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Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian

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Abstract. If an integrable classical Hamiltonian H describing bound motion depends on parameters which are changed very slowly then the adiabatic theorem states that the action variables I of the motion are conserved. Here the fate of the *angle variables* is analysed. Because of the unavoidable arbitrariness in their definition, angle variables belonging to *distinct* initial and final Hamiltonians cannot generally be compared. However, they *can* be compared if the Hamiltonian is taken on a *closed excursion* in parameter space so that initial and final Hamiltonians are the same. The result shows that the angle variable change arising from such an excursion is not merely the time integral of the instantaneous frequency $\omega \equiv \mathrm{d}H/\mathrm{d}I$, but differs from it by a definite extra angle which depends only on the circuit in parameter space, not on the duration of the process. The 2-form which describes this angle variable holonomy is calculated.

1. Introduction

Holonomy effects, those governed by inexact 'differentials' (1-forms) or 'non-integrable or anholonomic connections' are familiar in physics. Ranging from heat in thermodynamics, to the parallel transport of a vector in a curved space, where they are a fundamental ingredient in a physical theory, they are characterised by the existence of non-zero change in traversing closed circuits. (Strictly, perhaps, this should be called an *an*holonomy but in current usage the reversed emphasis is understood.)

Recently an embarassingly elementary holonomy effect in quantum mechanics that had somehow escaped early discovery was pointed out by Berry (1984). It concerns a quantum Hamiltonian depending on parameters which are changed adiabatically (infinitely slowly) and shows that the *phase* of an eigenstate changes when the Hamiltonian is taken around a closed circuit in parameter space, not only by the dynamical amount $\hbar^{-1}\int E\,\mathrm{d}t$ depending on the duration of the process, but by an extra geometrical amount depending only on the circuit. The fundamental object governing Berry's phase is a phase 2-form in parameter space for which he obtains a formula. The theory was recast in formal mathematical language by Simon (1983).

The present paper arises out of a question posed by Berry, namely what is the semiclassical limit ($\hbar\to 0$) of his phase. In fact, that particular question will not be answered here, though a guess can be made on the basis of the theory below which turns out to be correct when the proper analysis is made (Berry 1985). Instead I shall analyse an *analogous* holonomy effect which arises purely classically. It is analogous in that it arises when a classical Hamiltonian is taken *adiabatically* around a closed circuit in parameter space. Unlike the quantum holonomy though, the classical one has restricted validity. It applies only if the Hamiltonian has just one freedom, or more generally if it has *integrable* classical motion, because it involves the *action angle* variables of the Hamiltonian which only exist in that case. As is recalled in § 2 the action variables are the classical adiabatic invariants of such a system—they are guaranteed *not* to change around the circuit. But the *angle variable*, like the quantum phase, is not constrained to be the same; it is anholonomic, and the object here is to quantify and illustrate this effect. The semiclassical connection between the classical angle holonomy and the quantum phase holonomy was established by Berry (1985) and indeed two of the examples introduced below are there reanalysed in a framework better suited to semiclassical mechanics.

The plan of the paper is this. Restricting attention to one degree of freedom, I shall first recall the classical principle of adiabatic invariance of action (§ 2), present the angle variable holonomy (§ 3), illustrate it (§ 4), and finally generalise the result to an N-freedom integrable system and discuss it (§ 5).

2. Review of classical adiabatic invariance

The standard example which is used to explain adiabatic invariance in classical mechanics is that of the shortening pendulum (figure 1)—a bob swinging on a string which is slowly being drawn up through a hole. Work is done on the pendulum so its energy is not conserved as it would be for a static Hamiltonian, but there is a substitute quantity, the action, which *is* conserved in the adiabatic limit of slow change.

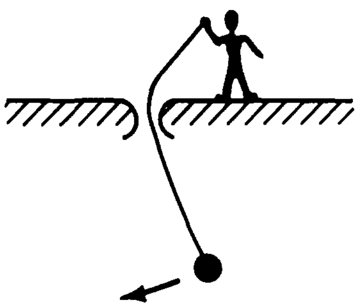

Figure 1. Adiabatic change in the length of a pendulum (from Arnold 1978).

