THE ROLE OF ORTHOGONAL AND SYMPLECTIC CLIFFORD ALGEBRAS IN QUANTUM FIELD THEORY

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Introduction

Persisting problem of quantum gravity and the unification of interactions.

A need to reformulate the conceptual foundations of physics and to employ a more evolved mathematical formalism.

In this talk I will consider Clifford algebras which provide very promising tools for description and generalization of geometry and physics.

Orthogonal Clifford algebra

$$\gamma_a \cdot \gamma_b \equiv \tfrac{1}{2}(\gamma_a \gamma_b + \gamma_b \gamma_a) = g_{ab}$$
$$\gamma_a \wedge \gamma_b \equiv \tfrac{1}{2}(\gamma_a \gamma_b - \gamma_b \gamma_a)$$

The inner, symmetric, product of basis vectors γ_a gives the orthogonal metric, g_{ab}.

The outer, antisymmetric, product of basis vectors gives the basis bivector.

Symplectic Clifford algebra

$$q_a \wedge q_b \equiv \tfrac{1}{2}(q_a q_b - q_b q_a) = J_{ab}$$
$$q_a \cdot q_b \equiv \tfrac{1}{2}(q_a q_b + q_b q_a)$$

The inner, antisymmetric, product of basis vectors q_a gives the symplectic metric, J_{ab}.

The outer, symmetric, product of basis vectors gives a symplectic bivector.

The generators of an orthogonal Clifford algebra can be transformed into a basis in which they behave as fermionic creation and annihilation operators.

The generators of a symplectic Clifford algebra behave as bosonic creation and annihilation operators.

We will show how both kinds of operators can be united into a single structure so that they form a basis of a `superspace'.
We will consider an action for a point particle in such superspace.

Instead of finite dimensional spaces, we can consider infinite dimensional spaces. Then we have description of a field theory in terms of fermionic and bosonic creation and annihilation operators.
The latter operators can be considered as being related to the basis vectors of the corresponding infinite dimensional space.

Spaces with orthogonal an symplectic forms

I. Orthogonal case

$$(a,b)_g = (a^a\gamma_a, b^b\gamma_b)_g = a^a(\gamma_a,\gamma_b)_g b^b = a^a g_{ab} b^b$$

$$(\gamma_a,\gamma_b)_g = g_{ab}$$ metric

Basis vectors

$$a = a^\mu \gamma_\mu$$

For a basis we can take generators of the orthogonal Clifford algebra:

$$(\gamma_a,\gamma_b)_g = g_{ab} = \frac{1}{2}(\gamma_a\gamma_b + \gamma_b\gamma_a) = \gamma_a\cdot\gamma_b = g_{ab}$$

Vectors are Clifford numbers:

$$(a,b)_g = \frac{1}{2}(ab + ba) = a\cdot b$$

Inner product of vectors *a* and *b*

II. Symplectic case

$$(z,z')_J = (z^a q_a, z'^b q_b)_J = z^a (q_a, q_b)_J z'^b = z^a J_{ab} z'^b$$

$$(q_a, q_b)_J = J_{ab}$$ symplectic metric

$$z = z^a q_a$$

Symplectic basis vectors

For a symplectic basis we can take generators of the symplectic Clifford algebra:

$$(q_a, q_b)_J = J_{ab} = \frac{1}{2}(q_a q_b - q_b q_a) = q_a \wedge q_b = J_{ab}$$

Vectors are now symplectic Clifford numbers:

$$(z,z')_J = \frac{1}{2}(z z' - z' z) = z \wedge z'$$

Inner product of symplectic vectors *z* and *z*’

Explicit notation with coordinates and momenta

$$z^a = (x^\mu, p^\mu)$$

$$z = z^a q_a = x^\mu q_\mu^{(x)} + p^\mu q_\mu^{(p)}$$

Symplectic vector

$$(z, z')_J = z^a (q_a, q_b)_J z'^b = z^a \tfrac{1}{2}(q_a q_b - q_b q_a) z'^b$$

$$= z^a J_{ab} z'^b$$

$$= (x^\mu p'^\nu - p^\nu x'^\nu)\eta_{\mu\nu}$$

$$J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix}$$

Relations $\tfrac{1}{2}[q_a, q_b] = J_{ab}$ give

$$\tfrac{1}{2}[q_\mu^{(x)}, q_\nu^{(x)}] = 0, \qquad \tfrac{1}{2}[q_\mu^{(p)}, q_\nu^{(p)}] = 0$$

$$\tfrac{1}{2}[q_\mu^{(x)}, q_\nu^{(p)}] = \eta_{\mu\nu}$$

Heisenberg commutation relations

Poisson bracket (symplectic case)

$$\{f,g\}_{\mathrm{PB}} \equiv \frac{\partial f}{\partial z^a} J^{ab} \frac{\partial g}{\partial z^b}$$

By introducing the symplectic basis vectors,
we can rewrite the above expression as

$$\boxed{\frac{1}{2}\left[\frac{\partial f}{\partial z^a} q^a, \frac{\partial g}{\partial z^b} q^b\right] = \frac{\partial f}{\partial z^a} J^{ab} \frac{\partial g}{\partial z^b}}$$

If we take $f = z^c, \qquad g = z^d$

$$\boxed{\frac{1}{2}[q^c, q^d] = J^{cd}}$$

These are the Heisenberg commutation relations for \`operators’ q^c and q^d.

q^a are thus \`quantized’ phase space coordinates z^a.

