THE ROLE OF ORTHOGONAL AND SYMPELECTIC CLIFFORD ALGEBRAS IN QUANTUM FIELD THEORY

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Introduction

Persisting problem of quantum gravity and the unification of interactions.

A need to reformulate the conceptual foundations of physics and to employ a more evolved mathematical formalism.

In this talk I will consider Clifford algebras which provide very promising tools for description and generalization of geometry and physics.

Orthogonal Clifford algebra

\[ \gamma_a \cdot \gamma_b \equiv \frac{1}{2} (\gamma_a \gamma_b + \gamma_b \gamma_a) = g_{ab} \]
\[ \gamma_a \wedge \gamma_b \equiv \frac{1}{2} (\gamma_a \gamma_b - \gamma_b \gamma_a) \]

The inner, symmetric, product of basis vectors gives the orthogonal metric, \( g_{ab} \).

The outer, antisymmetric, product of basis vectors gives the basis bivector.

Symplectic Clifford algebra

\[ q_a \wedge q_b \equiv \frac{1}{2} (q_a q_b - q_b q_a) = J_{ab} \]
\[ q_a \cdot q_b \equiv \frac{1}{2} (q_a q_b + q_b q_a) \]

The inner, antisymmetric, product of basis vectors gives the symplectic metric, \( J_{ab} \).

The outer, symmetric, product of basis vectors gives a symplectic bivector.
The generators of an orthogonal Clifford algebra can be transformed into a basis in which they behave as fermionic creation and annihilation operators.

The generators of a symplectic Clifford algebra behave as bosonic creation and annihilation operators.

We will show how both kinds of operators can be united into a single structure so that they form a basis of a `superspace'. We will consider an action for a point particle in such superspace.

Instead of finite dimensional spaces, we can consider infinite dimensional spaces. Then we have a description of a field theory in terms of fermionic and bosonic creation and annihilation operators. The latter operators can be considered as being related to the basis vectors of the corresponding infinite dimensional space.
I. Orthogonal case

\[(a, b)_g = (a^a \gamma_a, b^b \gamma_b)_g = a^a (\gamma_a, \gamma_b)_g b^b = a^a g_{ab} b^b \]

\[\gamma_a, \gamma_b = g_{ab} \text{ metric} \]

For a basis we can take generators of the orthogonal Clifford algebra:

\[(\gamma_a, \gamma_b)_g = g_{ab} = \frac{1}{2} (\gamma_a \gamma_b + \gamma_b \gamma_a) = \gamma_a \gamma_b = g_{ab} \]

Vectors are Clifford numbers:

\[(a, b)_g = \frac{1}{2} (ab + ba) = a \cdot b \]
II. Symplectic case

\[(z, z')_J = (z^a q_a, z'^b q_b)_J = z^a (q_a, q_b)_J z'^b = z^a J_{ab} z'^b\]

\[(q_a, q_b)_J = J_{ab}\] symplectic metric

For a symplectic basis we can take generators of the symplectic Clifford algebra:

\[(q_a, q_b)_J = J_{ab} = \frac{1}{2} (q_a q_b - q_b q_a) = q_a \land q_b = J_{ab}\]

Vectors are now symplectic Clifford numbers:

\[(z, z')_J = \frac{1}{2} (z z' - z' z) = z \land z'\]
Explicit notation with coordinates and momenta

\[ z^a = (x^\mu, p^\mu) \]
\[ z = z^a q_a = x^\mu q^{(x)}_\mu + p^\mu q^{(p)}_\mu \]

\[ (z, z')_J = z^a (q_a, q_b)_J z'^b = z^a \frac{1}{2} (q_a q_b - q_b q_a) z'^b \]

\[ = z^a J_{ab} z'^b \]
\[ = (x^\mu p^{\nu} - p^\nu x^{\nu}) \eta_{\mu \nu} \]

Symplectic vector

\[ J_{ab} = \begin{pmatrix} 0 & \eta_{\mu \nu} \\ -\eta_{\mu \nu} & 0 \end{pmatrix} \]

Relations \[ \frac{1}{2} [q_a, q_b] = J_{ab} \]

give

\[ \frac{1}{2} [q^{(x)}_\mu, q^{(x)}_\nu] = 0, \quad \frac{1}{2} [q^{(p)}_\mu, q^{(p)}_\nu] = 0 \]
\[ \frac{1}{2} [q^{(x)}_\mu, q^{(p)}_\nu] = \eta_{\mu \nu} \]