In the phase plane (q, p), the initial, long pendulum has a Hamiltonian $H(q, p, L_{\text{long}})$ with concentric horizontal ellipse shaped contours around one of which the point representing the bob runs depending on its amplitude of oscillation. As the pendulum length $L(t)$ is shortened the contours become squashed horizontally so that finally, the short pendulum has a Hamiltonian $H(q, p, L_{\text{short}})$ with vertical ellipses instead. The bob will again be running around one of them—the question is which one? The answer is not that with the same energy value as initially but that with the same *area*. It is the *action* function $I(q, p, L(t))$, by definition $(2\pi)^{-1}$ times the instantaneous area of the Hamiltonian $H(q, p, L(t))$ contour on which (q, p) lies, which is adiabatically conserved.

The demonstration of this follows immediately from two principles which are most easily expressed not in terms of the individual bob concerned, but of the continuum of 'virtual' bobs chasing each other around the swinging cycle with different phases—a continuous train of dots around the contour of the initial Hamiltonian in the phase plane.

(i) *The adiabatic principle.* If the imposed change is slow enough the bobs will all be affected in the same way and will therefore finally *still be chasing* each other around their new swinging cycle. So the final train of dots once again lies around a *single* contour of the (new) Hamiltonian (rather than being slewed across a range of them as it would in a fast change). The adiabatic principle is otherwise known as the principle of *angle-averaging* (Arnold 1978); each point is supposed to experience the disturbance averaged over all of them.

(ii) *Liouville's theorem.* The area enclosed by any loop of particles in phase space is conserved under all circumstances (whether the Hamiltonian is static, slowly changing or fast changing).

Although the adiabatic principle (i) is well defined and widely realised physically (Landau and Lifshitz 1976), it appears (Arnold 1978) to be surprisingly difficult to eliminate the mathematical loopholes which prevent the simple statement that it holds rigorously in the limit of slow change. It remains an 'assertion of a physical character, i.e. it is untrue without further assumptions'. I shall take it for granted.

3. Angle variable holonomy

The prescription above applies equally well to the slow change of any form of Hamiltonian function $H(q, p)$ in the phase plane (provided, at least, that contours are not forced to change their topology by passage through a saddle point (Robnik and Hannay 1985)). The particle in phase space races around a track (contour of the instantaneous Hamiltonian) of fixed area $2\pi I$ but slowly changing shape. Given this rule of the conservation of action for which contour the particle lies on, it seems natural to ask about the development of the complementary variable, the angle variable, which describes whereabouts on the contour the particle is; to ask, that is, how many circuits it has made.

The instantaneous frequency of traversals' i.e. that which it would obtain if the Hamiltonian was 'frozen' is given by the derivative $(2\pi)^{-1}\,\mathrm{d}H/\mathrm{d}I$, so it is tempting to write the total angle traversed in a time T as simply

$$\int_0^T \frac{\mathrm{d}H(q(t), p(t), t)}{\mathrm{d}I}\,\mathrm{d}t = \int_0^T \frac{\mathrm{d}H(I, t)}{\mathrm{d}I}\,\mathrm{d}t \tag{1}$$

where in the last form H is considered as a function $H(I, t)$ of the area of its contours and adiabatic invariance, $I(t) = I$ constant, is invoked.

The reason (1) is incomplete is of course that the angle variable also changes by virtue of the changing (I, θ) coordinate system in phase space. To reveal the true structure of the situation it is necessary to consider the time dependence of the Hamiltonian function, and (I, θ) coordinate system, as being produced by carrying them along a path $\boldsymbol{R}(t)$ in a parameter space $\boldsymbol{R} \equiv (X, Y, Z \ldots)$ of two or more dimensions in which the functions $H(q, p, \boldsymbol{R})$, $I(q, p, \boldsymbol{R})$, $\theta(q, p, \boldsymbol{R})$ are uniquely defined. The point of making $\boldsymbol{R}$ more than one dimensional is that we shall wish to consider closed excursions $\boldsymbol{R}(T) = \boldsymbol{R}(0)$ in which $\boldsymbol{R}(t)$ executes a loop. With just one parameter, the length of the shortening pendulum for intance, the only way to restore the original length is to reverse the shortening and the holonomy effect is not realised.