Symplectic metric

$$J^{ab} = \begin{pmatrix} 0 & -\eta_{\mu\nu} \\ \eta_{\mu\nu} & 0 \end{pmatrix}$$

$$J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix}$$

z^a are real coordinates

Poisson bracket (orthogonal case)

$$\{f,g\}_{\mathrm{PB}} \equiv \frac{\partial f}{\partial \lambda^a} g^{ab} \frac{\partial g}{\partial \lambda^b}$$

By introducing the basis vectors
we can rewrite the above expression as

$$\boxed{\frac{1}{2}\left\{ \frac{\partial f}{\partial \lambda^a}\gamma^a , \frac{\partial g}{\partial \lambda^b}\gamma^b \right\} = \frac{\partial f}{\partial \lambda^a} g^{ab} \frac{\partial g}{\partial \lambda^b}}$$

If we take $f = \lambda^c , \quad g = \lambda^d$

$$\boxed{\frac{1}{2}\{\gamma^c , \gamma^d\} \equiv \frac{1}{2}(\gamma^c\gamma^d + \gamma^d\gamma^c) = g^{cd}}$$

These are the anticommutation relations for
\`operators' γ^c and γ^d.

γ^a are thus \`quantized' λ^a.

Orthogonal metric

$$g^{ab} = \begin{pmatrix} \eta^{\mu\nu} & 0 \\ 0 & \eta^{\mu\nu} \end{pmatrix}$$

$$g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}$$

λ^a are real anticommuting coordinates

Representation of operators

I. Orthogonal Clifford algebra

$$\boxed{\gamma_a \cdot \gamma_b \equiv \frac{1}{2}(\gamma_a \gamma_b + \gamma_b \gamma_a) = g_{ab}}$$

$$g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}$$

In even dimensions we can write:

$$\gamma_a = (\gamma_\mu, \bar{\gamma}_\mu), \qquad \mu = 0,1,2,3$$

We can introduce Witt basis:

$$\theta_\mu = \frac{1}{\sqrt{2}}(\gamma_\mu + i\bar{\gamma}_\mu)$$

$$\bar{\theta}_\mu = \frac{1}{\sqrt{2}}(\gamma_\mu - i\bar{\gamma}_\mu)$$

$$\boxed{\theta_\mu \cdot \bar{\theta}_\nu = \frac{1}{2}(\theta_\mu \bar{\theta}_\nu + \bar{\theta}_\nu \theta_\mu) = \eta_{\mu\nu}, \qquad \theta_\mu \cdot \theta_\nu = 0, \qquad \bar{\theta}_\mu \cdot \bar{\theta}_\nu = 0}$$

$\gamma_\mu, \bar{\gamma}_\mu, \theta_\mu, \bar{\theta}_\mu$ can be represented:

1) as 4 x 4 matrices,
2) in terms of Grassmann coordinates:

$$\theta^\mu \to \sqrt{2}\xi^\mu, \qquad \bar{\theta}_\mu \to \sqrt{2}\frac{\partial}{\partial \xi^\mu}$$

$$\theta^\mu = \eta^{\mu\nu}\theta_\nu$$

$$\boxed{\xi^\mu \xi^\nu + \xi^\nu \xi^\mu = 0}$$

II. Symplectic Clifford algebra

$$q_a \wedge q_b \equiv \frac{1}{2}(q_a q_b - q_b q_a) = J_{ab}$$

$$J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix}$$

We can write:

$$q_a = (q_\mu^{(x)}, q_\mu^{(p)}), \qquad \mu = 0,1,2,3$$

$$q_\mu^{(x)} \wedge q_\nu^{(p)} = \frac{1}{2}(q_\mu^{(x)} q_\nu^{(p)} - q_\mu^{(p)} q_\nu^{(x)}) = \eta_{\mu\nu}$$

$$q_\mu^{(x)} \wedge q_\nu^{(x)} = 0, \qquad q_\mu^{(p)} \wedge q_\nu^{(p)} = 0$$

$q_\mu^{(x)}$, $q_\mu^{(p)}$ can be represented:

1) as 4 x 4 matrices,
2) in terms of commuting coordinates:

Then the operators cannot be cast into Hermitian form

$$q^{\mu(x)} \to \sqrt{2}x^\mu, \qquad q_\mu^{(p)} \to \sqrt{2}\frac{\partial}{\partial x^\mu}$$

$$q^{\mu(x)} = \eta^{\mu\nu} q_\nu^{(x)}$$

$$x^\mu x^\nu - x^\nu x^\mu = 0$$

Heisenberg equations as equations of motion for basis vectors

Let us now consider the action

$$I = \tfrac{1}{2}\int d\tau\,(\dot{z}^a J_{ab} z^b + z^a K_{ab} z^b)$$

$$\delta z^a$$

$$\dot{z}^a = J^{ab}\frac{\partial H}{\partial z^b}$$

$$H = \tfrac{1}{2} z^a K_{ab} z^b$$

$$z^a = (x^\mu, p^\mu)$$

Let us consider trajectories $z^a(\tau)$ as components of an infinite dimensional vector:

$$z = z^{a(\tau)} q_{a(\tau)} \equiv \int d\tau\, z^a(\tau)\, q_a(\tau)$$

$$\dot{z}^{a(\tau)} q_{a(\tau)} = -z^{a(\tau)} \dot{q}_{a(\tau)}$$

$$q_{a(\tau)} \wedge q_{b(\tau')} = J_{a(\tau)b(\tau')} = J_{ab}\,\delta(\tau - \tau')$$

and write the action in the form

$$I = \tfrac{1}{2}\dot{z}^{a(\tau)} J_{a(\tau)b(\tau')} z^{b(\tau')} + \tfrac{1}{2} z^{a(\tau)} K_{a(\tau)b(\tau')} z^{b(\tau')}$$

$$\mathcal{H} = \tfrac{1}{2}\int d\tau\, z^a(\tau)\, K_{ab}\, z^b(\tau) = \tfrac{1}{2} z^{a(\tau)} K_{a(\tau)b(\tau')}\, z^{b(\tau')}$$

The equations of motion are:

$$K_{a(\tau)b(\tau')} = K_{ab}\,\delta(\tau - \tau')$$

$$\dot{z}^{a(\tau)} q_{a(\tau)} = -z^{a(\tau)} \dot{q}_{a(\tau)} = q_{a(\tau)} J^{a(\tau)b(\tau')} \frac{\partial \mathcal{H}}{\partial z^{b(\tau')}} = -q^{a(\tau)}\, K_{a(\tau)b(\tau')} z^{b(\tau')}$$

$$\boxed{z^{b(\tau')}\dot{q}_{b(\tau')} = q^{a(\tau)} K_{a(\tau)b(\tau')} z^{b(\tau')}}$$

This equation holds for any $z^{b(\tau)}$

$$\boxed{\dot{q}_{b(\tau')} = q^{a(\tau)} K_{a(\tau)b(\tau')}}$$

Equations of motion for operators $q^{a(\tau)}$

$$K_{a(\tau)b(\tau')} = K_{ab}\delta(\tau - \tau')$$

$$\boxed{\dot{q}_b(\tau) = q^a(\tau) K_{ab}}$$

$$\left\{\begin{array}{l} \hat{H} = \frac{1}{2} q^a K_{ab} q^b \\ [q_a, \hat{H}] = K_{ab} q^b \end{array}\right.$$

$$\boxed{\dot{q}_a = [q_a, \hat{H}]}$$

Heisenberg equations of motion

Symplectic inner product is given by commutator

We see that the basis vectors of phase space satisfy the Heisenberg equations of motion for `quantum' operators

$$q^{a(\tau)} \equiv q^a(\tau)$$

$$K_{ab} = K_{ba}$$

$$q^{a(\tau)} \equiv q^a(\tau)$$

$$z^{a(\tau)} \equiv z^a(\tau)$$

$$z^{b(\tau')}\dot{q}_{b(\tau')} = q^{a(\tau)} K_{a(\tau)b(\tau')} z^{b(\tau')}$$

This equation hold for any $z^{b(\tau)}$

$$\dot{q}_{b(\tau')} = q^{a(\tau)} K_{a(\tau)b(\tau')}$$

Equations of motion for operators $q^{a(\tau)}$

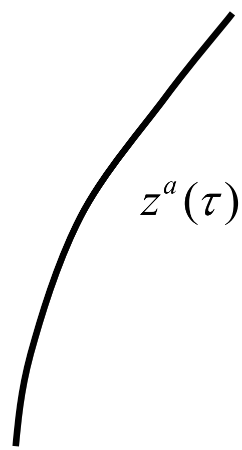
$K_{a(}$

$\dot{q}_b(\tau) = q^{a}$

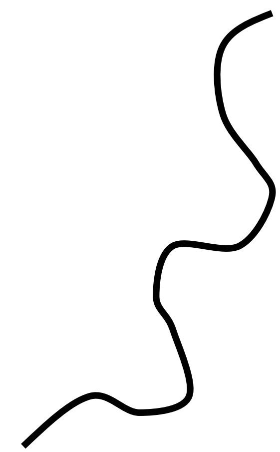
$\dot{q}_a = [q_a, \hat{H}$

We see that the basis ve
satisfy the Heisenberg eq
`quantum' operators

That the operator equations hold for any $z^{a(\tau)}$ means that coordinates and momenta are undetermined.



$$z^a(\tau) = (x^\mu(\tau), p^\mu(\tau))$$



Classical solution

Any trajectory

Point particle action in `superspace'

We introduce the generalized vector space whose elements are:

$$Z = z^A q_A$$

$$z^A = (z^a, \lambda^a), \qquad z^a = (x^\mu, \bar{x}^\mu), \quad \lambda^a = (\lambda^\mu, \bar{\lambda}^\mu)$$

$$q_A = (q_a, \gamma_a), \qquad q_a = (q_\mu, \bar{q}_\mu), \quad \gamma_a = (\gamma_\mu, \bar{\gamma}_\mu)$$

coordinates

basis elements

symplectic part

orthogonal part

$$\langle q_A q_B \rangle_S = G_{AB} = \begin{pmatrix} J_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix}$$

$$q_a \wedge q_b = J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix}$$

symplectic metric

$$\gamma_a \cdot \gamma_b = g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}$$

orthogonal metric

Let us consider a particle moving in such space. Its worldline is given by:

$$z^A = Z^A(\tau)$$

parameter on the worldline

Example of a possible action

$$I = \tfrac{1}{2}\int d\tau \langle Z^A q_A q_B \dot{Z}^B \rangle_S = \tfrac{1}{2}\int d\tau\, Z^A G_{AB} \dot{Z}^B$$

$$z^a = (z^\mu, \bar{z}^\mu)$$

$$\lambda^a = (\lambda^\mu, \bar{\lambda}^\mu)$$

$$\mu = 0, 1, 2, 3$$

$$G_{AB} = \begin{pmatrix} J_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix}, \qquad z^A = (z^{a.}, \lambda^a)$$

$$I = \tfrac{1}{2}\int d\tau \left(\underbrace{z^a J_{ab} \dot{z}^b} + \underbrace{\lambda^a g_{ab} \dot{\lambda}^b} \right)$$

$$J_{ab} = -J_{ba}$$

$$g_{ab} = g_{ba}$$

In this term, z^a are commuting coordinates.

In this term, λ^a are anticommuting (Grassmann) coordinates.

Canonical momenta:

$$p_a^{(z)} = \frac{\partial L}{\partial \dot{z}^a} = \tfrac{1}{2} J_{ab} z^b,$$

$$p_a^{(\lambda)} = \frac{\partial L}{\partial \dot{\lambda}^a} = \tfrac{1}{2} g_{ab} \lambda^a$$

$$I = \tfrac{1}{2}\int d\tau\,(z^a J_{ab}\dot{z}^b + \lambda^a g_{ab}\dot{\lambda}^b)$$

$$J_{ab} = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ -\eta_{\mu\nu} & 0 \end{pmatrix}, \qquad g_{ab} = \begin{pmatrix} \eta_{\mu\nu} & 0 \\ 0 & \eta_{\mu\nu} \end{pmatrix}$$

$$I = \tfrac{1}{2}\int d\tau\,(\dot{x}^\mu \eta_{\mu\nu}\bar{x}^\nu - \dot{\bar{x}}^\mu \eta_{\mu\nu} x^\nu + \dot{\lambda}^\mu \eta_{\mu\nu}\lambda^\nu + \dot{\bar{\lambda}}^\mu \eta_{\mu\nu}\bar{\lambda}^\nu)$$

$x^\mu,\ \bar{x}^\mu$ are commuting $\boxed{[x^\mu, x^\nu] = 0,\ \ [\bar{x}^\mu, \bar{x}^\nu] = 0}$

$\lambda^\mu,\ \bar{\lambda}^\mu$ are Grassmannian $\boxed{\{\lambda^\mu, \lambda^\nu\} = 0,\ \ \{\bar{\lambda}^\mu, \bar{\lambda}^\nu\} = 0}$

Canonical momenta:

$$p_\mu^{(x)} = \frac{\partial L}{\partial \dot{x}^\mu} = \tfrac{1}{2}\eta_{\mu\nu}\bar{x}^\nu, \qquad p_\mu^{(\bar{x})} = \frac{\partial L}{\partial \dot{\bar{x}}^\mu} = -\tfrac{1}{2}\eta_{\mu\nu}x^\nu$$

$$p_\mu^{(\lambda)} = \frac{\partial L}{\partial \dot{\lambda}^\mu} = \tfrac{1}{2}\eta_{\mu\nu}\lambda^\nu, \qquad p_\mu^{(\bar{\lambda})} = \frac{\partial L}{\partial \dot{\bar{\lambda}}^\mu} = \tfrac{1}{2}\eta_{\mu\nu}\bar{\lambda}^\nu$$

Quantization

$$x^\mu,\ p_\mu^{(x)} \to \hat{x}^\mu,\ \hat{p}_\mu^{(x)}$$

$$\lambda^\mu,\ p_\mu^{(\lambda)} \to \hat{\lambda}^\mu,\ \hat{p}_\mu^{(\lambda)}$$

Operators

Similar relations hold for barred quantities..

where

$$[\hat{x}^\mu, \hat{p}_\nu^{(x)}] = i\,\delta^\mu{}_\nu\,,$$

$$[\hat{x}^\mu, \hat{x}^\nu] = 0, \quad [\hat{p}_\mu^{(x)}, \hat{p}_\nu^{(x)}] = 0$$

$$\{\hat{\lambda}^\mu, \hat{p}_\nu^{(\lambda)}\} = i\,\delta^\mu{}_\nu\,, \quad \{\hat{\bar{\lambda}}^\mu, \hat{p}_\nu^{(\bar{\lambda})}\} = i\,\delta^\mu{}_\nu$$

$$\{\hat{\lambda}^\mu, \hat{\bar{\lambda}}^\nu\} = 0, \qquad \{\hat{p}_\mu^{(\lambda)}, \hat{p}_\nu^{(\bar{\lambda})}\} = 0$$

