Journal of Physics A: Mathematical and Theoretical

PAPER • OPEN ACCESS

Extending third quantization with commuting observables: a dissipative spin-boson model

To cite this article: Luka Medic *et al* 2024 *J. Phys. A: Math. Theor.* **57** 085301

View the article online for updates and enhancements.

OPEN ACCESS
IOP Publishing

Journal of Physics A: Mathematical and Theoretical

J. Phys. A: Math. Theor. **57** (2024) 085301 (13pp) https://doi.org/10.1088/1751-8121/ad2297

Extending third quantization with commuting observables: a dissipative spin-boson model

Luka Medic[1,2,*], **Anton Ramšak**[1,2] **and Tomaž Prosen**[2]

[1] Jožef Stefan Institute, Jamova 39, Ljubljana, Slovenia
[2] Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, Ljubljana, Slovenia

E-mail: luka.medic@ijs.si

Received 9 August 2023; revised 29 December 2023
Accepted for publication 25 January 2024
Published 8 February 2024

Abstract

We consider the spectral and initial value problem for the Lindblad–Gorini–Kossakowski–Sudarshan master equation describing an open quantum system of bosons and spins, where the bosonic parts of the Hamiltonian and Lindblad jump operators are quadratic and linear respectively, while the spins couple to bosons via mutually commuting spin operators. Needless to say, the spin degrees of freedom can be replaced by any set of finite-level quantum systems. A simple, yet non-trivial example of a single open spin-boson model is worked out in some detail.

Keywords: third quantization, quadratic Hamiltonian, spin-boson model, Lindblad master equation, decoherence

1. Introduction

Exact solutions of simple but nontrivial models describing characteristic physical phenomena are paramount for understanding statistical physics in nonequilibrium. While there is an abundance of such exactly solvable nonequilibrium models in classical statistical mechanics (see, e.g. a review paper [1]), very few explicit analytic approaches are known in the quantum realm (e.g. review papers [2, 3]).

* Author to whom any correspondence should be addressed.

A broad route to quantum nonequilibrium physics leads via the theory of open quantum systems, especially in the many-body realm. Considering a large (many-body) quantum system, one can often describe the dynamics of its (so-called central) parts within the framework of the so-called Markov approximation, neglecting the back information flow from the rest of the system (the so-called environment) to its central part. The differential equation describing the density matrix of the central system within the *Markov* approximation and the *rotating-wave* approximation [4] is the unique mathematical evolution law that preserves the hermeticity, complete positivity, and trace of the density matrix, called the Lindblad–Gorini–Kossakowski–Sudarshan [5, 6] equation, or Lindblad equation for short. We note that the many-body Lindblad equation provides a perfect mathematical platform for preparing engineered quantum states or quantum phases of matter within cold atom and ion trap setups [7, 8].

Some time ago, one of the authors developed *canonical formalism of quantization over operator spaces* for the diagonalization of the generator of the many-body Lindblad equation—the so-called Liouvillian superoperator, or completely solving the Lindbladian initial value problem, for a general *quadratic* Hamiltonian and a set of Lindblad jump operators which are *linear* in canonical creation/annihilation operators. The original proposal for fermionic systems [9, 10] was later extended to bosonic systems [11] (see also [12] for a more abstract discussion of quantization over operator spaces), and further developed by other authors [13–15]. Specifically, the latter reference extended the technique to include quadratic Hermitian jump operators, allowing the analytical treatment of nonequilibrium phase transitions [16] and crossovers between ballistic and diffusive as well as quantum and classical transport [17, 18]. Note, however, that even within the class of linear jump operators, one can discuss nontrivial critical phenomena in translationally invariant [19, 20] and so-called boundary-driven systems [3, 21, 22] (in the latter, the jump processes are confined to the boundaries of the system).

Experience has shown that the kinds of systems that can be treated efficiently under the closed system (2nd quantization) formulation can also be treated efficiently under the open system (3rd quantization) formulation. However, there is one type of system with very important applications, e.g. in quantum optics, that has somehow been left out so far, namely spin-boson systems. Some aspects of integrability and exact solvability of these systems in the closed system framework have been extensively discussed in the literature (see, e.g. [23] for the Rabi model and [24] for the Jaynes–Cummings and Dicke models). A recent case is the exactly solvable model of an electron in a driven harmonic oscillator with Rashba coupling [25, 26]. The exact solution of this model enables reliable treatment of the system coupled to a thermal bath [27]. It turns out that such a coupling of the time-dependent harmonic operator with spin degrees of freedom generates different Lindblad operators combined with bosonic and spin operators. The problem can be treated numerically [27], but an analytical approach to such problems is possible, and this is precisely the goal of the present work.