In this framework then the exact rates of change of a particle's action and angle are:

$$\dot{I} = -\partial H/\partial\theta(=\text{Zero}) + \dot{\boldsymbol{R}}\cdot\partial I/\partial\boldsymbol{R} \tag{2}$$

$$\dot{\theta} = \partial H/\partial I + \dot{\boldsymbol{R}}\cdot\partial\theta/\partial\boldsymbol{R} \tag{3}$$

where the last terms are the rates of change of action and angle coordinates at a fixed point (q, p) in phase space (figure 2). For non-adiabatic excursion of the Hamiltonian these equations will lead to changes in both I and θ which will depend on the trajectory selected i.e. on the initial values of both I and θ. For adiabatic excursion the equations become

$$\dot{I} = \text{Zero} + \dot{\boldsymbol{R}}\cdot\langle\partial I/\partial\boldsymbol{R}\rangle(=\text{Zero}) \tag{4}$$

$$\dot{\theta} = \partial H/\partial I + \dot{\boldsymbol{R}}\cdot\langle\partial\theta/\partial\boldsymbol{R}\rangle \tag{5}$$

where the average brackets stand for the average around the Hamiltonian contour on which the point lies. Specifically for any function $f(q, p)$ we define a function $\langle f\rangle$ of action I by

$$\langle f\rangle = (2\pi)^{-1}\oint_{\text{contour through }(q,p)} f\,\mathrm{d}\theta \equiv (2\pi)^{-1}\int f(q, p)\delta(I(q, p) - I)\,\mathrm{d}q\,\mathrm{d}p. \tag{6}$$

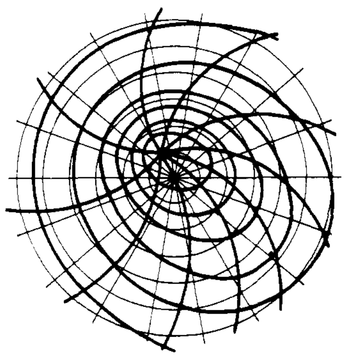

Figure 2. Action-angle coordinates changed with a parameter.

The average in (4) vanishes identically by Liouville's theorem and yields therefore $\dot{I} = 0$ as required. There is no reason however why the average in (5) should vanish and integration of this equation therefore yields the dynamical angle change anticipated in (1) *plus* the extra angle change we are interested in

$$\Delta\theta = \int \dot{\boldsymbol{R}}\cdot\langle\partial\theta(q(t), p(t), R(t))/\partial\boldsymbol{R}\rangle\,\mathrm{d}t = \int\langle\partial\theta/\partial\boldsymbol{R}\rangle\cdot\mathrm{d}\boldsymbol{R}. \tag{7}$$

In the last expression time has been completely eliminated because by definition (6) the average is a function of the conserved, and therefore initial, action I. There is a different field $\langle\partial\theta/\partial\boldsymbol{R}\rangle$ for each I, on which $\Delta\theta$ therefore depends. It does not depend on the initial angle.