Heisenberg commutation relations
Poisson bracket (symplectic case)

\[
\{f,g\}_{\text{PB}} = \frac{\partial f}{\partial z^a} J^{ab} \frac{\partial g}{\partial z^b}
\]

By introducing the symplectic basis vectors, we can rewrite the above expression as

\[
\frac{1}{2} \left[ \frac{\partial f}{\partial z^a} q^a, \frac{\partial g}{\partial z^b} q^b \right] = \frac{\partial f}{\partial z^a} J^{ab} \frac{\partial g}{\partial z^b}
\]

If we take \( f = z^c \), \( g = z^d \)

\[
\frac{1}{2} [q^c, q^d] = J^{cd}
\]

These are the Heisenberg commutation relations for `operators’ \( q^c \) and \( q^d \).

\( q^a \) are thus `quantized’ phase space coordinates \( z^a \).

\( z^a \) are real coordinates
Poisson bracket (orthogonal case)

\[ \{ f, g \}_{\text{PB}} \equiv \frac{\partial f}{\partial \lambda^a} g^{ab} \frac{\partial g}{\partial \lambda^b} \]

By introducing the basis vectors we can rewrite the above expression as

\[ \frac{1}{2} \left\{ \frac{\partial f}{\partial \lambda^a} \gamma^a, \frac{\partial g}{\partial \lambda^b} \gamma^b \right\} = \frac{\partial f}{\partial \lambda^a} g^{ab} \frac{\partial g}{\partial \lambda^b} \]

Orthogonal metric

\[ g^{ab} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \eta^{\mu\nu} \end{pmatrix} \]

If we take \( f = \lambda^c \), \( g = \lambda^d \)

\[ \frac{1}{2} \{ \gamma^c, \gamma^d \} \equiv \frac{1}{2} (\gamma^c \gamma^d + \gamma^d \gamma^c) = g^{cd} \]

These are the anticommutation relations for ‘operators’ \( \gamma^c \) and \( \gamma^d \).

\( \gamma^a \) are thus ‘quantized’ \( \lambda^a \).

\( \lambda^a \) are real anticommuting coordinates
Representation of operators

I. Orthogonal Clifford algebra

\[ \gamma_a \cdot \gamma_b \equiv \frac{1}{2} (\gamma_a \gamma_b + \gamma_b \gamma_a) = g_{ab} \]

In even dimensions we can write:

\[ \gamma_a = (\gamma_\mu, \overline{\gamma}_\mu), \quad \mu = 0, 1, 2, 3 \]

We can introduce Witt basis:

\[ \theta_\mu = \frac{1}{\sqrt{2}} (\gamma_\mu + i \overline{\gamma}_\mu) \]
\[ \overline{\theta}_\mu = \frac{1}{\sqrt{2}} (\gamma_\mu - i \overline{\gamma}_\mu) \]

\[ \theta_\mu \cdot \overline{\theta}_\nu = \frac{1}{2} (\theta_\mu \overline{\theta}_\nu + \overline{\theta}_\nu \theta_\mu) = \eta_{\mu\nu}, \quad \theta_\mu \cdot \theta_\nu = 0, \quad \overline{\theta}_\mu \cdot \overline{\theta}_\nu = 0 \]

\[ \gamma_\mu, \overline{\gamma}_\mu, \theta_\mu, \overline{\theta}_\mu \] can be represented:

1) as 4 x 4 matrices,
2) in terms of Grassmann coordinates:

\[ \theta^\mu \rightarrow \sqrt{2} \xi^\mu, \quad \overline{\theta}_\mu \rightarrow \sqrt{2} \frac{\partial}{\partial \xi^\mu} \]

\[ \xi^\mu \xi^\nu + \xi^\nu \xi^\mu = 0 \]
II. Symplectic Clifford algebra

\[ q_a \wedge q_b \equiv \frac{1}{2} (q_a q_b - q_b q_a) = J_{ab} \]

We can write:

\[ q_a = (q^{(x)}_\mu, q^{(p)}_\mu), \quad \mu = 0, 1, 2, 3 \]

\[ q^{(x)}_\mu \wedge q^{(p)}_\nu = \frac{1}{2} (q^{(x)}_\mu q^{(p)}_\nu - q^{(p)}_\mu q^{(x)}_\nu) = \eta_{\mu\nu} \]

\[ q^{(x)}_\mu \wedge q^{(x)}_\nu = 0, \quad q^{(p)}_\mu \wedge q^{(p)}_\nu = 0 \]

\[ q^{(x)}_\mu, q^{(p)}_\mu \text{ can be represented:} \]