Here, we essentially provide a small extension of the third quantization method that allows us to include additional quantum degrees of freedom with a finite-dimensional Hilbert space, provided that these degrees of freedom enter the Hamiltonian and jump operators only through commuting operators. Further, a concrete example, a 1-dimensional spin-boson model, is worked out in detail. The time evolution of the expectation value of spin is derived in the limit of weakly coupled bosonic and spin degrees of freedom. Remarkably, numerical results suggest a closed-form expression for the time evolution which is universally valid even for strong couplings; however, analytical proof of this conjecture remains an open question.

2. Formal solution

This paper aims to solve the following Lindblad–Gorini–Kossakowski–Sudarshan master equation for n particles:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \hat{\mathcal{L}}\rho := -i\,[H,\rho] + \sum_{\mu}\left(2L_\mu\rho L_\mu^\dagger - \left\{L_\mu^\dagger L_\mu,\rho\right\}\right), \tag{1}$$

where H and L_μ are the Hamiltonian and Lindblad operators, respectively. ρ is the density operator describing the state of the n particles. The operators H and L_μ are given by:

$$H = \underline{a}^\dagger\cdot\mathbf{H}\underline{a} + \underline{a}\cdot\mathbf{K}\underline{a} + \underline{a}^\dagger\cdot\bar{\mathbf{K}}\underline{a}^\dagger + \underline{\sigma}\cdot\mathbf{\Omega}\underline{a} + \underline{a}^\dagger\cdot\mathbf{\Omega}^\dagger\underline{\sigma}, \tag{2}$$

$$L_\mu = \underline{l}_\mu\cdot\underline{a} + \underline{k}_\mu\cdot\underline{a}^\dagger + \underline{w}_\mu\cdot\underline{\sigma}, \tag{3}$$

where $\underline{a}$ and $\underline{a}^\dagger$ are n-dimensional vectors of canonical bosonic annihilation/creation operators, and $\underline{\sigma}$ is an n-dimensional vector of mutually commuting Hermitian operators, with a finite discrete spectrum. We also assume that $[a_j,\sigma_{j'}]=0$. $\mathbf{H}$, $\mathbf{K}$, and $\mathbf{\Omega}$ are $n\times n$ matrices with complex entries. The matrix $\mathbf{H}$ is Hermitian ($\mathbf{H}^\dagger=\mathbf{H}$), the matrix $\mathbf{K}$ is symmetric ($\mathbf{K}=\mathbf{K}^T$), and the matrix $\mathbf{\Omega}$ is arbitrary. The vectors $\underline{l}_\mu$, $\underline{k}_\mu$, and $\underline{w}_\mu$ are n-dimensional vectors of generally complex constants.

Below we follow the notation and formalism developed in [11] extending n bosonic degrees of freedom with additional n finite level quantum systems (e.g. spins where σ_j can be considered as their $z-$projections). We introduce operators $\hat{a}_{\nu,j}$ and $\hat{a}'_{\nu,j}$, sometimes referred to as superoperators, acting over a linear vector space of density matrices, where $\nu=0,1$ and $j=1,\ldots,n$:

$$\begin{aligned}
&\hat{a}_{0,j} = \hat{a}_j^{\;L}, \qquad \hat{a}'_{0,j} = \hat{a}_j^{\dagger L} - \hat{a}_j^{\dagger R},\\
&\hat{a}_{1,j} = \hat{a}_j^{\dagger R}, \qquad \hat{a}'_{1,j} = \hat{a}_j^{\;R} - \hat{a}_j^{\;L}.
\end{aligned} \tag{4}$$

We stress that $\hat{a}'_{\nu,j}$ is not related to the Hermitian adjoint $\hat{a}^\dagger_{\nu,j}$ (not even with a possible redefinition of an inner product). Here, the superscripts L and R indicate the left- and right-multiplication maps

$$\hat{b}^L|\rho\rangle = |b\rho\rangle, \quad \hat{b}^R|\rho\rangle = |\rho b\rangle, \quad \text{where} \quad b\equiv a_j, a_j^\dagger, \text{ or } \sigma_j\,. \tag{5}$$

The operators (4) satisfy the *almost-canonical* commutation relations:

$$\left[\hat{a}_{\nu,j},\hat{a}'_{\mu,k}\right] = \delta_{\nu,\mu}\delta_{j,k}, \quad \left[\hat{a}_{\nu,j},\hat{a}_{\mu,k}\right] = \left[\hat{a}'_{\nu,j},\hat{a}'_{\mu,k}\right] = 0. \tag{6}$$

