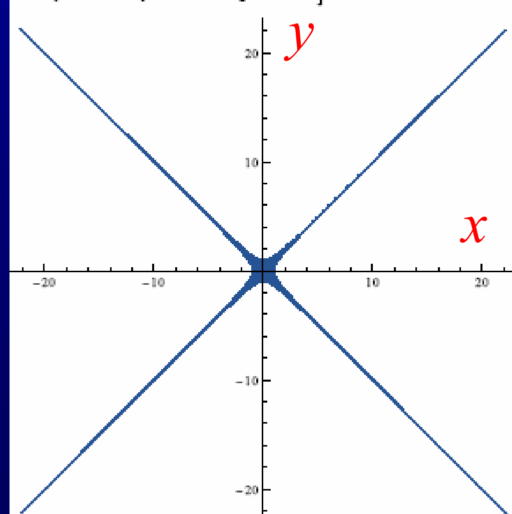



```
sol = NDSolve[{x''[t] + 1.0001 x[t] + 1.0001 * 0.1 * x[t] * (x[t]^2 - y[t]^2) == 0,
   y''[t] + y[t] + 0.1 * y[t] * (x[t]^2 - y[t]^2) == 0, x'[0] == 1,
   y'[0] == 0, x[0] == 0, y[0] == 1}, {x, y}, {t, 1000}]
```

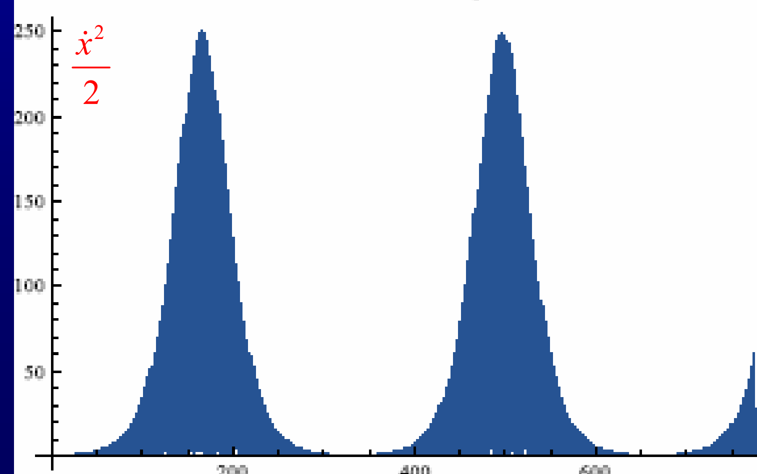
```
ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
   {t, 0, 400}, PlotRange -> All]
```




```
ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
   {t, 0, 777}, PlotRange -> All]
```




```
Plot[Evaluate[x'[t]^2/2 /. sol],
   {t, 0, 777}, PlotRange -> All]
```



Stueckelberg action in higher dimensions (also the Schild action)

$$I = \tfrac{1}{2}\int d\tau \, g_{\mu\nu} \, \dot{X}^\mu \dot{X}^\nu$$

$$\mu, \nu = 0, 1, 2, \ldots, D-1$$

$$g_{\mu\nu}\dot{X}^\mu \dot{X}^\nu = \gamma_{ab}\dot{X}^a \dot{X}^b + \frac{\dot{X}_0^2}{g_{00}}$$

$$\gamma_{ab} = g_{ab} - \frac{g_{0a} g_{0b}}{g_{00}}$$

$$I = \tfrac{1}{2}\left(\int d\tau \, \gamma_{ab}\dot{X}^a \dot{X}^b + \frac{\dot{X}_0^2}{g_{00}} \right)$$

If $g_{\mu\nu,0} = 0$, then $\dot{X}_0^2$ is a constant of motion

$a, b = 1, 2, \ldots, D-1$

Signature (r, s)

$r + s = D - 1$

We take $r = s$

Equations of motion

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{X}^a}\right) - \frac{\partial L}{\partial X^r} = 0$$

$$\gamma_{ab,c} = 0$$

$$\ddot{X}^a + \tfrac{1}{2}\frac{C}{g_{00}^2} g_{00,b}\,\gamma^{ab} = 0$$

$$V = -\tfrac{1}{2}\frac{\dot{X}_0^2}{g_{00}} = -\tfrac{1}{2}\frac{C}{g_{00}}$$

$$\ddot{X}^a + V_{,b}\,\gamma^{ab} = 0$$

This corresponds to the equations of motion on the previous slide

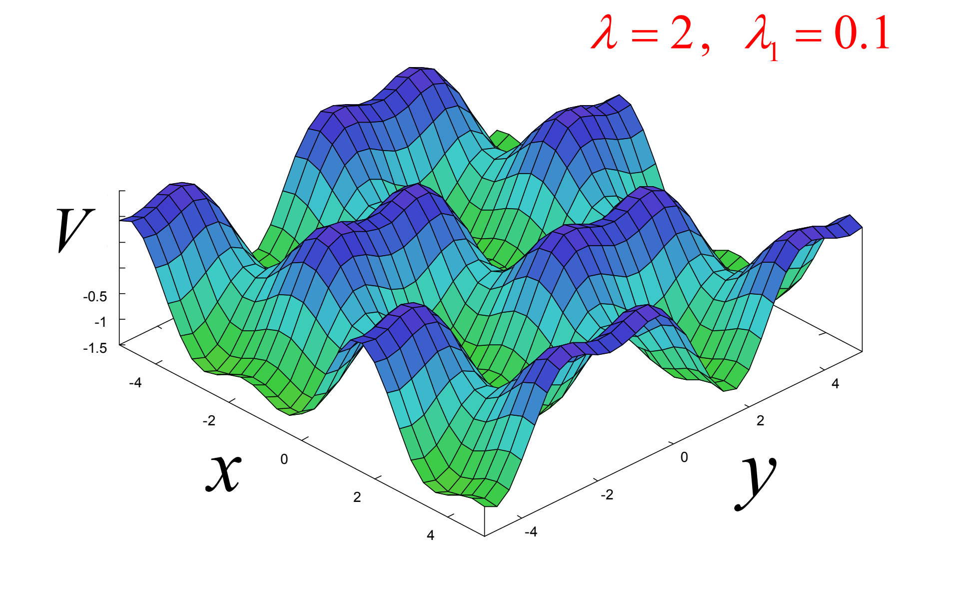
A realistic potential is bounded from above and below. It does not go to infinity.

As an example let us consider the system:

$$L = \frac{1}{2}(\dot{x}^2 - \dot{y}^2) - V, \qquad V = \frac{\lambda}{2}(\sin^2 x - \sin^2 y + \lambda_1 \sin x \sin y)$$

Positive, negative

kinetic energy



Equations of motion:

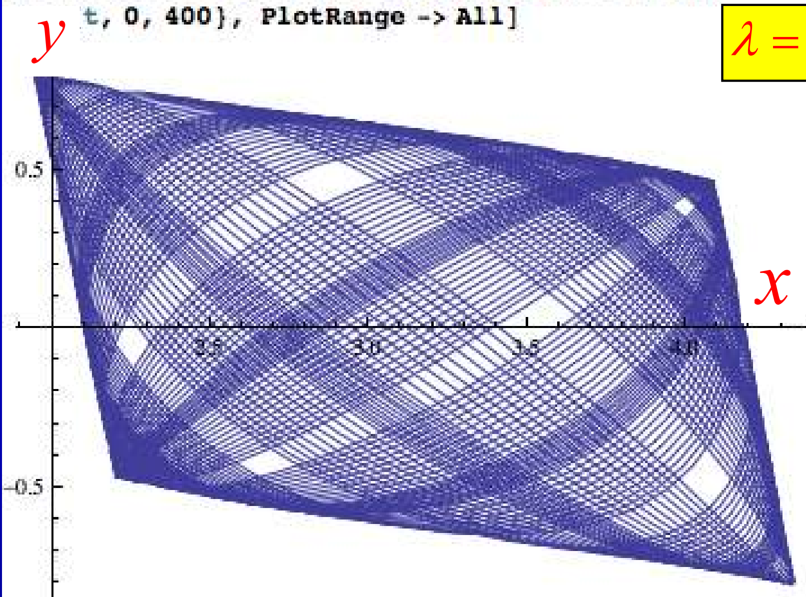
$$\ddot{x} + \frac{\lambda}{2}(2\sin x \cos x + \lambda_1 \cos x \sin y) = 0$$

$$\ddot{y} - \frac{\lambda}{2}(-2\sin y \cos y + \lambda_1 \sin x \cos y) = 0$$

```
sol = NDSolve[{x''[t] + 1 * (2 Cos[x[t]] Sin[x[t]] + 0.1 * Cos[x[t]] Sin[y[t]]) == 0,
   y''[t] - 1 * (0.1 * Cos[y[t]] Sin[x[t]] - 2 Cos[y[t]] Sin[y[t]]) == 0, x'[0] == 0.8, y'[0] == 0.1,
   x[0] == 2.50, y[0] == -0.50}, {x, y}, {t, 1000}]
```

```
ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
  {t, 0, 400}, PlotRange -> All]
```