1) as 4 x 4 matrices,
2) in terms of commuting coordinates:

\[ q^{\mu(x)} \rightarrow \sqrt{2} x^\mu, \quad q^{(p)}_\mu \rightarrow \sqrt{2} \frac{\partial}{\partial x^\mu} \]

\[ x^\mu x^\nu - x^\nu x^\mu = 0 \]

Then the operators cannot be cast into Hermitian form

\[ J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix} \]
Let us now consider the action

\[ I = \frac{1}{2} \int d\tau (\dot{z}^a J_{ab} z^b + z^a K_{ab} z^b) \]

where \( \delta z^a \)

\[ \dot{z}^a = J_{ab} \frac{\partial H}{\partial z^b} \]

Let us consider trajectories \( z^a(\tau) \) as components of an infinite dimensional vector:

\[ z = z^a(\tau) q_{a(\tau)} \equiv \int d\tau z^a(\tau) q_{a(\tau)} \]

\[ \dot{z}^a(\tau) q_{a(\tau)} = -z^a(\tau) \dot{q}_{a(\tau)} \]

and write the action in the form

\[ I = \frac{1}{2} \dot{z}^a(\tau) J_{a(\tau)b(\tau')} z^b(\tau') + \frac{1}{2} z^a(\tau) K_{a(\tau)b(\tau')} z^b(\tau') \]

The equations of motion are:

\[ \dot{z}^a(\tau) q_{a(\tau)} = -z^a(\tau) \dot{q}_{a(\tau)} = q_{a(\tau)} J_{a(\tau)b(\tau')} \frac{\partial H}{\partial z^b(\tau')} = -q^a(\tau) K_{a(\tau)b(\tau')} z^b(\tau') \]
Symplectic inner product is given by commutator

\[ z^b(\tau') \dot{q}^b(\tau') = q^a(\tau) K_{ab(\tau)} z^b(\tau') \]

This equation holds for any \( z^b(\tau) \)

Equations of motion for operators \( q^a(\tau) \)

\[ \dot{q}_b(\tau) = q^a(\tau) K_{ab} \]

\[ K_{a(\tau)b(\tau')} = K_{ab} \delta(\tau - \tau') \]

\[ \dot{q}_a = [q_a, \hat{H}] \]

Heisenberg equations of motion

\[ \hat{H} = \frac{1}{2} q^a K_{ab} q^b \]

\[ [q_a, \hat{H}] = K_{ab} q^b \]

We see that the basis vectors of phase space satisfy the Heisenberg equations of motion for `quantum' operators

\[ q^a(\tau) \equiv q^a(\tau) \]

\[ K_{ab} = K_{ba} \]
Symplectic inner product is given by commutator

\[ z^b(\tau') \dot{q}_b(\tau') = q^a(\tau) K_{a(\tau)b(\tau')} z^b(\tau') \]

This equation holds for any \( z^b(\tau) \)

\[ \dot{q}_b(\tau') = q^a(\tau) K_{a(\tau)b(\tau')} \]

Equations of motion for operators \( q^a(\tau) \)

\[ \dot{q}_a = [q_a, \hat{H}] \]

We see that the basis vector \( z^a(\tau) \),\( z^b(\tau) \) satisfy the Heisenberg equations of motion for ‘quantum’ operators

\[ z^a(\tau) = (x^\mu(\tau), p^\mu(\tau)) \]

Classical solution

Any trajectory

\[ q^a(\tau') = q^a(\tau) \]

\[ z^a(\tau') = z^a(\tau) \]
We introduce the generalized vector space whose elements are:

\[ Z = z^A q_A \]

coordinates
basis elements

\[ \langle q_A q_B \rangle_S = G_{AB} = \begin{pmatrix} J_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix} \]

\[ z^A = (z^a, \lambda^a), \quad z^a = (x^\mu, \bar{x}^\mu), \quad \lambda^a = (\lambda^\mu, \bar{\lambda}^\mu) \]

\[ q_A = (q_a, \gamma_a), \quad q_a = (q_\mu, \bar{q}_\mu), \quad \gamma_a = (\gamma_\mu, \bar{\gamma}_\mu) \]

\[ q_a \wedge q_b = J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix} \]

\[ \gamma_a \cdot \gamma_b = g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix} \]

Let us consider a particle moving in such space. Its worldline is given by:

\[ z^A = Z^A(\tau) \]

parameter on the worldline
Example of a possible action

\[ I = \frac{1}{2} \int d\tau \langle Z^A q_A q_B \dot{Z}^B \rangle_S = \frac{1}{2} \int d\tau Z^A G_{AB} \dot{Z}^B \]

where

\[ G_{AB} = \begin{pmatrix} J_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix}, \quad z^A = (z^a, \lambda^a) \]

\[ I = \frac{1}{2} \int d\tau (z^a J_{ab} \dot{z}^b + \lambda^a g_{ab} \dot{\lambda}^b) \]

\[ J_{ab} = -J_{ba} \]

\[ g_{ab} = g_{ba} \]

In this term, \( z^a \) are commuting coordinates.