Fixing I, then, we now examine the field $\langle\partial\theta/\partial\boldsymbol{R}\rangle$. It depends on the angle variable coordinatisation $\theta(q, p, \boldsymbol{R})$ which is to some extent arbitrary. Unlike the lines of constant action $I(q, p, \boldsymbol{R})$ which for fixed $\boldsymbol{R}$ are fully determined as the contours of the Hamiltonian $H(q, p, \boldsymbol{R})$, the lines of constant angle are only specified once one of them (say $\theta = 0$) is chosen. This one, and thus all the others, may be arbitarily twisted—into a spiral for instance (figure 2). So the angle variable change $\Delta\theta$ inevitably depends upon the angle coordinates chosen for the initial and final parameters $\boldsymbol{R}(0)$ and $\boldsymbol{R}(T)$. Only if they are *identical* coordinate systems, which in turn requires $\boldsymbol{R}(0) = \boldsymbol{R}(T)$ (barring specially favourable circumstances) can one expect to make coordinate independent statements about $\Delta\theta$. The excursions must be closed loops.

That $\Delta\theta$ is generally non-zero for such excursions is the central point of this paper expressing the anholonomy or non-integrability of the angle variable under adiabatic excursions of the Hamiltonian. The equivalent statement in local terms is that while the field $\langle \partial\theta/\partial\boldsymbol{R}\rangle$ is angle coordinate dependent, its 'curl'

$$\partial/\partial\boldsymbol{R} \wedge \langle \partial\theta/\partial\boldsymbol{R}\rangle \tag{8}$$

(or the appropriate generalisation below) is the primitive angle-coordinate-independent object. The assertion is that because the averaging is $\boldsymbol{R}$ dependent, the curl does not vanish identically as it would if the average brackets were absent. Physically speaking then, the field and its curl are rather like the vector potential $\boldsymbol{A}$ and the magnetic field $\boldsymbol{B}$ with their respective dependence on, and independence of, gauge transformations.

The manifestly angle-coordinate-independent expression for the curl is quoted below (9) and derived in the appendix. It is convenient there, and from here on to use the differential form notation appropriate to a structure free parameter space of arbitrary dimensionality. The gradient field $\partial\theta/\partial\boldsymbol{R}$ becomes the 1-form field $\boldsymbol{d}\theta$, and (8) is replaced by the exterior derivative 2-form $\boldsymbol{d}\langle\boldsymbol{d}\theta\rangle$.

The result for this angle 2-form is

$$\boldsymbol{d}\langle \boldsymbol{d}\theta\rangle = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}I^2}\int_0^{2\pi} \mathrm{d}\theta' \int_0^{2\pi} \mathrm{d}\chi\, \chi[\boldsymbol{d}I(\theta'-\chi)\wedge \boldsymbol{d}I(\theta')]. \tag{9}$$

Since the angle variables θ' and χ are integrated over and the bracket in (9) is periodic, this expression is obviously independent of the angle origin chosen, as required. Actually in practice it is usually simplest to choose a specific $\theta(q, p, \boldsymbol{R})$ arbitrarily, and obtain the 2-form by direct evaluation of the left hand side of (9) rather than carry out the integrals on the right.

4. Illustrations

Two illustrations of the geometrical angle holonomy originated in the idea that the simplest realisation of the effect might be to take a particle racing around an elliptical curve either in phase space or in real space, and to rotate the curve slowly and rigidly through one complete turn. The object would then be to demonstrate that the particle had not made the same number of circuits as it would have around the stationary curve, and not merely one less or one more either. It turns out, as we now see, that both the phase space and real space illustrations admit a rather more general analysis.

An important class of Hamiltonians (particularly for quantum mechanical purposes) are quadratic ones—the generalised simple harmonic oscillator,

$$H(q, p) = \tfrac{1}{2}(Xq^2 + 2Yqp + Zp^2) \tag{10}$$

whose contours are concentric oblique ellipses (for $XZ > Y^2$) or hyperbolae ($YZ < Y^2$)

in the phase plane. One could pick an elliptical one and ask that X, Y and Z be changed slowly in such a way as to effect exactly one rotation in the phase plane. It follows from (13) below that the consequent angle change is π times the sum of the ellipse axis ratio and its reciprocal. But the analysis is easily pursued to yield the angle 2-form and hence the result for an arbitrary closed excursion of the Hamiltonian in X, Y, Z space. Let the ellipse be that obtained from a circle by stretching by a factor α along a direction at angle ϕ to the q axis in the phase plane. Then the point on the ellipse at angle ϕ' to the q axis in the phase plane comes from the direction $-\theta+\phi$ on the circle with θ given by