$\lambda = 2, \quad \lambda_1 = 0.1$

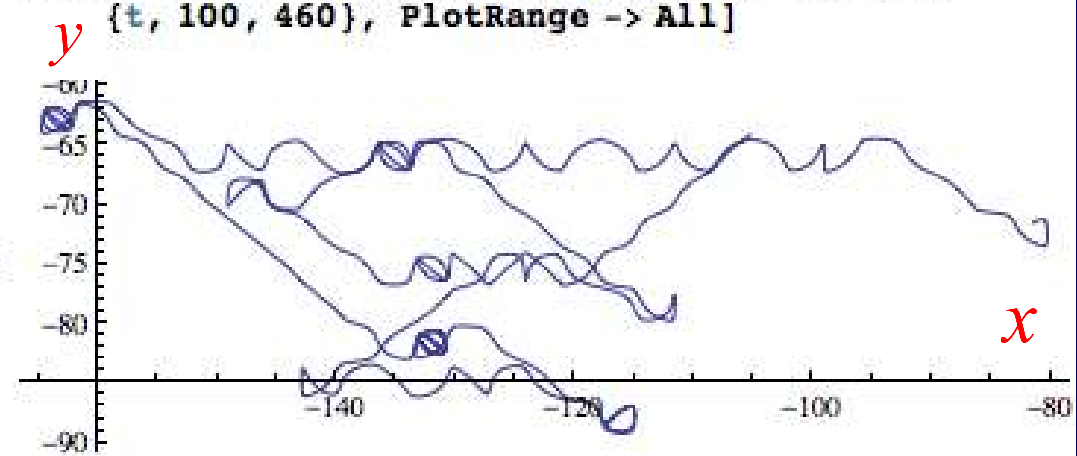


A trajectory in (x,y)-space when the initial speed is not too high

```
sol = NDSolve[{x''[t] + 1 * (2 Cos[x[t]] Sin[x[t]] + 0.1 * Cos[x[t]] Sin[y[t]]) == 0,
   y''[t] - 1 * (0.1 * Cos[y[t]] Sin[x[t]] - 2 Cos[y[t]] Sin[y[t]]) == 0, x'[0] == 1, y'[0] == 0.9,
   x[0] == 2.50, y[0] == -0.50}, {x, y}, {t, 1000}]
```

```
In[10]:= ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
           {t, 100, 460}, PlotRange -> All]
```

Out[10]=

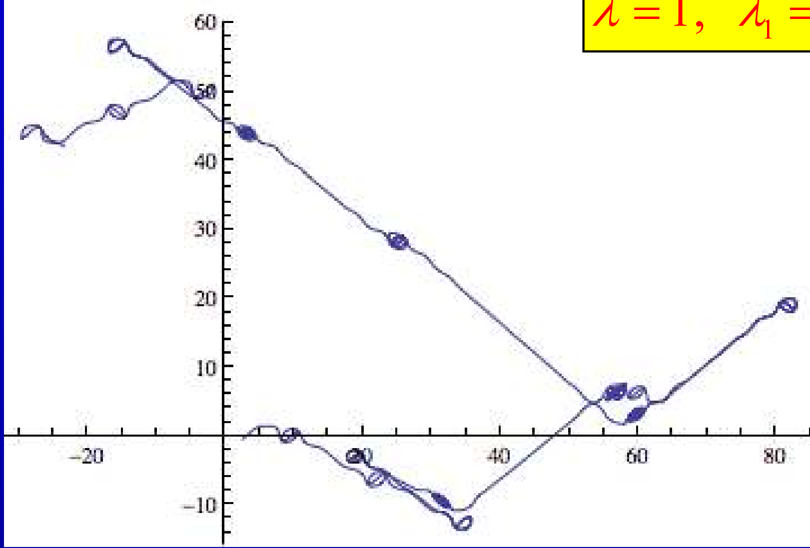


A trajectory for an increased speed

```
sol = NDSolve[{x''[t] + 1 * (Cos[x[t]] Sin[x[t]] + 0.1 * Cos[x[t]] Sin[y[t]]) == 0,
   y''[t] - 1 * (0.1 * Cos[y[t]] Sin[x[t]] - Cos[y[t]] Sin[y[t]]) == 0, x'[0] == 1, y'[0] == 0.9,
   x[0] == 2.5, y[0] == -0.505}, {x, y}, {t, 1000}]
```

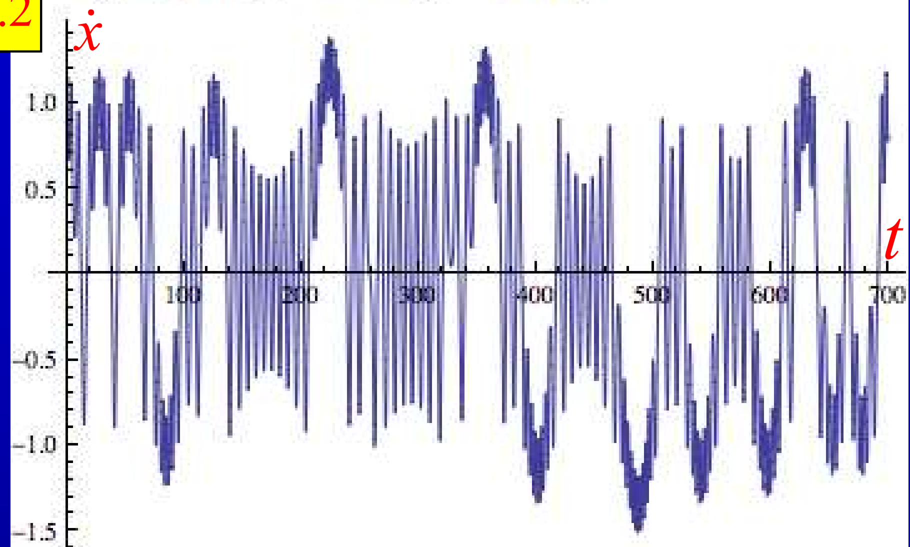
```
ParametricPlot[Evaluate[{x[t], y[t]} /. sol],
   {t, 0, 700}, PlotRange -> All]
```

$\lambda = 1, \quad \lambda_1 = 0.2$



Trajectory in the (x,y)-space

```
Plot[Evaluate[{x'[t]} /. sol],
   {t, 0, 700}, PlotRange -> All]
```



Time dependence of velocity

The position is not bounded, the trajectory can run into infinity.
The velocity remains finite.

A system of scalar fields

Action

$$\boxed{I[\phi^a] = \frac{1}{2}\int d^4x \sqrt{-g}\,(g^{\mu\nu}\partial_\mu\phi^a\,\partial_\nu\phi^b - m^2\phi^a\phi^b)\gamma_{ab}}$$

Metric in the space of fields ϕ^a

$a = 1, 2, \ldots, n$

$\mu, \nu = 0, 1, 2, 3$

$$\pi_a = \frac{\partial L}{\partial\partial_0\phi^a} = \partial^0\phi_a = \partial_0\phi_a \equiv \dot{\phi}_a$$ canonical momenta

Upon quantization, the following equal time commutation relations are satisfied:

$$[\phi^a(\mathbf{x}), \pi_b(\mathbf{x}')] = i\delta^3(\mathbf{x}-\mathbf{x}')\delta^a{}_b$$

The Hamiltonian is

$$\boxed{H = \frac{1}{2}\int d^3x\,(\dot{\phi}^a\dot{\phi}^b - \partial_i\phi^a\partial^i\phi^b + m^2\phi^a\phi^b)\gamma_{ab}}$$

$$\left\{\begin{array}{l} \phi^a = \int \frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_\mathbf{k}}(a^a(\mathbf{k})\mathrm{e}^{-ikx} + a^{a\dagger}(\mathbf{k})\mathrm{e}^{ikx}) \\ [a^a(\mathbf{k}), a^{b\dagger}(\mathbf{k}')] = (2\pi)^3\,2\omega_\mathbf{k}\,\delta^3(\mathbf{k}-\mathbf{k}')\gamma^{ab} \end{array}\right.$$

$$\boxed{H = \frac{1}{2}\int\frac{d^3\mathbf{k}}{(2\pi)^3}\frac{\omega_\mathbf{k}}{2\omega_\mathbf{k}}(a^{a\dagger}(\mathbf{k})a^b(\mathbf{k}) + a^a(\mathbf{k})a^{b\dagger}(\mathbf{k}))\gamma_{ab}}$$

metric with signature (r, s)

$$a^{\bar{a}}(\mathbf{k})|0\rangle = 0, \qquad a^{\underline{a}\dagger}(\mathbf{k})|0\rangle = 0$$

$a^a = (a^{\bar{a}}, a^{\underline{a}})$

$r = \delta^{\bar{a}}{}_{\bar{a}}\,, \qquad s = \delta^{\underline{a}}{}_{\underline{a}}$

$$\boxed{H = \int\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\frac{\omega_\mathbf{k}}{2\omega_\mathbf{k}}(a^{\bar{a}\dagger}(\mathbf{k})a_{\bar{a}}(\mathbf{k}) + a^{\underline{a}}(\mathbf{k})a_{\underline{a}}{}^\dagger) + \frac{1}{2}\int \mathrm{d}^3\mathbf{k}\,\omega_\mathbf{k}\delta^3(0)(r-s)}$$

If signature has equal number of plus and minus signs, i.e., if $r = s$, then the zero point energies cancel out from the Hamiltonian[1].

Generalization to Clifford Space

$$\phi = \phi^A \gamma_A$$ Clifford algebra-valued field

$$I = \frac{1}{2}\int d^4 x \sqrt{-g}\,(g^{\mu\nu}\partial_\mu \phi^A \partial_\nu \phi^B - m^2 \phi^A \phi^B) G_{AB}$$

Signature (R,S)
with $R = S$

Using the Cangemi-Jackiw-Zwiebach definition of vacuum, and following the same procedure as before, we obtain that the zero point energies cancel out:

Vacuum energy vanishes.

See illuminating discussion by R. P. Woodard,
Lect. Notes Phys. 720, 403 (2007)
[astro-ph/0601672].

Therefore, in such theory there is no cosmological constant problem. The small observed cosmological constant could be a residual effect of something else.

Cancellation of vacuum energies in this theory does not exclude the existence of the well known vacuum effects, such as the Casimir effect.

Interacting quantum fields

Example: scalar fields

$$\boxed{I = \int dx^4 \, \tfrac{1}{2}[g^{\mu\nu}\partial_\mu \varphi^a \partial_\nu \varphi^b \gamma_{ab} - V(\varphi)]}$$

Fock space basis

$$|P\rangle = |p_1, p_2, \ldots, p_n\rangle$$

Upon quantization:

$$|\Psi\rangle = \sum |P\rangle\langle P|\Psi\rangle$$

$$|\Psi(t)\rangle = e^{-iH(t-t_0)}|\Psi(t_0)\rangle$$

H is the Hamilton operator corresponding to the field action

$$\boxed{\langle P|\Psi(t)\rangle = \sum_{P'} \langle P|e^{-iH(t-t_0)}|P'\rangle\langle P'|\Psi(t_0)\rangle}$$

$$|\Psi(t_0)\rangle = |0\rangle$$

vacuum

$$\boxed{\langle P|\Psi(t)\rangle = \langle P|e^{-iH(t-t_0)}|0\rangle}$$

Such transition is possible, because $\langle P|$ contains particles with positive and negative energies.