In this term, \( \lambda^a \) are anticommuting (Grassmann) coordinates.

Canonical momenta:

\[ p^{(z)}_a = \frac{\partial L}{\partial \dot{z}^a} = \frac{1}{2} J_{ab} z^b, \]

\[ p^{(\lambda)}_a = \frac{\partial L}{\partial \dot{\lambda}^a} = \frac{1}{2} g_{ab} \dot{\lambda}^b, \]

\[ z^a = (z^\mu, \bar{z}^\mu) \]

\[ \lambda^a = (\lambda^\mu, \bar{\lambda}^\mu) \]

\( \mu = 0, 1, 2, 3 \)
\[ I = \frac{1}{2} \int d\tau \left( z^a J_{ab} \dot{z}^b + \lambda^a g_{ab} \dot{\lambda}^b \right) \]

\[ J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix}, \quad g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix} \]

\[ I = \frac{1}{2} \int d\tau \left( \dot{x}^\mu \eta_{\mu\nu} \bar{x}^\nu - \ddot{x}^\mu \eta_{\mu\nu} x^\nu + \dot{\lambda}^\mu \eta_{\mu\nu} \lambda^\nu + \ddot{\lambda}^\mu \eta_{\mu\nu} \bar{\lambda}^\nu \right) \]

\[ x^\mu, \bar{x}^\mu \text{ are commuting} \quad [x^\mu, x^\nu] = 0, \quad [ar{x}^\mu, \bar{x}^\nu] = 0 \]

\[ \lambda^\mu, \bar{\lambda}^\mu \text{ are Grassmannian} \quad \{\lambda^\mu, \lambda^\nu\} = 0, \quad \{\bar{\lambda}^\mu, \bar{\lambda}^\nu\} = 0 \]

**Canonical momenta:**

\[ p_{(x)}^\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{1}{2} \eta_{\mu\nu} \bar{x}^\nu, \quad p_{(\bar{x})}^\mu = \frac{\partial L}{\partial \dot{\bar{x}}^\mu} = -\frac{1}{2} \eta_{\mu\nu} x^\nu \]

\[ p_{(\lambda)}^\mu = \frac{\partial L}{\partial \dot{\lambda}^\mu} = \frac{1}{2} \eta_{\mu\nu} \lambda^\nu, \quad p_{(\bar{\lambda})}^\mu = \frac{\partial L}{\partial \dot{\bar{\lambda}}^\mu} = \frac{1}{2} \eta_{\mu\nu} \bar{\lambda}^\nu \]
Quantization

\[ x^\mu, p^{(x)}_\mu \rightarrow \hat{x}^\mu, \hat{p}^{(x)}_\mu \]

where

\[ \lambda^\mu, p^{(\lambda)}_\mu \rightarrow \hat{\lambda}^\mu, \hat{p}^{(\lambda)}_\mu \]

\[
\begin{align*}
[\hat{x}^\mu, \hat{p}^{(x)}_\nu] &= i \delta^\mu_\nu, \\
[\hat{x}^\mu, \hat{x}^\nu] &= 0, \\
[\hat{p}^{(x)}_\mu, \hat{p}^{(x)}_\nu] &= 0
\end{align*}
\]

\[
\begin{align*}
\{\hat{\lambda}^\mu, \hat{p}^{(\lambda)}_\nu\} &= i \delta^\mu_\nu, \\
\{\hat{\lambda}^\mu, \hat{p}^{(\lambda)}_\nu\} &= i \delta^\mu_\nu \\
\{\hat{\lambda}^\mu, \hat{\lambda}^\nu\} &= 0, \\
\{\hat{p}^{(\lambda)}_\mu, \hat{p}^{(\lambda)}_\nu\} &= 0
\end{align*}
\]