$$\theta = -\tan^{-1}(\alpha \tan(\phi'-\phi)). \tag{11}$$

This is the angle variable on the ellipse measured from the long axis. The negative sign arises from the convention that the angle variable is measured in the opposite sense to ordinary angle. Exterior differentiation (in parameter space α, ϕ) and averaging around the ellipse yields

$$\langle \mathrm{d}\theta\rangle = \frac{-1}{2\pi}\int_0^{2\pi} \mathrm{d}\phi' \frac{\partial\theta}{\partial\phi'}\left(\frac{\tan(\phi'-\phi)\boldsymbol{d}\alpha - \alpha\sec^2(\phi'-\phi)\boldsymbol{d}\phi}{1+\alpha^2\tan^2(\phi'-\phi)}\right) \tag{12}$$

$$= \text{Zero } \boldsymbol{d}\alpha - \tfrac{1}{2}(\alpha+\alpha^{-1})\boldsymbol{d}\phi \tag{13}$$

$$= -\frac{1}{2}\frac{(X+Z)}{(XZ-Y^2)^{1/2}}\boldsymbol{d}\tfrac{1}{2}\tan^{-1}\left(\frac{+2Y}{X-Z}\right) \tag{14}$$

where standard ellipse geometry has been used to relate α (ratio of principal axes) and ϕ to X, Y, Z. Taking the exterior derivative (and using $\boldsymbol{d}\,\boldsymbol{d}\equiv 0$) gives

$$\boldsymbol{d}\langle\boldsymbol{d}\theta\rangle = -\tfrac{1}{4}\boldsymbol{d}\left(\frac{X+Z}{(XZ-Y^2)^{1/2}}\right)\wedge \boldsymbol{d}\tan^{-1}\left(\frac{2Y}{X-Z}\right) \tag{15}$$

$$= \frac{-1}{4(XZ-Y^2)^{3/2}}[(XZ-Y^2)(\boldsymbol{d}X+\boldsymbol{d}Z)-\tfrac{1}{2}(X+Z)(Z\boldsymbol{d}X+X\boldsymbol{d}Z-2Y\boldsymbol{d}Y)]$$
$$\wedge\frac{2(X-Z)\boldsymbol{d}Y-2Y(\boldsymbol{d}X-\boldsymbol{d}Z)}{(X-Z)^2+4Y^2} \tag{16}$$

$$= \frac{-1}{4(XZ-Y^2)^{3/2}}[X(\boldsymbol{d}Y\wedge\boldsymbol{d}Z)+Y(\boldsymbol{d}Z\wedge\boldsymbol{d}X)+Z(\boldsymbol{d}X\wedge\boldsymbol{d}Y)]. \tag{17}$$

The symmetry of the bracket here with respect to the cyclic permutation of X, Y, Z shows that the curl vector field representing the 2-form is a radial one. Moreover since it decays as the inverse square of the 'radius', the phase change around a circuit actually depends only on the circuit of *directions* (i.e. on the radial projection of the circuit onto the unit sphere).

The second illustration involves the rotation of a curve in real space rather than phase space. A wire 'hoop' of area A and perimeter L in a plane with a bead of unit mass sliding frictionlessly around it is slowly turned through one revolution. There is now no need to restrict attention to an elliptical hoop—the shape can be kept general. The analysis of this system is easy in the frame rotating with the hoop. If Ω is the hoop's angular velocity and $\boldsymbol{q}(t)$ the position of the bead in the hoop frame, the canonical momentum $\boldsymbol{p}(t)$ is given by

$$\boldsymbol{p}=\dot{\boldsymbol{q}}+\boldsymbol{\Omega}\wedge\boldsymbol{q} \tag{18}$$