Vacuum decays into a superposition of many particle states:

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty} |p_1, p_2, \ldots, p_n\rangle \underbrace{\langle p_1, p_2, \ldots, p_n|\Psi(t)\rangle}$$

The amplitude that we will measure the multi particle state $|p_1, p_2, \ldots, p_n\rangle$

Generalized field action

We will write the usual field action

$$\boxed{I=\int dx^4[\tfrac{1}{2}g^{\mu\nu}\partial_\mu\varphi^a\partial_\nu\varphi^b\gamma_{ab}-V(\varphi)]}$$

in a more compact notation:

$$\boxed{I=\tfrac{1}{2}\partial_\mu\varphi^{a(x)}\partial_\nu\varphi^{b(x')}\gamma^{\mu\nu}{}_{a(x)b(x')}-U[\varphi]}$$

Integration over the repeated continuous "index" (x) is implied here.

This comes from a higher dimensional action:

$$\boxed{I_\varphi=\tfrac{1}{2}\partial_\mu\varphi^{A(x)}\partial_\nu\varphi^{B(x')}\,G^{\mu\nu}{}_{A(x)B(x')}}$$

Kaluza-Klein split of the metric

$$G^{\mu\nu}{}_{AB}=\begin{pmatrix}\gamma^{\mu\nu}{}_{ab}+A_a{}^{\bar A}A_b{}^{\bar B}\phi^{\mu\nu}{}_{\bar A\bar B}, & A_a{}^{\bar B}\phi^{\mu\nu}{}_{\bar A\bar B}\\ A_b{}^{\bar B}\phi^{\mu\nu}{}_{\bar A\bar B}, & \phi^{\mu\nu}{}_{\bar A\bar B}\end{pmatrix}$$

$$\boxed{I_\varphi=\tfrac{1}{2}\partial_\mu\varphi^{a(x)}\partial_\nu\varphi^{b(x')}\gamma^{\mu\nu}{}_{a(x)b(x')}+\underbrace{\tfrac{1}{2}\partial_\mu\varphi_{\bar A}\,\partial_\nu\varphi_{\bar B}\,\phi^{\mu\nu\bar A\bar B}}_{-U[\varphi]}}$$

Total action:

$$\boxed{I[\varphi,G]=I_\varphi+I_G}$$

Since the metric $G^{\mu\nu}{}_{AB}$ is dynamical, the potential $U[\varphi]$ is not fixed, but it changes with evolution of the system.

Clifford algebra and spinors in Minkowski space

$$\gamma_a = (\gamma_0, \gamma_1, \gamma_2, \gamma_3)$$

Witt basis

$$\theta_1 = \frac{1}{2}(\gamma_0 + \gamma_3), \quad \theta_2 = \frac{1}{2}(\gamma_1 + i\gamma_2),$$

$$\bar{\theta}_1 = \frac{1}{2}(\gamma_0 - \gamma_3), \quad \bar{\theta}_2 = \frac{1}{2}(\gamma_1 - i\gamma_2)$$

The new basis vectors satisfy

Fermionic anticommutation relations

$$\boxed{\{\theta_a, \bar{\theta}_b\} = \eta_{ab}, \quad \{\theta_a, \theta_b\} = 0, \quad \{\bar{\theta}_a, \bar{\theta}_b\} = 0}$$

We now observe that the product

$$f = \bar{\theta}_1 \bar{\theta}_2$$

satisfies

$$\bar{\theta}_a f = 0, \qquad a = 1,2$$

f can be interpreted as `**vacuum**', and $\bar{\theta}_a$ can be interpreted as operators that annihilate f.

An object constructed as a superposition

$$\Psi = (\psi^0 \underline{1} + \psi^1 \theta_1 + \psi^2 \theta_2 + \psi^{12} \theta_1 \theta_2) f$$

is a 4-component spinor.

Four independent spinors

Four different possible vacua:

$$f_1 = \bar{\theta}_1\bar{\theta}_2\,, \qquad f_2 = \theta_1\theta_2\,, \qquad f_3 = \theta_1\bar{\theta}_2\,, \qquad f_4 = \bar{\theta}_1\theta_2$$

$$\bar{\theta}_1 f_1 = 0, \qquad \bar{\theta}_2 f_1 = 0,$$
$$\theta_1 f_2 = 0, \qquad \theta_2 f_2 = 0, \quad \text{etc.}$$

Four different kinds of spinors:

$$\Psi^1 = (\psi^{11}\underline{1} + \psi^{21}\theta_1\theta_2 + \psi^{31}\theta_1 + \psi^{41}\theta_2) f_1$$
$$\Psi^2 = (\psi^{12}\underline{1} + \psi^{22}\bar{\theta}_1\bar{\theta}_2 + \psi^{32}\bar{\theta}_1 + \psi^{42}\bar{\theta}_2) f_2$$
$$\Psi^3 = (\psi^{13}\bar{\theta}_1 + \psi^{23}\theta_2 + \psi^{33}\underline{1} + \psi^{43}\bar{\theta}_1\theta_2) f_3$$
$$\Psi^4 = (\psi^{14}\theta_1 + \psi^{24}\bar{\theta}_2 + \psi^{34}\underline{1} + \psi^{44}\theta_1\bar{\theta}_2) f_4$$

Each of those spinors lives in a different minimal left ideal of $Cl(1,3)$.

In general, complexified version

An arbitrary element of $Cl(1,3)$ is the sum:

$$\Phi = \Psi^1 + \Psi^2 + \Psi^3 + \Psi^4 = \psi^{\alpha i}\xi_{\alpha i} \equiv \psi^{\tilde{A}}\xi_{\tilde{A}}$$

$\alpha = 1,2,3,4; \quad i = 1,2,3,4$

$$\xi_{\tilde{A}} \equiv \xi_{\alpha i} = \{\underline{1} f_1,\ \theta_1\theta_2 f_1, \ldots, \theta_1 f_4,\ \bar{\theta}_2 f_4,\ \underline{1} f_4,\ \bar{\theta}_1\theta_2 f_4\},$$

Spinor basis of $Cl(1,3)$

Matrix notation:

$$\psi^{\alpha i} = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}, \qquad \xi_{\tilde{A}} \equiv \xi_{\alpha i} = \begin{pmatrix} f_1 & f_2 & \bar{\theta}_1 f_3 & \theta_1 f_4 \\ \theta_1\theta_2 f_1 & \bar{\theta}_1\bar{\theta}_2 f_2 & \theta_2 f_3 & \bar{\theta}_2 f_4 \\ \theta_1 f_1 & \bar{\theta}_1 f_2 & f_3 & f_4 \\ \theta_2 f_1 & \bar{\theta}_2 f_2 & \bar{\theta}_1\theta_2 f_3 & \theta_1\bar{\theta}_2 f_4 \end{pmatrix}$$

Spinor of the 2nd left ideal

Basis of the 2nd left ideal

Generalized Dirac equation (Dirac-Kähler equation*)

$$(i\gamma^\mu\partial_\mu - m)\Phi = 0$$

$$\Phi = \phi^A\gamma_A = \psi^{\tilde{A}}\xi_{\tilde{A}} = \psi^{\alpha i}\xi_{\alpha i}$$

Spinor basis of $Cl(1,3)$

α is spinor index of a left minimal ideal.
i runs over four left ideals of $Cl(1,3)$

$$\langle(\xi^{\tilde{A}})^{\ddagger}\gamma^\mu\xi_{\tilde{B}}\rangle_S \equiv (\gamma^\mu)^{\tilde{A}}{}_{\tilde{B}}$$

$$\left(i(\gamma^\mu)^{\tilde{A}}{}_{\tilde{B}}\partial_\mu - m\delta^{\tilde{A}}{}_{\tilde{B}}\right)\psi^{\tilde{B}} = 0$$

$$(\gamma^\mu)^{\tilde{A}}{}_{\tilde{B}} = (\gamma^\mu)^\alpha{}_\beta\delta^i{}_j$$

$$\left(i(\gamma^\mu)^\alpha{}_\beta\partial_\mu - m\delta^\alpha{}_\beta\right)\psi^{\beta i} = 0$$

$$(i\gamma^\mu\partial_\mu - m)\psi^i = 0$$

Here we omit spinor index α

Metric in spinor space

$$(\xi_{\tilde{A}})^{\ddagger} * \xi_{\tilde{B}} = z_{\tilde{A}\tilde{B}} = z_{(\alpha i)(\beta j)} = z_{\alpha\beta}z_{ij}$$

$$z_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad z_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Action

$$I = \int \mathrm{d}^4x\, \bar{\psi}^i(i\gamma^\mu\partial_\mu - m)\psi^j z_{ij}$$

*E. Kähler, Rendiconti di Matematica 21 (1962) 425;
S.I. Kruglov, Dirack-Kähler Equation, arXiv: hep-th/0110251 (and many references therein)
D. Spehled, and G.C. Marques, Eur. Phys. J. 61 (2009) 75

Hamiltonian

$$\boxed{H = \int \mathrm{d}^3 x \, \bar{\psi}^i (-i\gamma^r \partial_r + m)\psi^j z_{ij}}$$

We expand ψ^i in terms of the annihilation and creation operators

$$\boxed{H = \sum_{n=1}^{2} \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} m \left(b_n^{i\dagger}(\boldsymbol{p}) b_n^j(\boldsymbol{p}) - d_n^i(\boldsymbol{p}) d_n^{j\dagger} \right) z_{ij}}$$

Index i distinguishes the spinors of different left ideals of $Cl(1,3)$.
Index $n=1,2$ is the usual one that distinguishes `spin up' and `spin down' states.