Altogether, we have

\[ z^a, p^{(z)}_a \rightarrow \hat{z}^a, \hat{p}^{(z)}_a \]

\[ \lambda^a, p^{(\lambda)}_a \rightarrow \hat{\lambda}^a, \hat{p}^{(\lambda)}_a \]

where the operators satisfy

\[
\begin{align*}
[\hat{z}^a, \hat{p}^{(z)}_b] &= i \delta^a_b, \\
\{\hat{\lambda}^a, \hat{p}^{(\lambda)}_b\} &= i \delta^a_b
\end{align*}
\]

Operators

Similar relations hold for barred quantities.

\[ z^a = (z^\mu, \bar{z}^\mu) \]

\[ \lambda^a = (\lambda^\mu, \bar{\lambda}^\mu) \]

\[ \mu = 0, 1, 2, 3 \]

Commutators

Anticommutators
But we see that the above operator equations are just the relations for the basis vectors of the orthogonal and symplectic Clifford algebra, provided that we identify:

\[
\begin{align*}
\hat{z}^a &= (q^\mu, i\bar{q}^\mu) \\
\hat{\lambda}^a &= (\gamma^\mu, i\gamma^\mu)
\end{align*}
\]

We see that `quantization’ is in fact the replacements of the coordinates \( z^a, \lambda^a \) with the corresponding basis vectors.

The only difference is in the factor \( i \) in front of \( \bar{q}_\mu \).
\[ I = \frac{1}{2} \int d\tau (z^a J_{ab} \dot{z}^b + \lambda^a g_{ab} \dot{\lambda}^b) \]
\[ = \frac{1}{2} \int d\tau \langle z^a q_a q_b \dot{z}^b + \lambda^a \gamma_a \gamma_b \dot{\lambda}^b \rangle_S \]

Basis vectors, entering the action, are 'quantum operators', apart from the \( i \) in the relations

\[ \hat{z}^a = (q^\mu, i \bar{q}^\mu), \quad \hat{\lambda}^a = (\gamma^\mu, i \bar{\gamma}^\mu) \]

\[ q_a = (q_\mu, \bar{q}_\mu), \quad \gamma_a = (\gamma_\mu, \bar{\gamma}_\mu) \]
The above action is not complete. An additional term is needed.

\[ I = \frac{1}{2} \int d\tau (z^a J_{ab} \dot{z}^b + \lambda^a g_{ab} \dot{\lambda}^b) \]
\[ = \frac{1}{2} \int d\tau \left( z^a q_a q_b \dot{z}^b + \lambda^a \gamma_a \gamma_b \dot{\lambda}^b \right)_S \]
\[ = \frac{1}{2} \int d\tau \left( z^a J_{ab} \dot{z}^b + \lambda'^a g'_{ab} \dot{\lambda}'^b \right) \]

\[ \xi^\mu \eta_{\mu\nu} \dot{\xi}^\nu + \bar{\xi}^\mu \eta_{\mu\nu} \dot{\bar{\xi}}^\nu \]

\[ q_a = (q_\mu, \bar{q}_\mu), \]
\[ \gamma_a = (\gamma_\mu, \bar{\gamma}_\mu) \]

\[ \lambda'^a = (\lambda'^\mu, \bar{\lambda}'^\mu), \quad \lambda'^\mu \equiv \xi^\mu = \frac{1}{\sqrt{2}} (\lambda^\mu - i \bar{\lambda}^\mu) \]
\[ \bar{\lambda}'^\mu \equiv \bar{\xi}^\mu = \frac{1}{\sqrt{2}} (\lambda^\mu + i \bar{\lambda}^\mu) \]

\[ g'_{ab} = \gamma'_a \gamma'_b = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ \eta_{\mu\nu} & 0 \end{pmatrix} \]

\[ G_{AB} = \begin{pmatrix} J_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix} \]

The generalized Bars action is (invariant under \( \tau \)-dependent rotations of \( Z^A \))

In particular, this term gives:

\[ \alpha p_\mu p^\mu + \beta \lambda^\mu p_\mu \]

Lagrange multipliers (contained in \( A^A_C \))

Mass comes from extra dimensions
Upon quantization, the classical constraint

\[ \lambda^\mu p_\mu = 0 \]

becomes the Dirac equation:

\[ \hat{\lambda}^\mu \hat{p}_\mu \Psi = 0 \]

where \( \hat{\lambda}^\mu = \gamma^\mu \).