(which is equal to its ordinary momentum in the non-rotating frame). The action is the line integral of $\boldsymbol{p}$ around the hoop

$$I=\frac{1}{2\pi}\oint\boldsymbol{p}\cdot\mathrm{d}\boldsymbol{q} \tag{19}$$

$$=\frac{1}{2\pi}\left(\oint\dot{\boldsymbol{q}}\cdot\mathrm{d}\boldsymbol{q}+2\Omega A\right). \tag{20}$$

This says that the adiabatically conserved quantity is the (spatial) average speed of the bead around the hoop plus $2\Omega A/L$. If Ω is sufficiently small the bead's fluctuations in speed around the length of the hoop are small and the spatial average speed can be taken as the temporal average speed. If Ω rises from zero and falls back again with time integral 2π corresponding to one revolution of the hoop, then the time integral of the average speed is $-4\pi A/L$ greater than it would have been had Ω remained zero. This, then, is the extra distance travelled, or, multiplied by $2\pi/L$, the extra angle change

$$\Delta\theta=-8\pi^2A/L^2. \tag{21}$$

This result reduces to the expected one $\Delta\theta=-2\pi$ for the circular hoop, where the hoop has simply slipped around, catching up on the bead by one revolution, so that the bead has made one fewer turns in the hoop frame.

The result (21) can be obtained by a variety of alternative means. One of the two used by Berry (1985) is direct integration of the acceleration due to the pseudo-forces in the rotating frame. There the effect emerges entirely as a result of the 'Euler force' $m(\dot{\boldsymbol{\Omega}}\wedge\boldsymbol{q})$. In an analysis of ring gyroscopes (Forder 1984), the result arises from the relativistic lack of synchrony of clocks in the rotating frame.

As a final illustration I consider a rather obvious realisation of the extra angle change. It is a spinning symmetric top with fixed base point whose axis is pulled around a closed circuit enclosing a solid angle Ω. (Gravity plays no role since the axis is driven, not free.) The component of angular velocity ω of the top along its axis is invariant (this time rigorously, not merely adiabatically). So the dynamical part of the top's rotation angle after the circuit is simply ω times the circuit duration. But there is an extra angle which would be present even if the top were not spinning ($\omega=0$), namely that caused by the familiar parallel transport holonomy of a vector on a curved surface, in this case the unit sphere. The angle turned through by a vector, and therefore the extra angle for the top, is the integral of the Gaussian curvature over the area enclosed—that is, the solid angle Ω. In this example the angle 2-form is just the curvature 2-form of the unit sphere.

5. Generalisation and discussion

While the presentation has referred to a Hamiltonian system with one freedom only, it is straightforward, by adding indices in the appendix, to generalise to any N freedom system admitting action angle variables I_j, θ_j $(1\leqslant j\leqslant N)$ that is any *integrable* system. All N of the action functions I_j are then adiabatic invariants and each angle variable θ_j suffers extra change on closed adiabatic excursion of the Hamiltonian. The 2-form for this extra change is

$$d\langle d\theta_j\rangle = \frac{1}{2}\frac{\partial^2}{\partial I_j\,\partial I_k}\int_0^{2\pi} d^N\boldsymbol{\theta}\int_0^{2\pi} d^N\boldsymbol{\chi}\,\chi_l[dI_l(\boldsymbol{\theta}-\boldsymbol{\chi})\wedge dI_k(\boldsymbol{\theta})]. \tag{22}$$

As emphasised in the introduction, while 1-freedom systems are necessarily integrable, N-freedom systems are not. There is no generalisation of (22) to N-freedom systems with the generic mixed structure of chaos and tori, indeed there are *no* known adiabatic invariants in that case.