$$i = (\bar{i}, \underline{i}), \qquad \bar{i} = 1,2; \qquad \underline{i} = 3,4$$

We split the index

$$b_n^{\bar{i}} \, |0\rangle = 0, \qquad d_n^{\bar{i}} \, |0\rangle = 0$$

$$b_n^{\underline{i}\dagger} \, |0\rangle = 0, \qquad d_n^{\underline{i}\dagger} \, |0\rangle = 0$$

Annihilation operator acting on the generalized Dirac vacuum

$$\boxed{H = \sum_{n=1}^{2} \frac{\mathrm{d}^3 \boldsymbol{p}}{(2\pi)^3} m \Big(b_n^{\bar{i}\dagger}(\boldsymbol{p}) b_n^{\bar{j}}(\boldsymbol{p}) - b_n^{\underline{i}}(\boldsymbol{p}) b_n^{\underline{j}\dagger}(\boldsymbol{p}) + d_n^{\bar{i}\dagger}(\boldsymbol{p}) d_n^{\bar{j}} - d_n^{\underline{i}}(\boldsymbol{p}) d_n^{\underline{j}\dagger}(\boldsymbol{p}) + \underbrace{\delta(\boldsymbol{0})(z^{ij} - z^{ij})}_{\text{This term vanishes}} \Big) z_{ij}}$$

$$\boxed{\langle 0 | H | 0 \rangle = 0}$$

Vacuum expectation of this Hamiltonian is zero

Each fermion ψ^i couples to the corresponding gauge field. The Casimir force between two metalic plates, consisting of ψ^i, $i = 1$, is not expected to vanish in this general theory.

$\langle T^{00} \rangle = \langle H \rangle$ is the source of the gravitational field. Because $\langle 0 | H | 0 \rangle = \langle 0 | T^{00} | 0 \rangle = 0$, the cosmological constant vanishes. There is no problem of the huge cosmological constant.

Besides resolving the problem of the cosmological constant, the Dirac-Kähler equation and its generalization* may provide a theoretical framework that could be used for the unification of fundamental particles and forces.

*M. Pavšič, Int. J. Mod. Phys. 21 (2006) 5905 (and references therein)
M. Pavšič, Phys. Lett. B 614 (2005) 85
M. Pavšič, Phys. Lett. B 692 (2010) 212

Generalized Dirac field

$$\psi = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}$$

$$\text{Energy} = \begin{pmatrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ - & - & + & + \end{pmatrix}$$

I. Dirac spinors of the usual energy type

II. Dirac spinors of the opposite energy type

Positive and negative energy states of the usual Dirac spinors do not mix in our Universe. Even if they did mix, the evolution of the Universe has led to the current situation with no mixing.

Dirac sea

This was not so clear when Dirac proposed his theory.

Here is what Fermi wrote:

It is well known that the most serious difficulty in Dirac's relativistic wave equation lies in the fact that it yields besides the normal positive states also negative ones, which have no physical significance. This would do no harm if no transition between positive and negative states were possible (as are, e.g., transitions between states with symmetrical and antisymmetrical wave function). But this is unfortunately not the case: Klein has shown by a very simple example that electrons impinging against a very high potential barrier have a finite probability of going over in a negative state.

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E. Fermi, Rev. Mod. Phys., **4**, 87 (1932)

This problem was resolved by the Dirac sea of negative energy particles.

Generalized Dirac field

$$\psi = \begin{pmatrix} \psi^{11} & \psi^{12} & \psi^{13} & \psi^{14} \\ \psi^{21} & \psi^{22} & \psi^{23} & \psi^{24} \\ \psi^{31} & \psi^{32} & \psi^{33} & \psi^{34} \\ \psi^{41} & \psi^{42} & \psi^{43} & \psi^{44} \end{pmatrix}$$

$$\text{Energy} = \begin{pmatrix} + & + & - & - \\ + & + & - & - \\ - & - & + & + \\ \underbrace{- \quad -}_{\text{I}} & & \underbrace{+ \quad +}_{\text{II}} & \end{pmatrix}$$

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During the evolution, the sea of type I negative and type II positive energy states have formed.

The existence of type I and type II Dirac spinors also should not be considered as a priori problematic.

Clifford algebra description of fermionic fields

$$\boxed{\Psi = \psi^{r(x)} h_{r(x)}}$$

$r = 1, 2; \quad x \in \mathbb{R}^3$ or $x \in \mathbb{R}^{1,3}$

particular case

$$h_{r(x)} \cdot h_{s(x')} = \rho_{r(x)s(x')}$$ metric $$\rho_{r(x)s(x')} = \delta_{rs}\,\delta(x - x')$$

New basis:

$$h_{(x)} = \tfrac{1}{\sqrt{2}}(h_{1(x)} + i\,h_{2(x)})$$
$$\bar{h}_{(x)} = \tfrac{1}{\sqrt{2}}(h_{1(x)} - i\,h_{2(x)})$$

Witt basis

$$h_{(x)} \cdot \bar{h}_{(x')} = \delta(x - x')$$
$$h_{(x)} \cdot h_{(x')} = \bar{h}_{(x)} \cdot \bar{h}_{(x')} = 0$$

Fermionic commutation relations

components

$$\boxed{\Psi = \psi^{(x)} h_{(x)} + \bar{\psi}^{(x)} \bar{h}_{(x)}}$$

basis vectors

A vector in infinite dimensional space

Vacuum

$$\Omega = \prod_x \bar{h}_{(x)} \qquad \bar{h}_{(x)}\Omega = 0$$

$$\boxed{\Psi\,\Omega = \psi^{(x)}\, h_{(x)}\,\Omega}$$

The second part of Ψ disappears

$$\Psi = \psi^{(x)} h_{(x)} + \bar{\psi}^{(x)} \bar{h}_{(x)}$$

Let us consider a more general case:

$$\Psi\,\Omega = (\psi_0 + \psi^{(x)}\, h_{(x)} + \psi^{(x)(x')} h_{(x)} h_{(x')} + \ldots)\Omega$$

This state is the infinite dimensional space analog of the spinor as an element of a left ideal of Clifford algebra

Other possible vacuums:

$$\Omega = \prod_x h_{(x)}\,, \qquad h_{(x)}\Omega = 0$$

$$\Omega = \left(\prod_{x\in R_1} \bar{h}_{(x)}\right)\left(\prod_{x\in R_2} h_{(x)}\right),$$

$$\bar{h}_{(x')}\Omega = 0\,, \qquad h_{(x')}\Omega = \left(\prod_{\substack{x\in R \\ x\neq x_1'}} \bar{h}_{(x)}\right)\left(\prod_{x\in R_2} h_{(x)}\right)$$

``hole'' at $x = x'$

Analogous holds in momentum representation.

In a notation that is closer to the usual one, we have

$$b_n^{\bar{i}}(p^0>0,\boldsymbol{p}),\qquad d_n^{\bar{i}}(p^0<0,\boldsymbol{p})$$

$$b_n^{\bar{i}\dagger}(p^0<0,\boldsymbol{p}),\qquad d_n^{\bar{i}\dagger}(p^0>0,\boldsymbol{p})$$

annihilate Ω

Vacuum

$$\Omega=\left(\prod_{n,\mathbf{p}} b_n^{\bar{i}}(p^0>0,\boldsymbol{p})\right)\left(\prod_{n,\mathbf{p}} d_n^{\bar{i}}(p^0<0,\boldsymbol{p})\right)\left(\prod_{n,\mathbf{p}} b_n^{\bar{i}\dagger}(p^0<0,\boldsymbol{p})\right)\left(\prod_{n,\mathbf{p}} d_n^{\bar{i}\dagger}(p^0>0,\boldsymbol{p})\right)$$

This is a generalized **bare** Dirac vacuum

One particle Fock states:

$$\underbrace{b_n^{\bar{i}\dagger}\Omega,\quad d_n^{\bar{i}\dagger}\Omega,}_{\text{Positive energies}}\quad \underbrace{b_n^{\bar{i}}\Omega,\quad d_n^{\bar{i}}\Omega}_{\text{Negative energies}}\quad ,\ \ldots,$$ and all many particle states

Positive energies

Negative energies

$$\Omega \xrightarrow{\text{decays}}$$ A superposition of positive and negative energy states

The final state with infinitely many positive and negative energy particles of all types, is:

$$\Psi(t)=b_{n_1}^{\bar{i}_1\dagger}(\boldsymbol{p}_1)b_{n_2}^{\bar{i}_2\dagger}(\boldsymbol{p}_2)\ldots d_{n_1}^{\bar{i}_1\dagger}(\boldsymbol{p}_1)d_{n_2}^{\bar{i}_2\dagger}(\boldsymbol{p}_2)\ldots d_{n_1}^{\bar{i}_1}(\boldsymbol{p}_1)d_{n_2}^{\bar{i}_2}(\boldsymbol{p}_2)\ldots\Omega$$

This state is the totally filled sea

The latter state is another kind of vacuum. It is also is `unstable' and can evolve into another state, with the holes in the sea.

In the Clifford algebra description, a vacuum is a sea formed with the field operators. A possible vacuum is the **Dirac vacuum.**

Partially filled sea

Conclusion

We have generalized spacetime to Clifford space. Its metric is neutral, i.e., (8,8).

The physics in spaces with extra time-like dimensions does not necessarily suffers from instabilities arising from the presence of negative energies.

We have demonstrated stability on the example of the classical oscillator $x^a(t)$ for two cases:

- unequal ``masses'' of positive and negative energy components
- a potential that is bounded from below and from above

Because of the correspondence principle this is also true for the quantized oscillator.

Field theories should be suitable generalized, so to include the kinetic term for the metric in the field space.
Then the corresponding field potential is not fixed, but changes during the evolution of the system.

Clifford algebra formulation of fermionic fields and vacuums brings novel insight into the evolution of such systems. A vacuum is a sea of operators.

Our findings have consequences for quantum gravity, if in higher derivative theories we take a realistic potential that does not go into positive or negative infinity.
Then there are no instabilities or ghosts.