\( \Psi \) can be represented

1) as a column \( \psi^\alpha \)
2) as a function \( \psi(x^\mu, \xi^\mu) \)

\( \hat{\lambda}^\mu = \gamma^\mu \) can be represented

1) as matrices
2) as \( \xi_\mu + \frac{\partial}{\partial \xi_\mu} \)

We also have \( \hat{\lambda}^\mu = i \bar{\gamma}^\mu \) which can be represented

1) as matrices
2) as \( i(\xi_\mu - \frac{\partial}{\partial \xi_\mu}) \)
From \( \theta_\mu = \frac{1}{\sqrt{2}} (\gamma_\mu + i \gamma_\mu) \)

\( \bar{\theta}_\mu = \frac{1}{\sqrt{2}} (\gamma_\mu - i \gamma_\mu) \)

we can build up spinors by taking a `vacuum'

\( \Omega = \prod_\mu \bar{\theta}_\mu \) which satisfies \( \bar{\theta}_\mu \Omega = 0 \)

and acting on it by `creation' operators \( \theta^\mu \).

So we obtain a `Fock space' basis for spinors:

\( \mathcal{G}_A = (1\Omega, \theta_\mu \Omega, \theta_\mu \theta_\nu \Omega, \theta_{\mu
u\rho} \Omega, \theta_{\mu
u\rho\sigma} \Omega) \)

in terms of which any state can be expanded:

\[
\Psi = \sum \psi^{\tilde{A}} \mathcal{G}_{\tilde{A}}
\]

With operators \( \theta_\mu, \bar{\theta}_\mu \) defined above, we can construct the spinors as the elements of a minimal left ideal of \( \text{Cl}(8) \).

Taking all possible vacua, such as

\( \Omega = \theta_{\mu_1} \theta_{\mu_2} ... \theta_{\mu_r} \bar{\theta}_{\mu_{r+1}} \bar{\theta}_{\mu_{r+2}} ... \bar{\theta}_{\mu_n}, \quad r = 0, 1, 2, 3, 4 \)

we obtain the Fock space basis for the whole \( \text{Cl}(8) \).

\[
\theta_\mu \cdot \bar{\theta}_\nu = \frac{1}{2} \{ \theta_\mu, \bar{\theta}_\nu \} = \eta_{\mu \nu},
\]

\( \theta_\mu \cdot \theta_\nu = 0, \quad \bar{\theta}_\mu \cdot \bar{\theta}_\nu = 0 \)

\( \mu = 0, 1, 2, 3 \)

\( \tilde{A} = 1, 2, ..., 16 \)

\[
\xi^\mu = \frac{1}{\sqrt{2}} (\lambda^\mu - i \bar{\lambda}^\mu)
\]

\[
\bar{\xi}^\mu = \frac{1}{\sqrt{2}} (\lambda^\mu + i \bar{\lambda}^\mu)
\]
Description of fields

\[ \Psi = \psi^{i(x)} h_{i(x)} \tag{1} \]

\[ i = 1, 2; \quad x \in \mathbb{R}^3 \quad \text{or} \quad x \in \mathbb{R}^{1,3} \]

\[ h_{i(x)} \cdot h_{j(x')} = \rho_{i(x)j(x')} \quad \text{metric} \quad \rho_{i(x)j(x')} = \delta_{ij} \delta(x - x') \]

New basis:

\[ h_{(x)} = \frac{1}{\sqrt{2}} (h_{1(x)} + i h_{2(x)}) \]

\[ h_{*(x)} = \frac{1}{\sqrt{2}} (h_{1(x)} - i h_{2(x)}) \]

Witt basis

\[ \psi^{(x)} h_{(x)} + \psi^{*(x)} h_{*(x)} \]

\[ h_{(x)} \cdot h_{*(x')} = \rho_{(x) *(x')} \]

\[ h_{(x)} \cdot h_{(x')} = h_{(x)} \cdot h_{*(x')} = 0 \]

Scalar product:

\[ \langle \Psi \Psi \rangle_S = \psi^{(x)} \rho_{(x) *(x')} \psi^{*(x')} + \psi^{*(x)} \rho_{*(x) (x')} \psi^{(x')} \]

\[ \psi^{(x)} h_{(x)} \rightarrow | \Psi \rangle \]

Both vectors bring the same information about the state

\[ \psi^{*(x)} h_{*(x)} \rightarrow \langle \psi | \]

\[ \langle \Psi | \Psi \rangle = \psi^{*(x)} h_{*(x)} \cdot h_{(x')} \psi^{(x')} = \psi^{*(x)} \rho_{*(x)(x')} \psi^{(x')} = \int dx \psi^{*(x)} \psi(x) \]