While it is likely that angle variable changes have long been required and calculated correctly in particular contexts, in celestial mechanics for example, they have probably been found by rather *ad hoc* methods like that used in the rotating hoop illustration earlier. This paper has sought to present the effect in its generality. The other (indeed the original) motivation, the possible connection with the semiclassical limit of Berry's phase $\Delta\phi$ has not been further mentioned, but the standard relationship for the *time* evolution of θ and ϕ in a *static* Hamiltonian

$$\frac{d\theta}{dt} = \frac{dH}{dI} = -\frac{d}{dI}\left(\hbar\frac{d\phi}{dt}\right) \tag{23}$$

suggests the relationship

$$\Delta\theta = -\hbar\, d\phi/dI. \tag{24}$$

This is indeed the conclusion of semiclassical analysis (Berry 1985).

Another connection between the present classical analysis and quantum mechanics, for 1-freedom systems, arises from the trick of treating the wavefunction $\psi(x)$ and its derivative $\psi'(x)$ as 'position' and 'momentum' in a bogus 'phase space'. The time independent Schrödinger equation for given energy E then becomes a Newton's equation for ψ, with x playing the role of 'time', and an x dependent (i.e. 'time' dependent) linear restoring force $(E-V)\psi$. If $V(x)$ is a slowly varying function of x then the quadratic Hamiltonian analysis of the second illustration earlier, is directly applicable. This is one way of viewing a recent analysis of phase holonomy arising not from slow time variation of a quantum Hamiltonian but from slow space variation—that is, phase holonomy in WKB theory (Wilkinson 1984).

Acknowledgments

I am grateful to Dr P J Richens whose elementary explanation of adiabatic invariance I have reproduced in § 2. Also to Professor M V Berry for discussing with me his parallel analyses.

Appendix

The formula (9) for the angle 2-form $d\langle d\theta\rangle$ which is manifestly independent of the arbitrarinesss in the definition of angle variables may be derived as follows. As in the main text, q, p are fixed phase plane coordinates and $I(q, p, \boldsymbol{R})$ and $\theta(q, p, \boldsymbol{R})$ are parameter dependent functions of them. The symbol $\boldsymbol{d}$ stands for exterior differentiation in parameter space $\boldsymbol{R}$ (not phase space q, p). It is convenient for the derivation to distinguish the value, I_0 say, of the conserved action from the action function I.

$$d\langle d\theta\rangle = d\int (d\theta)\delta(I(q,p,\boldsymbol{R})-I_0)\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi} \tag{A1}$$

$$= \int (dI\wedge d\theta)\frac{\mathrm{d}\delta}{\mathrm{d}I}\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi} \tag{A2}$$

$$= \frac{\mathrm{d}}{\mathrm{d}I_0}\int (d\theta\wedge dI)\delta\frac{\mathrm{d}q\,\mathrm{d}p}{2\pi} \tag{A3}$$

$$= \frac{\mathrm{d}}{\mathrm{d}I_0}\langle d\theta\wedge dI\rangle. \tag{A4}$$

The task is now to eliminate $d\theta$ from this average. An expression for $\partial d\theta/\partial\theta$ can be obtained and used. (This derivative is not equal to $d(\partial\theta/\partial\theta)\equiv 0$ because d and $\partial/\partial\theta$ do not commute.) The expression derives from the fact that the variables I, θ remain canonical; $\{I,\theta\}=1$ for all $\boldsymbol{R}$ so that by exterior differentiation

$$0=\{dI,\theta\}+\{I,d\theta\}=\partial dI/\partial I+\partial d\theta/\partial\theta. \tag{A5}$$

Integrating this with respect to θ supplies $d\theta$, which inserted into the average $\langle d\theta\wedge dI\rangle$ gives

$$\langle d\theta\wedge dI\rangle = -\int_0^{2\pi}\left[\left(\int_0^{\theta}\frac{\partial dI}{\partial I}\mathrm{d}\theta'\right)\wedge dI(\theta)\right]\frac{\mathrm{d}\theta}{2\pi} \tag{A6}$$

$$= \int_0^{2\pi}\left(\frac{\partial dI(\theta)}{\partial I}\wedge\int_0^{\theta} dI(\theta')\theta'\right)\frac{\mathrm{d}\theta}{2\pi} \tag{A7}$$