This was demonstrated on Pais-Uhlenbeck oscillator which is a model for a higher derivative theory:

Comments on the dynamics of the Pais-Uhlenbeck oscillator
A.V. Smilga (SUBATECH, Nantes). Aug 2008. 15 pp.
Published in **SIGMA 5 (2009) 017**

Stable Self-Interacting Pais-Uhlenbeck Oscillator
Matej Pavšič. Feb 21, 2013. 15 pp.
Published in **Mod.Phys.Lett. A28 (2013) 1350165**

Pais-Uhlenbeck Oscillator with a Benign Friction Force
Matej Pavšič (Stefan Inst., Ljubljana). Apr 4, 2013. 3 pp.
Published in **Phys.Rev. D87 (2013) 10, 107502**

Some Comments on Ghosts and Unitarity: The Pais-Uhlenbeck Oscillator Revisited
Ibrahim Burak Ilhan, Alex Kovner (Connecticut U.). Jan 2013. 12 pp.
Published in **Phys.Rev. D88 (2013) 044045**

Auxiliar slides (not presented in the talk)

The Hamilton form of the equations of motion[1]

$$\dot{x} = \{x, H\} = \frac{\partial H}{\partial p_x} = p_x\,, \qquad \dot{y} = \{y, H\} = \frac{\partial H}{\partial p_y} = -p_y$$

$$\dot{p}_x = \{p_x, H\} = -\frac{\partial H}{\partial x} = -\omega^2 x\,, \qquad \dot{p}_y = \{p_y, H\} = -\frac{\partial H}{\partial y} = \omega^2 y$$

Poisson brackets are defined as usual $\{x, p_x\} = 1\,, \quad \{y, p_y\} = 1$

In the quantized theory we have commutators $[x, p_x] = i\,, \quad [y, p_y] = i$

Introducing[1]

$$c_x = \frac{1}{\sqrt{2}}(\sqrt{\omega}\,x + \frac{i}{\sqrt{\omega}}\,p_x), \quad c_x^\dagger = \frac{1}{\sqrt{2}}(\sqrt{\omega}\,x - \frac{i}{\sqrt{\omega}}\,p_x)$$

$$c_y = \frac{1}{\sqrt{2}}(\sqrt{\omega}\,y + \frac{i}{\sqrt{\omega}}\,p_y), \quad c_y^\dagger = \frac{1}{\sqrt{2}}(\sqrt{\omega}\,y - \frac{i}{\sqrt{\omega}}\,p_y)$$

we have $[c_x, c_x^\dagger] = 1, \quad [c_y, c_y^\dagger] = 1,$

$$[c_x, c_y] = [c_x^\dagger, c_y^\dagger] = 0$$

$$\boxed{H = \frac{1}{2}\omega(c_x^\dagger c_x + c_x c_x^\dagger - c_y^\dagger c_y - c_y c_y^\dagger)}$$

$$c_x\,|0\rangle = 0\,, \qquad c_y\,|0\rangle = 0$$

$$\boxed{H = \omega(c_x^\dagger c_x - c_y^\dagger c_y)}$$

Vacuum

Using $p_x = -i\partial/\partial x\,, \quad p_y = -i\partial/\partial y$

and writing $\langle x, y\,|\,0\rangle \equiv \psi_0(x, y)$

we have

$$\frac{1}{2}\left(\sqrt{\omega}x + \frac{1}{\sqrt{\omega}}\frac{\partial}{\partial x}\right)\psi_0(x, y) = 0$$

$$\frac{1}{2}\left(\sqrt{\omega}y + \frac{1}{\sqrt{\omega}}\frac{\partial}{\partial y}\right)\psi_0(x, y) = 0$$

$$\psi_0 = \frac{2\pi}{\omega}\,\mathrm{e}^{-\frac{1}{2}\omega(x^2+y^2)}$$

Normalization: $\int \psi_0^2\, dx\, dy = 1$

All states have positive norm, e.g.,:

$$\langle 0\,|\,c\,c^\dagger\,|\,0\rangle = \langle 0\,|\,[c, c^\dagger]\,|\,0\rangle = \langle 0\,|\,0\rangle = \int \psi^2\, dx\, dy = 1.$$

Generalization to $M_{r,s}$

signature (r,s), $a,b=1,2,\ldots,r+s$

$$L=\frac{1}{2}\dot{x}^a\dot{x}_a-\frac{1}{2}\omega^2x^ax_a$$

$$p_a=\frac{\partial L}{\partial\dot{x}^a}=\dot{x}_a=\eta_{ab}\dot{x}^b$$

Upon quantization:

$$[x^a,p_b]=i\delta^a{}_b \quad \text{or} \quad [x^a,p^b]=i\eta^{ab}$$

Procedure with generalizing the operators $c_x, c_x^\dagger, c_y, c_y^\dagger$ of the 2-dimensional case[1]:

$$H=\frac{1}{2}p^ap_a+\frac{1}{2}\omega^2x^ax_a$$

A.

$$c^a=\frac{1}{\sqrt{2}}\left(\sqrt{\omega}x^a+\frac{i}{\sqrt{\omega}}p_a\right)$$

$$c^{a\dagger}=\frac{1}{\sqrt{2}}\left(\sqrt{\omega}x^a-\frac{i}{\sqrt{\omega}}p_a\right)$$

$$H=\frac{1}{2}\omega(c_a^\dagger c^a+c^ac_a^\dagger)$$

$$\boxed{[c^a,c^{b\dagger}]=\delta^{ab}}$$

$$c^a|0\rangle=0$$

$$c^ac_a^\dagger=\eta_{ab}c^ac^{b\dagger}=\eta_{ab}(c^{b\dagger}c^a+\delta^{ab})$$
$$=c^{a\dagger}c_a+r-s$$

$$H=\omega(c_a^\dagger c^a+\frac{r}{2}-\frac{s}{2})$$

Procedure with an alternative definition of creation and annihilation operators[2]:

$$H=\frac{1}{2}p^ap_a+\frac{1}{2}\omega^2x^ax_a$$

B.

$$a^a=\frac{1}{2}\left(\sqrt{\omega}\,x^a+\frac{i}{\sqrt{\omega}}p^a\right)$$

$$a^{a\dagger}=\frac{1}{2}\left(\sqrt{\omega}\,x^a-\frac{i}{\sqrt{\omega}}p^a\right)$$

$$H=\frac{1}{2}\omega(a^{a\dagger}a_a+a_aa^{a\dagger})$$

$$\boxed{[a^a,a_b^\dagger]=\delta^a{}_b \quad \text{or} \quad [a^a,a^{b\dagger}]=\eta^{ab}}$$

I. $a^a|0\rangle=0$

$$H=\omega\,(a^{a\dagger}a_a+\frac{r}{2}+\frac{s}{2})$$

II. $a^a=(a^{\bar{a}},a^{\underline{a}})$

$a^{\bar{a}}|0\rangle=0$, $a^{\underline{a}\dagger}|0\rangle=0$

$$H=\omega\,(a^{\bar{a}\dagger}a_{\bar{a}}+a_{\underline{a}}a^{\underline{a}\dagger}+\frac{r}{2}-\frac{s}{2})$$

In *Case A*, the creation and annihilation operators are superpositions of the coordinates x^a and the covariant components of momenta p_a .

In *Case B*, the creation and annihilation operators are superpositions of the coordinates x^a and the contravariant components of momenta p^a.

In *Case B*, there are two possible definitions of vacuum:

Possibility I.
This is the usual definition $a^a \, |0\rangle = 0$. The eigenvalues of

$$H = \omega \, (a^{a\dagger} a_a + \frac{r}{2} + \frac{s}{2})$$

are all positive. There exist negative norm states or ghosts.

Possibility I
All energies are positive.
Negative norms.

Possibiilty II.
This is the Cangemi-Jackiw-Zwiebach definition[3]

$$a^{\bar{a}} \, |0\rangle = 0, \qquad a^{\underline{a}\dagger} \, |0\rangle = 0, \qquad \bar{a} = 1, 2, ..., r, \quad \underline{a} = 1, 2, ..., s$$

The eigenvalues of

$$H = \omega \, (a^{\bar{a}\dagger} a_{\bar{a}} + a_{\underline{a}} a^{\underline{a}\dagger} + \frac{r}{2} - \frac{s}{2})$$

can be positive or negative. The are no negative norm states. The presence of negative energies does not automatically imply instability of the system.

Possibility II
Positive and negative energies.
No negative norms.

If $r = s$, then the zero point energy vanishes.

[3]D. Cangemi, R. Jackiw, and B. Zwiebach, Ann. of Phys. 245 (1996) 408
R. Jackiw, and H.-J. Lee, Phys. Rev. D 54 (1996) 6213

Quantum oscillator

$$\boxed{i\frac{\partial\psi}{\partial t} = H\psi}$$

$$H = \frac{1}{2}\left(-\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right) + V(x,y)$$

$$\left\{\begin{array}{l} \psi = \sum_{mn} c_{mn}(t)\,\psi_{mn} \\ H_{mn;rs} = \int \mathrm{d}x\,\mathrm{d}y\,\psi^*_{mn} H \psi_{rs} \end{array}\right.$$