Orthogonal case

particular case

\[ \rho_{(x) *(x')} = \delta(x - x') \]

\[ \rho_{(x) *(x')} = \rho_{*(x)(x')} \]

Fermionic commutation relations
Vacuum

\[ \Omega = \prod_x h_{*}(x) \quad \Rightarrow \quad h_{*}(x)\Omega = 0 \]

\[ \Psi \Omega = \psi^{(x)} h_{(x)} \Omega \]

The second part of \( \Psi \) disappears

\[ \Psi = \psi^{(x)} h_{(x)} + \psi^{*(x)} h_{*}(x) \]

Let us consider a more general case:

\[ \Psi \Omega = (\psi_0 + \psi^{(x)} h_{(x)} + \psi^{(x)(x')} h_{(x)} h_{(x')} + ... )\Omega \]

This state is the infinite dimensional space analog of the spinor as an element of a left ideal of Clifford algebra

Reversed state:

\[ (\Psi \Omega)^\dagger = \Omega^\dagger \Psi^\dagger = \Omega^\dagger (\psi_0^* + \psi^{*(x)} h_{*}(x) + \psi^{*(x)(x')} h_{*}(x) h_{*}(x') + ... ) \]

Then

\[ (\Psi \Omega)^\dagger \Psi \Omega = \Omega^\dagger \Psi^\dagger \Psi \Omega \]

\[ = \Omega^\dagger \psi^{*(x)} h_{*}(x) h_{(x')} \psi^{(x')} \Omega + ... = 2\Omega^\dagger \psi^{*(x)} \delta_{(x)(x')} \psi^{(x')} \Omega + ... \]

This acting on vacuum gives 0
Vacuum

\[ \Omega = \prod_x h^*_x \]

\[ h^*_x \Omega = 0 \]

\[ \Psi \Omega = \psi^{(x)} h^{(x)} \Omega \]

The second part of \( \Psi \) disappears

\[ \Psi = \psi^{(x)} h^{(x)} + \psi^{*(x)} h^*_x \]

Let us consider a more general case:

\[ \Psi \Omega = (\psi_0 + \psi^{(x)} h^{(x)} + \psi^{(x)(x')} h^{(x)} h^{(x')} + ... ) \Omega \]

This state is the infinite dimensional space analog of the spinor as an element of a left ideal of Clifford algebra

We obtain a non vanishing form is we take the reversed state

\[ (\Psi \Omega)^\ddagger = \Omega^\ddagger \Psi^\ddagger = \Omega^\ddagger (\psi^*_0 + \psi^{*(x)} h^*_x + \psi^{*(x)(x')} \]

Then

\[ (\Psi \Omega)^\ddagger \Psi \Omega = \Omega^\ddagger \Psi^\ddagger \Psi \Omega \]

\[ = \Omega^\ddagger \psi^{*(x)} h^*_x h^{(x')} \psi^{(x')} \Omega + ... = 2 \Omega^\ddagger \psi^{*(x)} \delta_{(x)(x')} \psi^{(x')} \Omega + ... \]

\[ 2 \delta_{(x)(x')} - h^{(x')} h^*_x \]

\[ \langle \Omega^\ddagger \Psi^\ddagger \Psi \Omega \rangle_S = \psi^{*(x)} \delta_{(x)(x')} \psi^{(x')} + ... \]

This acting on vacuum gives 0

scalar part
\[ I = \int d\tau dx \left[ i \phi^*(\tau, x) \dot{\phi}(\tau, x) - H \right] \]

\[ \frac{\partial L}{\partial \dot{\phi}} = \Pi = i \phi^* \]

\[ I = \int d\tau dx \left[ \Pi \dot{\phi} - H \right] = \int d\tau dx \left[ \frac{1}{2} (\Pi \dot{\phi} - \phi \Pi) - H \right] \]

\[ \phi^{i(x)} = (\phi^{(x)}, \Pi^{(x)}), \quad i = 1, 2 \]

\[ J_{i(x)j(x')} = \begin{pmatrix} 0 & \delta_{(x)(x')} \\ -\delta_{(x)(x')} & 0 \end{pmatrix} \]

\[ \delta_{(x)(x')} \equiv \delta(x - x') \]

Symplectic vector:
\[ \Phi = \phi^{i(x)} k_{i(x)} = \phi^{1(x)} k_{1(x)} + \phi^{2(x)} k_{2(x)} \]
\[ \equiv \phi^{(x)} k_{\phi^{(x)}} + \Pi^{(x)} k_{\Pi^{(x)}} \]