Altogether, we have

$$z^a\,,\ p_a^{(z)} \to \hat{z}^a\,,\ \hat{p}_a^{(z)}$$

$$\lambda^a,\ p_a^{(\lambda)} \to \hat{\lambda}^a,\ \hat{p}_a^{(\lambda)}$$

where the operators satisfy

$$[\hat{z}^a\,,\ \hat{p}_b^{(z)}] = i\,\delta^a{}_b$$

$$\{\hat{\lambda}^a,\ \hat{p}_b^{(\lambda)}\} = i\delta^a{}_b$$

Commutators

Anticommutators

$$z^a = (z^\mu, \bar{z}^\mu)$$

$$\lambda^a = (\lambda^\mu, \bar{\lambda}^\mu)$$

$$\mu = 0,1,2,3$$

$$[\hat{z}^a\,,\;\hat{p}_b^{(z)}] = i\,\delta^a{}_b$$

$$\{\hat{\lambda}^a,\,\hat{p}_b^{(\lambda)}\} = i\,\delta^a{}_b$$

$$\left\{\begin{array}{l} p_a^{(z)} = \frac{1}{2} J_{ab} z^b, \\ p_a^{(\lambda)} = \frac{1}{2} g_{ab} \lambda^a \end{array}\right.$$

$$\frac{1}{2}[\hat{z}^a, \hat{z}^b] = i\,J^{ab}$$

$$\frac{1}{2}\{\hat{\lambda}^a, \hat{\lambda}^b\} = i\,g^{ab}$$

But we see that the above operator equations are just the relations for the basis vectors of the orthogonal and symplectic Clifford algebra, provided that we identify:

$$\hat{z}^a = (q^\mu, i\overline{q}^\mu)$$

$$\hat{\lambda}^a = (\gamma^\mu, i\overline{\gamma}^\mu)$$

We se that `quantization' is in fact the replacements of the coordinates z^a, λ^a with the corresponding basis vectors.

The only difference is in the factor i in front of $\overline{q}_\mu$

$$I = \tfrac{1}{2}\int d\tau\,(z^a J_{ab}\dot{z}^b + \lambda^a g_{ab}\dot{\lambda}^b)$$
$$= \tfrac{1}{2}\int d\tau \langle z^a q_a q_b \dot{z}^b + \lambda^a \gamma_a \gamma_b \dot{\lambda}^b \rangle_S$$

$$q_a = (q_\mu, \bar{q}_\mu),$$
$$\gamma_a = (\gamma_\mu, \bar{\gamma}_\mu)$$

Basis vectors, entering the action, are `quantum operators', apart from the i in the relations

$$\hat{z}^a = (q^\mu, i\,\bar{q}^\mu), \quad \hat{\lambda}^a = (\gamma^\mu, i\,\bar{\gamma}^\mu)$$

$$I = \tfrac{1}{2}\int d\tau \,(z^a J_{ab}\dot z^b + \lambda^a g_{ab}\dot\lambda^b)$$
$$= \tfrac{1}{2}\int d\tau \langle z^a q_a q_b \dot z^b + \lambda^a \gamma_a \gamma_b \dot\lambda^b\rangle_S$$
$$= \tfrac{1}{2}\int d\tau \,(z^a J_{ab}\dot z^b + \underbrace{\lambda'^a g'_{ab}\dot\lambda'^b}_{\xi^\mu \eta_{\mu\nu}\dot{\bar\xi}^\nu + \bar\xi^\mu \eta_{\mu\nu}\dot\xi^\nu})$$

$$q_a = (q_\mu, \bar q_\mu),$$
$$\gamma_a = (\gamma_\mu, \bar\gamma_\mu)$$

$$\lambda'^a = (\lambda'^\mu, \bar\lambda'^\mu), \quad \lambda'^\mu \equiv \xi^\mu = \tfrac{1}{\sqrt2}(\lambda^\mu - i\bar\lambda^\mu)$$
$$\bar\lambda'^\mu \equiv \bar\xi^\mu = \tfrac{1}{\sqrt2}(\lambda^\mu + i\bar\lambda^\mu)$$

$$g'_{ab} = \gamma'_a \cdot \gamma'_b = \begin{pmatrix} 0 & \eta_{\mu\nu} \\ \eta_{\mu\nu} & 0 \end{pmatrix}$$

The above action is not complete. An additional term is needed.

$$I = \tfrac{1}{2}\int d\tau \, Z^A G_{AB} \dot Z^B$$

$$G_{AB} = \begin{pmatrix} J_{ab} & 0 \\ 0 & g_{ab} \end{pmatrix}$$

$\dot Z^B \to \dot Z^B + A^B{}_C Z^C$ replace with covariant derivative

$$I = \tfrac{1}{2}\int d\tau \, Z^A G_{AB} (\dot Z^B + \underbrace{A^B{}_C Z^C})$$

Generalized Bars action (invariant under τ-dependent rotations of Z^A)

In particular, this term gives:

$$\alpha\, p_\mu p^\mu + \beta\, \lambda^\mu p_\mu$$

Lagrange multipliers (contained in $A^A{}_C$)

Mass comes from extra dimensions

Upon quantization, the classical constraint

$$\lambda^{\mu} p_{\mu} = 0$$

becomes the Dirac equation:

$$\boxed{\hat{\lambda}^{\mu} \hat{p}_{\mu} \Psi = 0}$$

where $\hat{\lambda}^{\mu} = \gamma^{\mu}$.