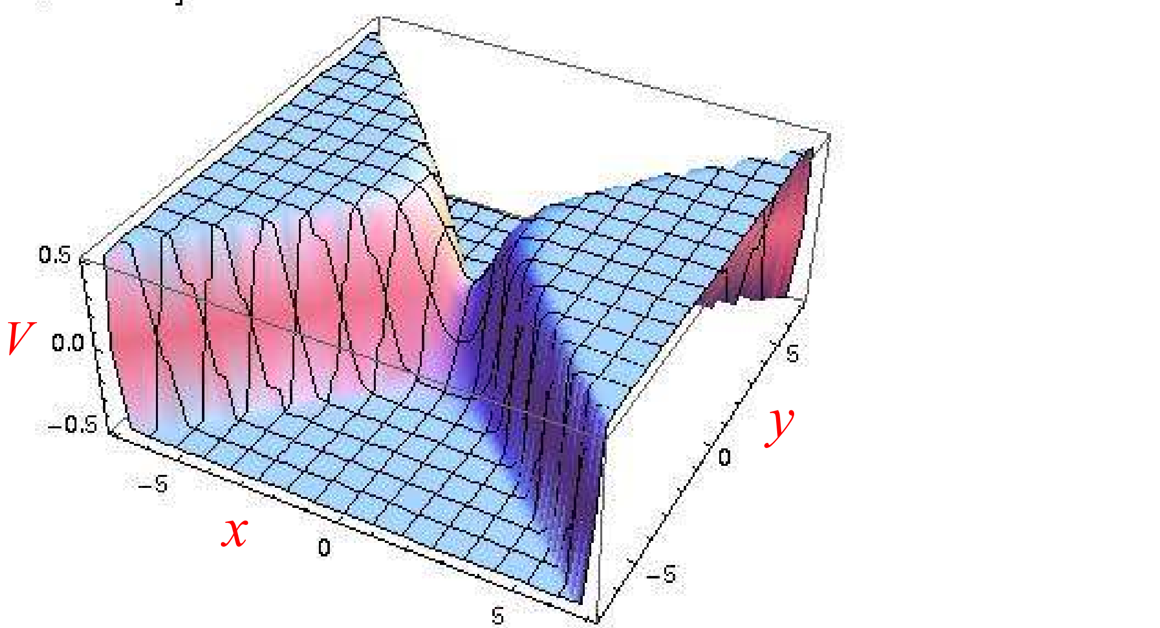
Basis functions of the 2D harmonic oscillator

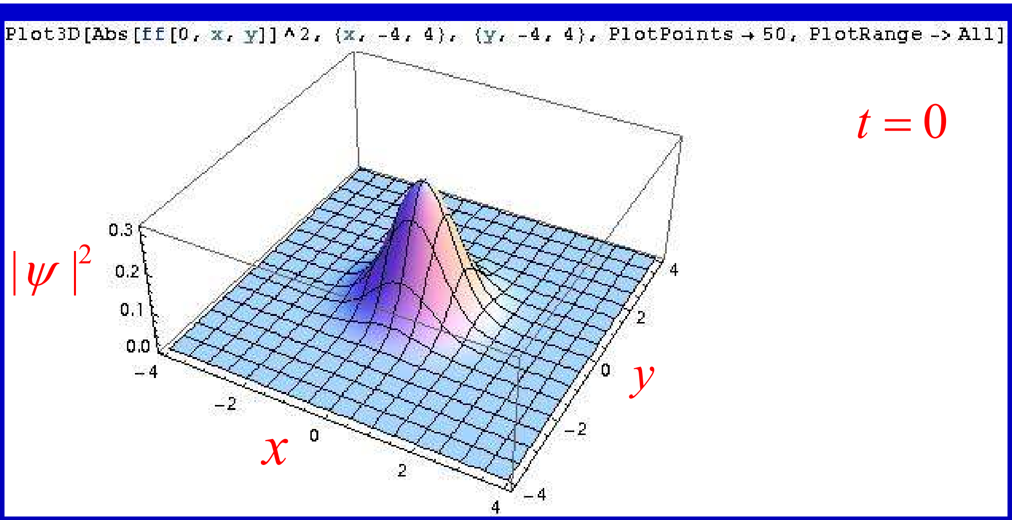
$$\boxed{i\dot{c}_{mn} = \sum_{rs} H_{mn;rs}\, c_{rs}}$$

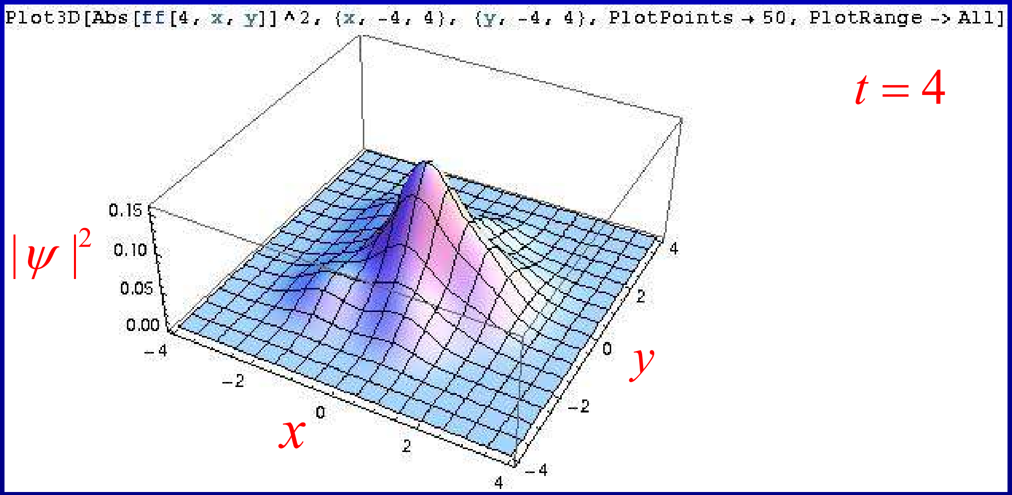
We will investigate the case:

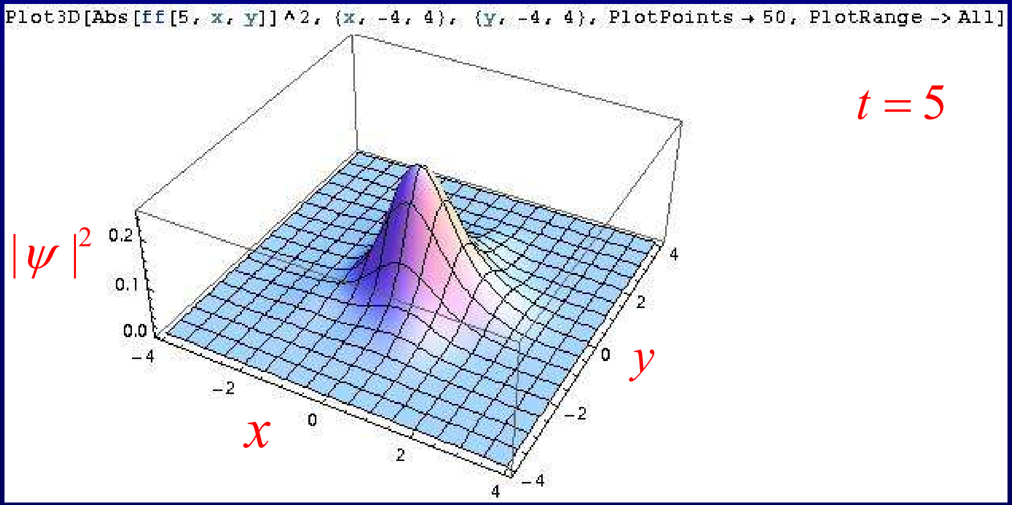
$$V(x,y) = \frac{1}{2}\varepsilon\left(1 - e^{-\varepsilon(x^2-y^2)}\right)$$

```
Plot3D[(1/2)*Sign[x^2 - y^2]*(1 - Exp[-Abs[x^2 - y^2]]), {x, -7, 7}, {y, -7, 7},
 PlotRange -> All]
```