Symplectic inner product:
\[ k_{i(x)} \wedge k_{j(x')} = J_{i(x)j(x')} \]

basis vectors
metric
\[ I = \int d\tau dx \left[ i \phi^*(\tau, x) \dot{\phi}(\tau, x) - H \right] \]

\[ \frac{\partial L}{\partial \phi} = \Pi = i \phi^* \]

\[ I = \int d\tau dx \left[ \Pi \dot{\phi} - H \right] = \int d\tau dx \left[ \frac{1}{2} (\Pi \dot{\phi} - \phi \dddot{\Pi}) - H \right] \]

\[ \phi^{(x)} = (\phi^{(x)}, \Pi^{(x)}), \quad i = 1, 2 \]

\[ I = \int d\tau (\phi^{(x)} J_{i(x) j(x')} \phi^{j(x')}) - H \]

**Symplectic vector:**

\[ \Phi = \phi^{i(x)} k_{i(x)} = \phi^{1(x)} k_{1(x)} + \phi^{2(x)} k_{2(x)} \equiv \phi^{(x)} k_{\phi^{(x)}} + \Pi \]

**Symplectic inner product:**

\[ k_{i(x)} \wedge k_{j(x')} = J_{i(x) j(x')} \]

**Schroedinger or scalar field**

**Stueckelberg field**

**Symplectic case**

**basis vectors**

**metric**

**H**

\[ H = \phi^{i(x)} K_{i(x) j(x')} \phi^{j(x')} \]

\[ K_{i(x) j(x')} = \begin{pmatrix} 0 & \delta_{(x)(x')} \\ -\delta_{(x)(x')} & 0 \end{pmatrix} \]

\[ g_{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
The Poisson bracket of two classical fields is equal to the symplectic metric.

On the other hand, the symplectic metric is equal to the wedge product of basis vectors.

In fact, the basis vectors are quantum operators, and satisfy the quantum canonical commutation relations:

\[
\frac{1}{2} [ k_\phi(x), k_\Pi(x') ] = \delta(x - x')
\]

or

\[
[ \hat{\phi}(x), \hat{\Pi}(x') ] = i \delta(x - x')
\]
`Matter` configuration

- a system of point particles $X^{\mu i}$
- a (system) of branes $X^{\mu(\xi)}$

A compact notation for a configuration $X^M$

Basis vectors $h_M$

Metric $h_M \cdot h_N = \eta_{MN}$

In Witt basis we have annihilation and creation operators:

- $h^+_M$
- $h^-_M$

$|\Psi\rangle = \text{(wavepacket profile)} (\prod_M h^+_M) |0\rangle$

Expectation value

$\langle \Psi \ h_M \ \Psi \rangle_1 \equiv \langle h_M \rangle$

Induced metric

$\langle h_M \rangle \cdot \langle h_M \rangle = g_{MN}$
I expect that in general we will obtain an induced metric with non vanishing curvature.

Since spacetime is a subspace of a configuration space, we will also obtain the metric of spacetime.

Curved spacetime metric originates from quantum configurations of many `particle’ systems.
Conclusion

An action for a physical system can be written in the phase space form, and it contains either the symplectic or the orthogonal form (or both).

The corresponding basis vectors satisfy either the fermionic anticommutation relations or the bosonic commutation relations, and satisfy the Heisenberg equations of motion.

Quantum operators are just the basis vectors of the phase space action.

The fact that basis vectors on the one hand are quantum operators, and on the other hand they give metric, can be exploited in the development of quantum gravity.

According to Feynman it is necessary to know several different representations of the same physics.

We have pointed out how `quantization' can be seen from yet another perspective.
The idea that basis vectors are quantum operators can be found in a book

M. Pavšič: The Landscape of Theoretical Physics: A Global view; From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle (Kluwer Academic, 2001)

where the orthogonal and symplectic cases are discussed.

Very promising is the description of gravity in terms of the Clifford algebra equivalent of the tetrad field which simplifies calculations significantly.

Some other related publications:

Class. Quant. Grav. 20, 2697-2714 (2003); gr-qc/0111092


Spin gauge theory of gravity in Clifford space: A Realization of Kaluza-Klein theory in 4- dimensional spacetime, Int. J. Mod. Phys. A21, 5905-5956 (2006); gr-qc/0507053


On the relativity in configurations space: A renewed physics in sight, 0912.3669 [gr-qc]