Ψ can be represented

1) as a column ψ^{α}

2) as a function $\psi(x^{\mu}, \xi^{\mu})$

$\hat{\lambda}^{\mu} = \gamma^{\mu}$ can be represented

1) as matrices

2) as $\xi_{\mu} + \dfrac{\partial}{\partial \xi^{\mu}}$

We also have $\hat{\bar{\lambda}}^{\mu} = i\,\bar{\gamma}^{\mu}$ which can be represented

1) as matrices

2) as $i\left(\xi_{\mu} - \dfrac{\partial}{\partial \xi^{\mu}}\right)$

From $\theta_\mu = \frac{1}{\sqrt{2}}(\gamma_\mu + i\,\bar{\gamma}_\mu)$

$$\bar{\theta}_\mu = \frac{1}{\sqrt{2}}(\gamma_\mu - i\,\bar{\gamma}_\mu)$$

we can build up spinors by taking a `vacuum'

$\Omega = \prod_\mu \bar{\theta}_\mu$ which satisfies $\bar{\theta}_\mu \Omega = 0$

and acting on it by `creation' operators θ^μ .

So we obtain a `Fock space' basis for spinors:

$$\vartheta_{\tilde{A}} = (\mathbf{1}\Omega,\ \ \theta_\mu\Omega,\ \ \theta_\mu\theta_\nu\Omega,\ \ \theta_{\mu\nu\rho}\Omega,\ \ \theta_{\mu\nu\rho\sigma}\Omega)$$

in terms of which any state can be expanded:

$$\boxed{\Psi = \sum \psi^{\tilde{A}} \vartheta_{\tilde{A}}}$$

With operators $\theta_\mu, \bar{\theta}_\mu$ defined above, we can construct the spinors as the elements of a minimal left ideal of $C\ell(8)$.

Taking all possible vacua, such as

$$\Omega = \theta_{\mu_1}\theta_{\mu_2}...\theta_{\mu_r}\,\bar{\theta}_{\mu_{r+1}}\bar{\theta}_{\mu_{r+2}}...\bar{\theta}_{\mu_n}\,, \qquad r = 0,1,2,3,4$$

we obtain the Fock space basis for the whole $Cl(8)$.

$$\theta_\mu \cdot \bar{\theta}_\nu \equiv \tfrac{1}{2}\{\theta_\mu, \bar{\theta}_\nu\} = \eta_{\mu\nu}\,,$$

$$\theta_\mu \cdot \theta_\nu = 0, \qquad \bar{\theta}_\mu \cdot \bar{\theta}_\nu = 0$$

$$\mu = 0,1,2,3$$

$$\tilde{A} = 1,2,...,16$$

$$\xi^\mu = \tfrac{1}{\sqrt{2}}(\lambda^\mu - i\,\bar{\lambda}^\mu)$$

$$\bar{\xi}^\mu = \tfrac{1}{\sqrt{2}}(\lambda^\mu + i\,\bar{\lambda}^\mu)$$

Description of fields Orthogonal case

$$\boxed{\Psi = \psi^{i(x)} h_{i(x)}}$$

$$i = 1,2; \quad x \in \mathbb{R}^3 \ \text{ or } \ x \in \mathbb{R}^{1,3}$$

particular case

$$h_{i(x)} \cdot h_{j(x')} = \rho_{i(x)j(x')} \quad \text{metric} \quad \rho_{i(x)j(x')} = \delta_{ij}\, \delta(x - x')$$

New basis:

$$h_{(x)} = \tfrac{1}{\sqrt{2}}(h_{1(x)} + i\, h_{2(x)})$$

$$h_{*(x)} = \tfrac{1}{\sqrt{2}}(h_{1(x)} - i\, h_{2(x)})$$

Witt basis

$$\boxed{\Psi = \psi^{(x)} h_{(x)} + \psi^{*(x)} h_{*(x)}}$$

$$\left.\begin{aligned} h_{(x)} \cdot h_{*(x')} &= \rho_{(x)*(x')} \\ h_{(x)} \cdot h_{(x')} &= h_{*(x)} \cdot h_{*(x')} = 0 \end{aligned}\right\}$$

$$\rho_{(x)*(x')} \equiv \delta(x - x')$$

$$\rho_{(x)*(x')} = \rho_{*(x)\,(x')}$$

Fermionic commutation relations

Scalar product:

$$\langle \Psi\, \Psi \rangle_S = \psi^{(x)} \rho_{(x)*(x')} \psi^{*(x')} + \psi^{*(x)} \rho_{*(x)\,(x')} \psi^{(x')}$$

$$\psi^{(x)} h_{(x)} \to |\Psi\rangle$$

$$\psi^{*(x)} h_{*(x)} \to \langle \psi|$$

Both vectors bring the same information about the state

$$\langle \Psi | \Psi \rangle = \psi^{*(x)} h_{*(x)} \cdot h_{(x')} \psi^{(x')} = \psi^{*(x)} \rho_{*(x)(x')} \psi^{(x')} = \int dx\, \psi^*(x) \psi(x)$$

Vacuum $\Omega = \prod_x h_{*(x)}$ $h_{*(x)}\Omega = 0$

$$\boxed{\Psi\,\Omega = \psi^{(x)}\, h_{(x)}\,\Omega}$$

The second part of Ψ disappears

$$\Psi = \psi^{(x)} h_{(x)} + \psi^{*(x)} h_{*(x)}$$

Let us consider a more general case:

$$\Psi\,\Omega = (\psi_0 + \psi^{(x)}\, h_{(x)} + \psi^{(x)(x')} h_{(x)} h_{(x')} + \ldots\ )\Omega$$

This state is the infinite dimensional space analog of the spinor as an element of a left ideal of Clifford algebra

Reversed state:

$$(\Psi\,\Omega)^{\ddagger} = \Omega^{\ddagger}\Psi^{\dagger} = \Omega^{\ddagger}(\psi_0^{*} + \psi^{*(x)}\, h_{*(x)} + \psi^{*(x)(x')} h_{*(x)} h_{*(x')} + \ldots\ )$$