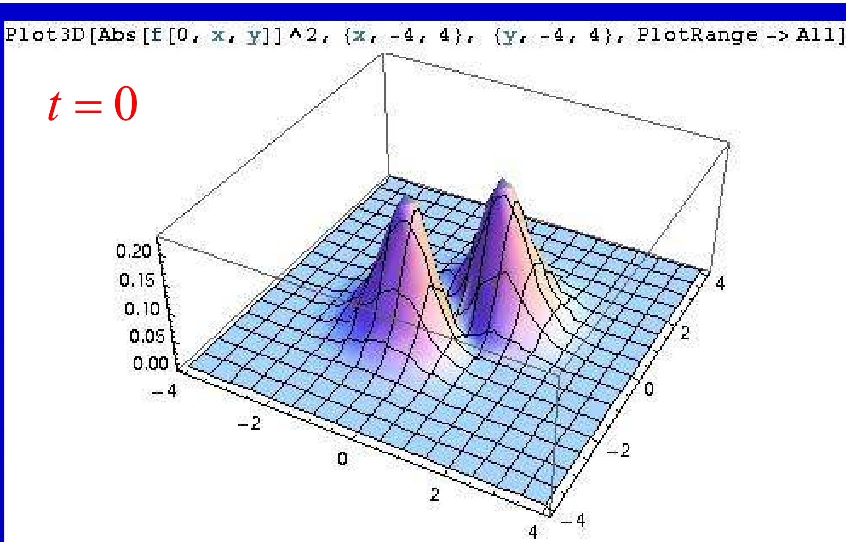


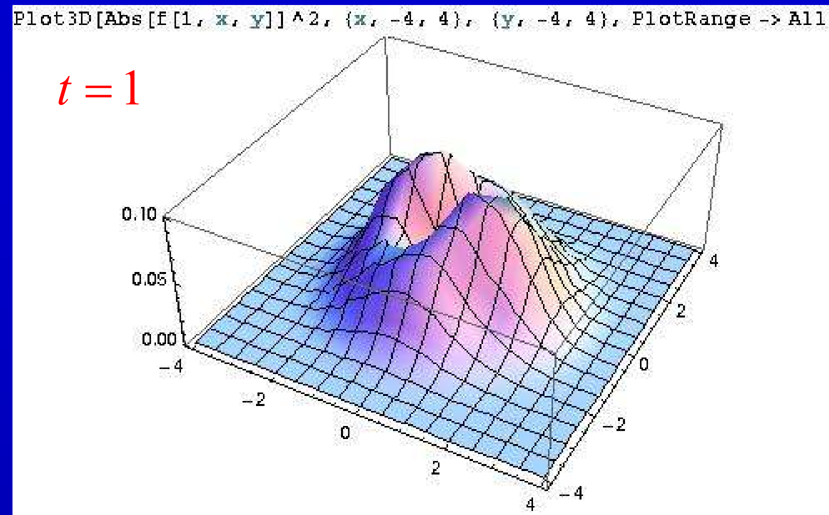


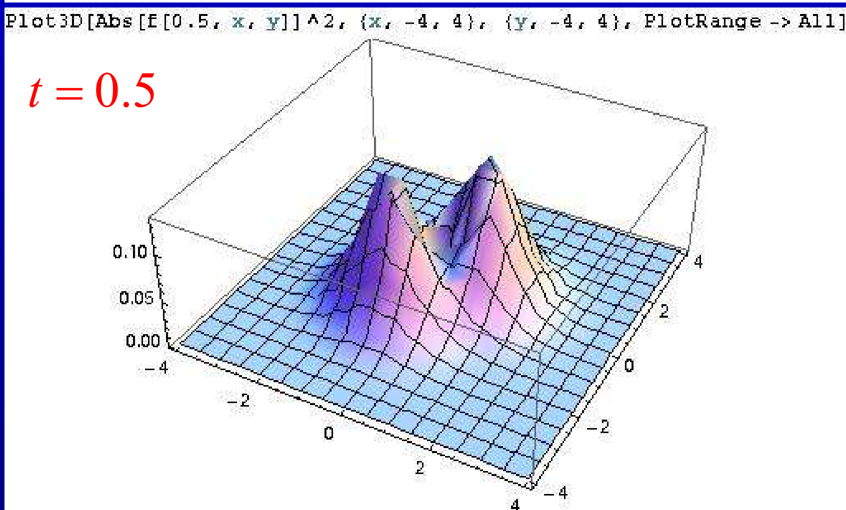
$$\psi = \sum_{m,n=0}^{4} c_{mn}(t)\psi_{mn}$$

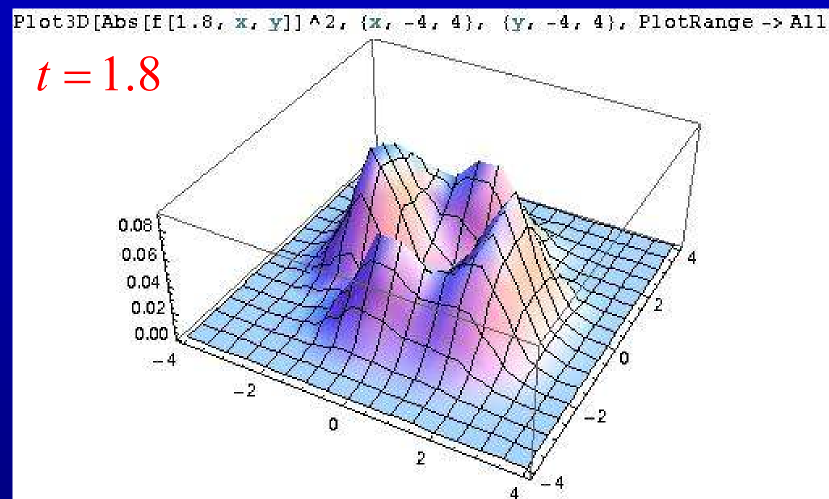
Initial condition

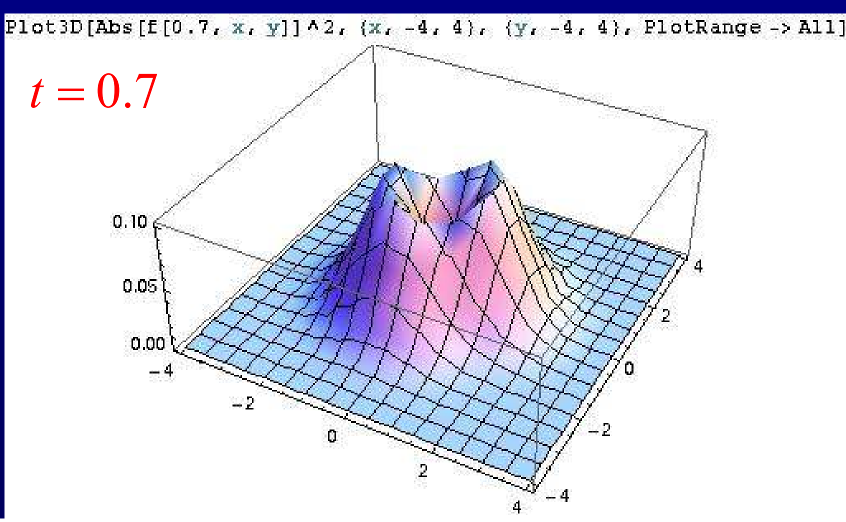
$$c_{00}(0) = 1, \text{ the other coefficients} = 0$$