In terms of these operators, we rewrite the Liouvillean as follows:

$$\hat{\mathcal{L}} = -i\hat{H}^L + i\hat{H}^R + \sum_\mu \left(2\hat{L_\mu}^L \hat{L_\mu^\dagger}^R - \hat{L_\mu^\dagger}^L \hat{L_\mu}^L - \hat{L_\mu}^R \hat{L_\mu^\dagger}^R\right) \tag{7}$$

$$\begin{aligned}
&= -i\,\underline{\hat{a}}_0' \cdot \mathbf{H}\underline{\hat{a}}_0 + i\,\underline{\hat{a}}_1' \cdot \bar{\mathbf{H}}\underline{\hat{a}}_1 + i\,\underline{\hat{a}}_1' \cdot \mathbf{K}\left(2\underline{\hat{a}}_0 + \underline{\hat{a}}_1'\right) - i\,\underline{\hat{a}}_0' \cdot \bar{\mathbf{K}}\left(2\underline{\hat{a}}_1 + \underline{\hat{a}}_0'\right)\\
&\quad - i\,\underline{\hat{a}}_0' \cdot \boldsymbol{\Omega}^\dagger \underline{\sigma}^L + i\,\underline{\sigma}^R \cdot \boldsymbol{\Omega}\underline{\hat{a}}_1' - i\left(\underline{\sigma}^L - \underline{\sigma}^R\right) \cdot \boldsymbol{\Omega}\underline{\hat{a}}_0 + i\,\underline{\hat{a}}_1 \cdot \boldsymbol{\Omega}^\dagger\left(\underline{\sigma}^R - \underline{\sigma}^L\right)\\
&\quad + \underline{\hat{a}}_0' \cdot \left(\mathbf{N} - \bar{\mathbf{M}}\right)\underline{\hat{a}}_0 + \underline{\hat{a}}_1' \cdot \left(\bar{\mathbf{N}} - \mathbf{M}\right)\underline{\hat{a}}_1 + \underline{\hat{a}}_0' \cdot \left(\bar{\mathbf{L}}^T - \bar{\mathbf{L}}\right)\underline{\hat{a}}_1 + \underline{\hat{a}}_1' \cdot \left(\mathbf{L}^T - \mathbf{L}\right)\underline{\hat{a}}_0\\
&\quad - \underline{\hat{a}}_0' \cdot \bar{\mathbf{L}}\underline{\hat{a}}_0 - \underline{\hat{a}}_1' \cdot \mathbf{L}\underline{\hat{a}}_1 + 2\underline{\hat{a}}_0' \cdot \mathbf{N}\underline{\hat{a}}_1' + 2\underline{\sigma}^L \cdot \mathbf{W}\underline{\sigma}^R - \underline{\sigma}^L \cdot \mathbf{W}\underline{\sigma}^L - \underline{\sigma}^R \cdot \mathbf{W}\underline{\sigma}^R\\
&\quad + \underline{\hat{a}}_0 \cdot \left(\bar{\mathbf{F}} - \mathbf{E}\right)\left(\underline{\sigma}^L - \underline{\sigma}^R\right) + \underline{\hat{a}}_1 \cdot \left(\mathbf{F} - \bar{\mathbf{E}}\right)\left(\underline{\sigma}^R - \underline{\sigma}^L\right) - \underline{\hat{a}}_0' \cdot \bar{\mathbf{E}}\underline{\sigma}^L - \underline{\hat{a}}_1' \cdot \mathbf{E}\underline{\sigma}^R\\
&\quad - \underline{\hat{a}}_0' \cdot \mathbf{F}\left(\underline{\sigma}^L - 2\underline{\sigma}^R\right) - \underline{\hat{a}}_1' \cdot \bar{\mathbf{F}}\left(\underline{\sigma}^R - 2\underline{\sigma}^L\right),
\end{aligned} \tag{8}$$

where we define $n \times n$ matrices:

$$\begin{aligned}
&\mathbf{M} := \sum_\mu \underline{l}_\mu \otimes \bar{\underline{l}}_\mu = \mathbf{M}^\dagger, \quad \mathbf{N} := \sum_\mu \underline{k}_\mu \otimes \bar{\underline{k}}_\mu = \mathbf{N}^\dagger, \quad \mathbf{L} := \sum_\mu \underline{l}_\mu \otimes \bar{\underline{k}}_\mu,\\
&\mathbf{W} := \sum_\mu \underline{w}_\mu \otimes \bar{\underline{w}}_\mu = \mathbf{W}^\dagger, \quad \mathbf{E} := \sum_\mu \underline{l}_\mu \otimes \bar{\underline{w}}_\mu, \quad \mathbf{F} := \sum_\mu \underline{k}_\mu \otimes \bar{\underline{w}}_\mu.
\end{aligned} \tag{9}$$