Then $(\Psi\,\Omega)^{\ddagger}\Psi\Omega = \Omega^{\ddagger}\Psi^{\ddagger}\,\Psi\Omega$

$$= \Omega^{\ddagger}\,\psi^{*(x)} \underbrace{h_{*(x)} h_{(x')}}_{2\,\delta_{(x)(x')} - \underbrace{h_{(x')} h_{*(x)}}_{\text{This acting on vacuum gives 0}}} \psi^{(x')}\,\Omega + \ldots = 2\Omega^{\ddagger}\,\psi^{*(x)}\delta_{(x)\,(x')}\,\psi^{(x')}\Omega + \ldots$$

Vacuum $\Omega = \prod_x h_{*(x)}$ $h_{*(x)}\Omega = 0$

$$\boxed{\Psi\,\Omega = \psi^{(x)}\, h_{(x)}\,\Omega}$$

The second part of Ψ disappears

$$\Psi = \psi^{(x)} h_{(x)} + \psi^{*(x)} h_{*(x)}$$

Let us consider a more general case:

$$\Psi\,\Omega = (\psi_0 + \psi^{(x)}\, h_{(x)} + \psi^{(x)(x')} h_{(x)} h_{(x')} + \dots)\Omega$$

This state is the infinite dimensional space analog of the spinor as an element of a left ideal of Clifford algebra

We obtain a non vanishing form is we take the reversed state

$$(\Psi\,\Omega)^{\ddagger} = \Omega^{\ddagger}\Psi^{\ddagger} = \Omega^{\ddagger}(\psi_0^{*} + \psi^{*(x)}\, h_{*(x)} + \psi^{*(x)(x'}$$

Then $(\Psi\,\Omega)^{\ddagger}\Psi\Omega = \Omega^{\ddagger}\Psi^{\ddagger}\,\Psi\Omega$

$$= \Omega^{\ddagger}\,\psi^{*(x)} \underbrace{h_{*(x)} h_{(x')}}_{2\,\delta_{(x)(x')} - \underbrace{h_{(x')} h_{*(x)}}_{\text{This acting on vacuum gives 0}}} \psi^{(x')}\,\Omega + \dots = 2\Omega^{\ddagger}\,\psi^{*(x)} \delta_{(x)\,(x')}\, \psi^{(x')}\Omega + \dots$$

$$\langle \Omega^{\ddagger}\Psi^{\ddagger}\,\Psi\Omega\rangle_S = \psi^{*(x)} \delta_{(x)\,(x')}\, \psi^{(x')} + \dots$$

scalar part

Symplectic case

$$I = \int d\tau\, dx \left[i\,\phi^*(\tau,x)\,\dot{\phi}(\tau,x) - H \right]$$

$x \in \mathbb{R}^3$ Schroedinger field

$x \in \mathbb{R}^{1,3}$ Stueckelberg field

$$\frac{\partial L}{\partial \dot{\phi}} = \Pi = i\,\phi^*$$

$$I = \int d\tau\, dx \left[\Pi\dot{\phi} - H \right] = \int d\tau\, dx \left[\tfrac{1}{2}(\Pi\dot{\phi} - \phi\dot{\Pi}) - H \right]$$

$$\phi^{i(x)} = (\phi^{(x)}, \Pi^{(x)}), \qquad i = 1,2$$

$$I = \int d\tau\, (\dot{\phi}^{i(x)} J_{i(x)j(x')} \phi^{j(x')} - H)$$

$$J_{i(x)j(x')} = \begin{pmatrix} 0 & \delta_{(x)(x')} \\ -\delta_{(x)(x')} & 0 \end{pmatrix}$$

$$\delta_{(x)(x')} \equiv \delta(x - x')$$

Symplectic vector:

$$\Phi = \phi^{i(x)} k_{i(x)} = \phi^{1(x)} k_{1(x)} + \phi^{2(x)} k_{2(x)}$$
$$\equiv \phi^{(x)} k_{\phi^{(x)}} + \Pi^{(x)} k_{\Pi^{(x)}}$$

Symplectic inner product:

$$k_{i(x)} \wedge k_{j(x')} = J_{i(x)j(x')}$$

Bosonic commutation relations

basis vectors

metric

Symplectic case

$$I=\int d\tau\,dx\Big[i\,\phi^*(\tau,x)\,\dot\phi(\tau,x)-H\Big]$$

$x\in\mathbb{R}^3$ — Schroedinger or scalar field

$x\in\mathbb{R}^{1,3}$ — Stueckelberg field

$$\frac{\partial L}{\partial\dot\phi}=\Pi=i\,\phi^*$$

$$I=\int d\tau\,dx\Big[\Pi\dot\phi-H\Big]=\int d\tau\,dx\Big[\tfrac{1}{2}(\Pi\dot\phi-\phi\dot\Pi)-H\Big]$$

$$\phi^{i(x)}=(\phi^{(x)},\Pi^{(x)}),\qquad i=1,2$$

$$I=\int d\tau\,(\dot\phi^{i(x)}J_{i(x)j(x')}\phi^{j(x')}-H)$$

$$J_{i(x)j(x')}=\begin{pmatrix}0 & \delta_{(x)(x')}\\ -\delta_{(x)(x')} & 0\end{pmatrix}$$