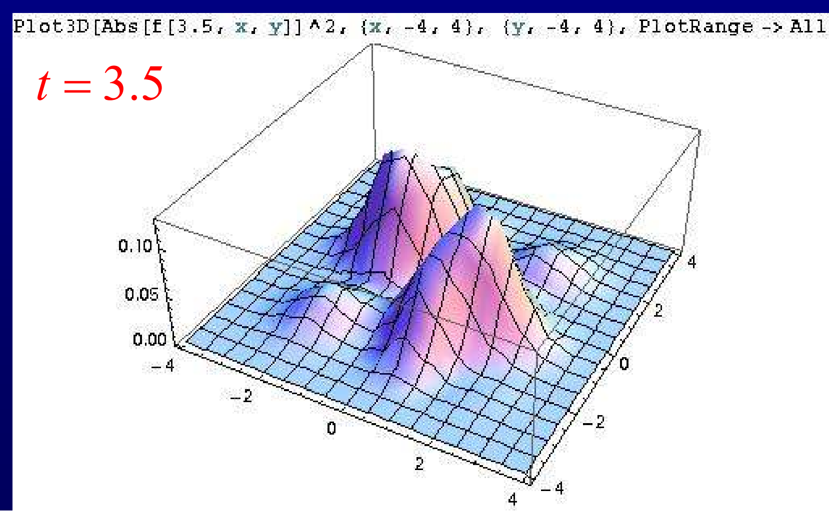






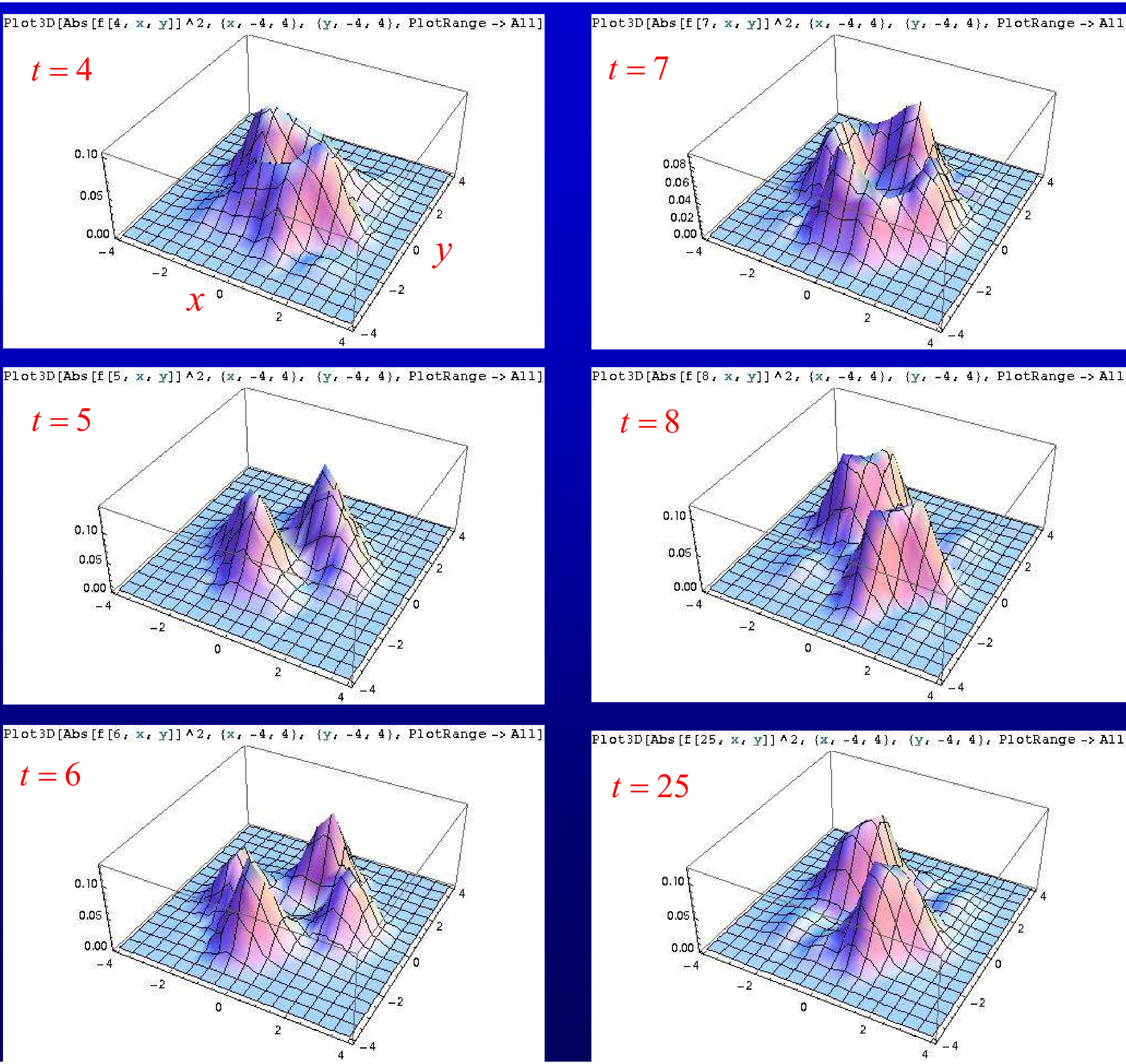


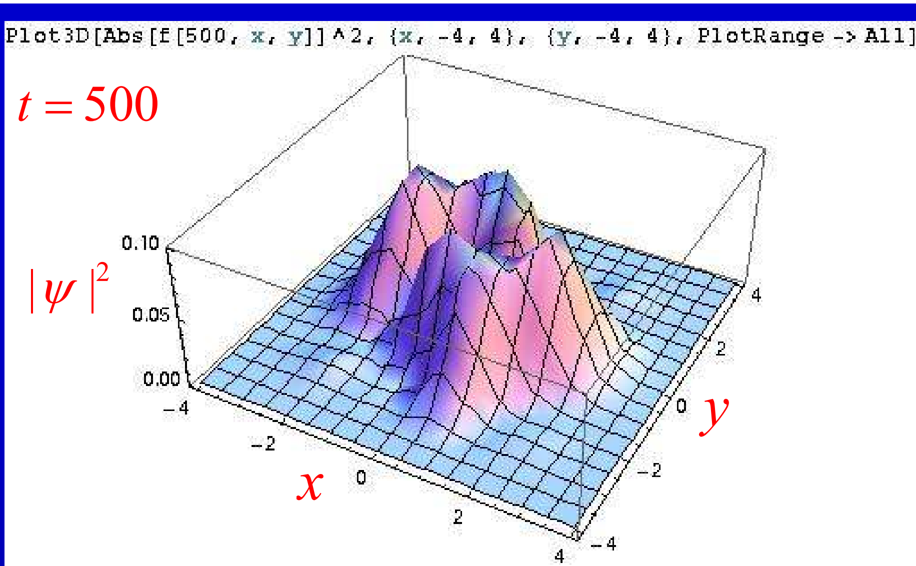


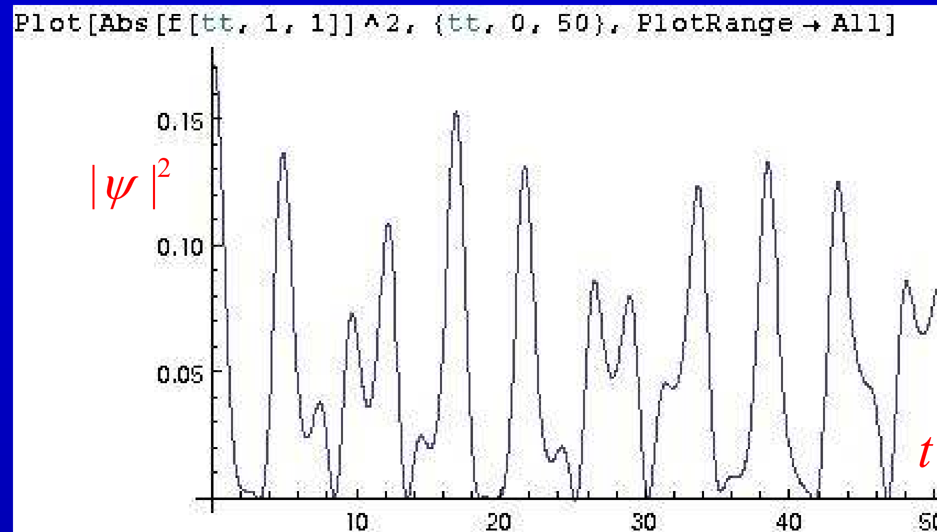


$$c_{01}(0) = \frac{1}{\sqrt{2}}$$

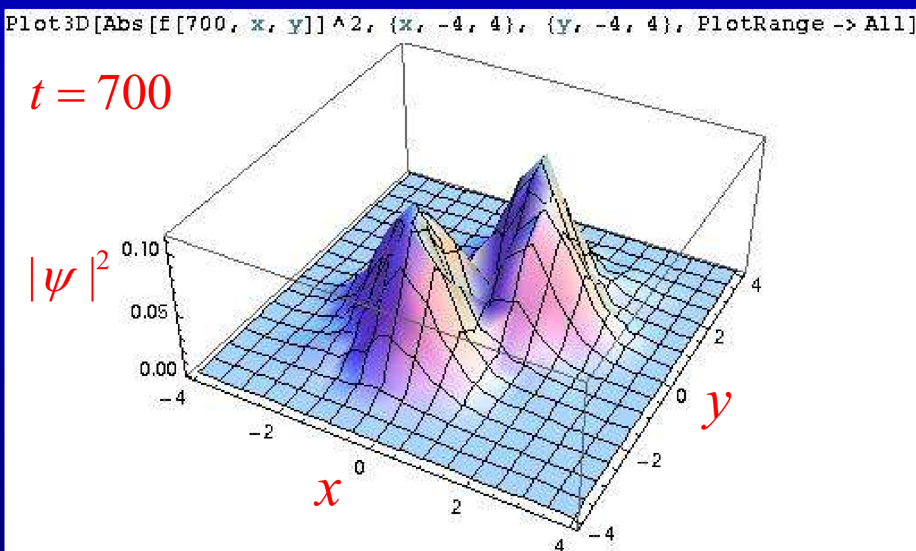
$$c_{10}(0) = \frac{1}{\sqrt{2}}$$

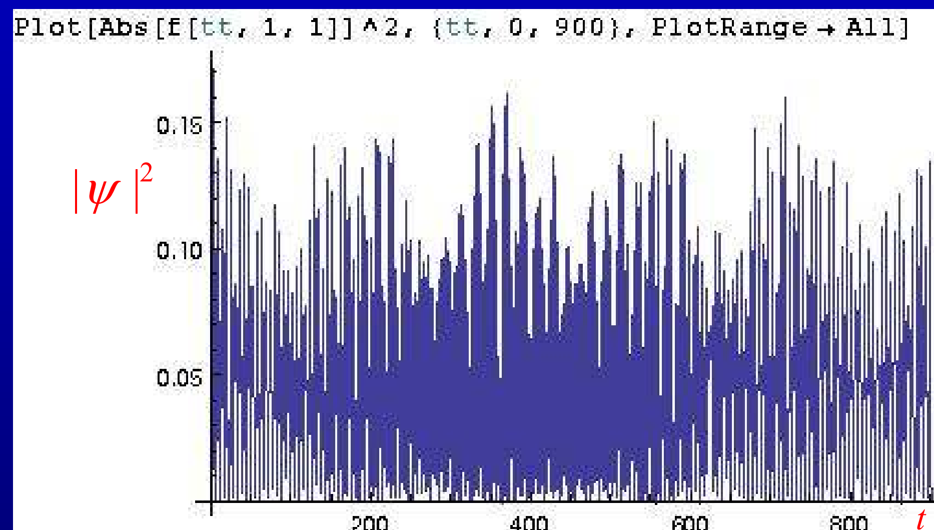

Plot3D[Abs[f[4, x, y]]^2, {x, -4, 4}, {y, -4, 4}, PlotRange -> All]
$t = 4$
x
y
Plot3D[Abs[f[7, x, y]]^2, {x, -4, 4}, {y, -4, 4}, PlotRange -> All]
$t = 7$
Plot3D[Abs[f[5, x, y]]^2, {x, -4, 4}, {y, -4, 4}, PlotRange -> All]
$t = 5$
Plot3D[Abs[f[8, x, y]]^2, {x, -4, 4}, {y, -4, 4}, PlotRange -> All]
$t = 8$
Plot3D[Abs[f[6, x, y]]^2, {x, -4, 4}, {y, -4, 4}, PlotRange -> All]
$t = 6$
Plot3D[Abs[f[25, x, y]]^2, {x, -4, 4}, {y, -4, 4}, PlotRange -> All]
$t = 25$


Plot3D[Abs[f[500, x, y]]^2, {x, -4, 4}, {y, -4, 4}, PlotRange -> All]
t = 500
|ψ|²
0.10
0.05
0.00
−4
−2
0
2
4
x
y


Plot[Abs[f[tt, 1, 1]]^2, {tt, 0, 50}, PlotRange → All]
|ψ|²
0.15
0.10
0.05
10
20
30
40
50
t

time


Plot3D[Abs[f[700, x, y]]^2, {x, -4, 4}, {y, -4, 4}, PlotRange -> All]
t = 700
|ψ|²
0.10
0.05
0.00
−4
−2
0
2
4
x
y


Plot[Abs[f[tt, 1, 1]]^2, {tt, 0, 900}, PlotRange → All]
|ψ|²
0.15
0.10
0.05
200
400
600
800
t

time

Interacting quantum fields

Example: scalar fields

$$\boxed{I = \int dx^4 \, \tfrac{1}{2}[g^{\mu\nu}\partial_\mu \varphi^a \partial_\nu \varphi^b \gamma_{ab} - V(\varphi)]}$$

Fock space basis

$$|P\rangle = |p_1, p_2, \ldots, p_n\rangle$$

Upon quantization:

$$|\Psi\rangle = \sum |P\rangle\langle P|\Psi\rangle$$

$$|\Psi(t)\rangle = e^{-iH(t-t_0)}|\Psi(t_0)\rangle$$

H is the Hamilton operator corresponding to the field action

$$\boxed{\langle P|\Psi(t)\rangle = \sum_{P'}\langle P|e^{-iH(t-t_0)}|P'\rangle\langle P'|\Psi(t_0)\rangle}$$

$$|\Psi(t_0)\rangle = |0\rangle$$

vacuum

$$\boxed{\langle P|\Psi(t)\rangle = \langle P|e^{-iH(t-t_0)}|0\rangle}$$

Such transition is possible, because $\langle P|$ contains particles with positive and negative energies.

Vacuum decays int

$$|\Psi(t)\rangle = \sum_{n=0}^{\infty}|p_1, p_2, .$$

$$\sum_{p_1}|\langle p_1|\Psi\rangle|^2 + \sum_{p_1,p_2}|\langle p_1,p_2|\Psi\rangle|^2 + \sum_{p_1,p_2,\ldots,p_n}|\langle p_1,p_2,\ldots,p_n|\Psi\rangle|^2 + \ldots = 1$$

Probabilities that vacuum decays into any of the states $|p_1\rangle,\ |p_1,p_2\rangle,\ \ |p_1,p_2,\ldots,p_n\rangle,\ldots,$ are not drastically different.