where an integration by parts has been performed, the integrated part vanishing by virtue of the (exact) Liouville relation used in (4)

$$\int_0^{2\pi} dI\frac{\mathrm{d}\theta}{2\pi}=\langle dI\rangle=0. \tag{A8}$$

The expression (A7) is, as can easily be verified; independent of the choice of origin of θ. It can be simplified by inspection. The integrand is the product of two functions evaluated at different angles. Keeping first their *separation* χ fixed and then integrating over all χ (invoking the independence of origin) we obtain

$$\langle d\theta\wedge dI\rangle = \int_0^{2\pi}\left\langle\frac{\partial dI(\theta)}{\partial I}\wedge dI(\theta-\chi)\right\rangle(2\pi-\chi)\,\mathrm{d}\chi. \tag{A9}$$

Some final manipulations now remain to remove the operation $\partial/\partial I$ which is still angle coordinate dependent. On the one hand the 2π term in (A9) can be seen to vanish because the average of the (wedge) product is then the product of the averages, each of which vanish by (A8). On the other hand by the substitution $\chi'=2\pi-\chi$ (A9) can be rewritten

$$\int_0^{2\pi}\left\langle\frac{\partial dI(\theta-\chi')}{\partial I}\wedge dI(\theta)\right\rangle\chi'\,\mathrm{d}\chi' \tag{A10}$$

using the cyclic nature of θ. Taking half the sum of the two expressions, (A9) without the 2π, and (A10), we obtain

$$\langle \boldsymbol{d}\theta \wedge \boldsymbol{d}I\rangle = \frac{1}{2}\int_0^{2\pi}\left\langle -\frac{\boldsymbol{d}I(\theta)}{\partial I}\wedge \boldsymbol{d}I(\theta-\chi) + \frac{\partial \boldsymbol{d}I(\theta-\chi)}{\partial I}\wedge \boldsymbol{d}I(\theta)\right\rangle \chi\,\mathrm{d}\chi \tag{A11}$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}I_0}\int_0^{2\pi}\langle \boldsymbol{d}I(\theta-\chi)\wedge \boldsymbol{d}I(\theta)\rangle \chi\,\mathrm{d}\chi \tag{A12}$$

where the antisymmetry of the wedge product has been used and $\partial/\partial I$ has been extracted from the average brackets and converted to $\mathrm{d}/\mathrm{d}I_0$ as is then legitimate.

Finally then we have the required result

$$\boldsymbol{d}\langle \boldsymbol{d}\theta\rangle = \frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}I_0^2}\int_0^{2\pi}\langle \boldsymbol{d}I(\theta-\chi)\wedge \boldsymbol{d}I(\theta)\rangle \chi\,\mathrm{d}\chi. \tag{A13}$$

The generalised analysis for an N-freedom system is a straightforward matter of adding indices to the above and yields (22). For example (A4) becomes

$$\boldsymbol{d}\langle \boldsymbol{d}\theta_j\rangle = \frac{\mathrm{d}}{\mathrm{d}I_{0l}}\langle \boldsymbol{d}\theta_j \wedge \boldsymbol{d}I_l\rangle. \tag{A14}$$

Since this intermediate expression is often useful in calculations it is worth noting that it can be rewritten, by use of the constancy of the Poisson brackets for all $\boldsymbol{R}$, $\boldsymbol{d}\{I_i, I_j\} = \boldsymbol{d}\{I_i, \theta_j\} = \boldsymbol{d}\{\theta_i, \theta_j\} = 0$, in the more symmetric form

$$\mathrm{d}\langle \boldsymbol{d}\theta_j\rangle = \frac{\mathrm{d}}{\mathrm{d}I_{0j}}\langle \boldsymbol{d}\theta_l \wedge \boldsymbol{d}I_l\rangle. \tag{A15}$$

This form corresponds most closely to that obtained by Berry (1985) who uses q and p as functions of fixed phase space coordinates I, θ instead of *vice versa*. Like (A14), (A15) yields the result (22) directly.

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