Symplectic vector:

$$\Phi=\phi^{i(x)}k_{i(x)}=\phi^{1(x)}k_{1(x)}+\phi$$

$$\equiv\phi^{(x)}k_{\phi^{(x)}}+\Pi$$

Symplectic inner product:

$$k_{i(x)}\wedge k_{j(x')}=J_{i(x)j(x')}$$

basis vectors

met

$$H=\phi^{i(x)}K_{i(x)j(x')}\phi^{j(x')}$$

$$K_{i(x)j(x')}=\left(-\frac{1}{2m}\partial^r\partial_r+V(x)\right)\delta(x-x')g_{ij}\,,$$

$$g_{ij}=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$$

$$K_{i(x)j(x')}=\begin{pmatrix}(m^2+\partial^r\partial_r)\delta(x-x') & 0\\ 0 & \delta(x-x')\end{pmatrix},$$

Poisson bracket

$$\left\{ f(\phi^{i(x)}), g(\phi^{i(x)}) \right\}_{\mathrm{PB}} = \frac{\partial f}{\partial \phi^{i(x)}} J^{i(x)j(x')} \frac{\partial g}{\partial \phi^{j(x')}}$$

$$J^{i(x)j(x')} = \begin{pmatrix} 0 & -\delta_{(x)(x')} \\ \delta_{(x)(x')} & 0 \end{pmatrix}$$

In particular: $f = \phi^{k(x'')}, \qquad g = \phi^{l(x''')}$

$$\left\{ \phi^{k(x'')}, \phi^{l(x''')} \right\}_{\mathrm{PB}} = J^{k(x'')l(x''')} = k^{k(x'')} \wedge k^{l(x''')}$$
$$\equiv \tfrac{1}{2} \left[k^{k(x'')}, k^{l(x''')} \right]$$

The Poisson bracket of two classical fields
is equal to the symplectic metric.

On the other hand, the symplectic metric
is equal to the wedge product of basis vectors.

In fact, the basis vectors are quantum operators, and
satisfy the quantum canonical commutation relations:

$$\tfrac{1}{2}[k_\phi(x), k_\Pi(x')] = \delta(x - x')$$

or

$$[\hat{\phi}(x), \hat{\Pi}(x')] = i\delta(x - x')$$

$$\hat{\phi}(x) = k_\phi(x) \,,$$
$$\hat{\Pi}(x) = i \tfrac{1}{2} q_\Pi(x)$$

Discussion: Prospects for quantum gravity

`Matter' configuration

a system of point particles $X^{\mu i}$

a (system) of branes $X^{\mu(\xi)}$

etc.

$$\left.\begin{matrix} X^{\mu i} \\ X^{\mu(\xi)} \end{matrix}\right\} X^M$$

A compact notation for a configuration

Basis vectors h_M

Metric $h_M \cdot h_N = \eta_{MN}$

In Witt basis we have annihilation and creation operators:

$$h_M^+ , \quad h_M^-$$

$$|\Psi\rangle = (\text{wavepacket profile})(\prod_M h_M^+)|0\rangle$$

Expectation value

$$\langle \Psi h_M \Psi \rangle_1 \equiv \langle h_M \rangle$$

Induced metric

$$\langle h_M \rangle \cdot \langle h_M \rangle = g_{MN}$$

I expect that in general we will obtain an induced metric
with non vanishing curvature.

Since spacetime is a subspace of a configuration space,
we will also obtain the metric of spacetime.

Curved spacetime metric originates from quantum configurations
of many `particle' systems.

Conclusion

An action for a physical system can be written in the phase space form, and it contains either the symplectic or the orthogonal form (or both).

The corresponding basis vectors satisfy either the fermionic anticommutation relations or the bosonic commutation relations, and satisfy the Heisenberg equations of motion.

Quantum operators are just the basis vectors of the phase space action.

The fact that basis vectors on the one hand are quantum operators, and on the other hand they give metric, can be exploited in the development of quantum gravity.

According to Feynman it is necessary to know several different representations of the same physics.

We have pointed out how \`quantization' can be seen from yet another perspective.

The idea that basis vectors are quantum operators can be found in a book

M. Pavšič: The Landscape of Theoretical Physics: A Global view;
From Point Particles to the Brane World and Beyond, in Search of a Unifying Principle
(Kluwer Academic, 2001)

where the orthogonal and symplectic cases are discussed.

Very promising is the description of gravity in terms of the Clifford algebra equivalent of the tetrad field which simplifies calculations significantly.

Some other related publications:

Class. Quant. Grav. 20, 2697-2714 (2003); gr-qc/0111092

Kaluza-Klein theory without extra dimensions: Curved Clifford space,
Phys. Lett. B614, 85-95 (2005); hep-th/0412255

Spin gauge theory of gravity in Clifford space: A Realization of Kaluza-Klein theory in 4- dimensional spacetime, Int. J. Mod. Phys. A21, 5905-5956 (2006); gr-qc/0507053

Beyond the relativistic point particle: A reciprocally invariant system,
Phys. Lett. B 680, 526-532 (2009)

On the relativity in configurations space: A renewed physics in sight,
0912.3669 [gr-qc]

Space inversion of spinors revisited: A possible explanation of
chiral behavior in weak interactions, Phys. Lett. B 692, 212-217 (2010)