To get rid of terms linear in $\underline{\hat{a}}$ and $\underline{\hat{a}}^\dagger$ we use a simple affine map

$$\underline{\hat{a}} \to \underline{\hat{a}} + \underline{s}, \quad \underline{\hat{a}}' \to \underline{\hat{a}}' + \underline{s}' \tag{10}$$

where $\underline{s}, \underline{s}'$ are constant n-vectors to be determined. Let us define $\underline{\hat{b}} = \left(\underline{\hat{a}}, \underline{\hat{a}}'\right)^T = \left(\underline{\hat{a}}_0, \underline{\hat{a}}_1, \underline{\hat{a}}_0', \underline{\hat{a}}_1'\right)^T$, $\underline{d} = \left(\underline{s}, \underline{s}'\right)^T = \left(\underline{s}_0, \underline{s}_1, \underline{s}_0', \underline{s}_1'\right)^T$ and $\underline{\tilde{\sigma}} = \left(\underline{\sigma}^L, \underline{\sigma}^R\right)^T$. The Liouvillean can be compactly written in a matrix form

$$\hat{\mathcal{L}} = \left(\underline{\hat{b}} - \underline{d}\right) \cdot \mathbf{S}\left(\underline{\hat{b}} - \underline{d}\right) + \mathrm{tr}\left(\mathbf{X}\right) - \Gamma \tag{11}$$

where

$$\mathbf{S} := \begin{pmatrix} 0 & -\mathbf{X} \\ -\mathbf{X}^T & \mathbf{Y} \end{pmatrix} = \mathbf{S}^T, \tag{12}$$

$$\mathbf{X} := \frac{1}{2}\begin{pmatrix} i\bar{\mathbf{H}} - \bar{\mathbf{N}} + \mathbf{M} & -2i\mathbf{K} - \mathbf{L} + \mathbf{L}^T \\ 2i\bar{\mathbf{K}} - \bar{\mathbf{L}} + \bar{\mathbf{L}}^T & -i\mathbf{H} - \mathbf{N} + \bar{\mathbf{M}} \end{pmatrix}, \tag{13}$$

$$\mathbf{Y} := \frac{1}{2}\begin{pmatrix} -2i\bar{\mathbf{K}} - \bar{\mathbf{L}} - \bar{\mathbf{L}}^T & 2\mathbf{N} \\ 2\mathbf{N}^T & 2i\mathbf{K} - \mathbf{L} - \mathbf{L}^T \end{pmatrix} = \mathbf{Y}^T. \tag{14}$$

In order to eliminate the linear terms, we must fulfill the following condition for the vector $\underline{d}$:

$$2\mathbf{S}\underline{d} = -\mathbf{G}\underline{\tilde{\sigma}}, \quad \text{where} \quad \mathbf{G} := \begin{pmatrix} \bar{\mathbf{F}} - \mathbf{E} - i\boldsymbol{\Omega}^T & -\bar{\mathbf{F}} + \mathbf{E} + i\boldsymbol{\Omega}^T \\ -\mathbf{F} + \bar{\mathbf{E}} - i\bar{\boldsymbol{\Omega}}^T & \mathbf{F} - \bar{\mathbf{E}} + i\bar{\boldsymbol{\Omega}}^T \\ -\mathbf{F} - \bar{\mathbf{E}} - i\bar{\boldsymbol{\Omega}}^T & 2\mathbf{F} \\ 2\bar{\mathbf{F}} & -\bar{\mathbf{F}} - \mathbf{E} + i\boldsymbol{\Omega}^T \end{pmatrix}. \tag{15}$$

To obtain $\underline{d}$ one has to solve the above linear system. If $\mathbf{S}$ is nonsingular, then we can express it explicitly as $\underline{d} = -\frac{1}{2}\mathbf{S}^{-1}\mathbf{G}\underline{\tilde{\sigma}}$. Further, we can calculate the last term in equation (11), which is related to the asymptotic rate of decoherence in our system:

$$\begin{aligned}\Gamma &= \underline{d}\cdot\mathbf{S}\underline{d} - \underline{\sigma}^L\cdot\mathbf{W}\left(\underline{\sigma}^R - \underline{\sigma}^L\right) - \left(\underline{\sigma}^L - \underline{\sigma}^R\right)\cdot\mathbf{W}\underline{\sigma}^R \\ &= \frac{1}{4}\underline{\tilde{\sigma}}\cdot\mathbf{G}^T\mathbf{S}^{-1}\mathbf{G}\underline{\tilde{\sigma}} - \left(\underline{\sigma}^L - \underline{\sigma}^R\right)\cdot\left(\mathbf{W}\underline{\sigma}^R - \bar{\mathbf{W}}\underline{\sigma}^L\right).\end{aligned} \tag{16}$$

The term $\mathrm{tr}(\mathbf{X})$ in equation (11) stems from reordering of operators $\hat{a}_{\nu,j}$ and $\hat{a}'_{\nu,j}$. Note that the assumed nonsingularity of the matrix $\mathbf{S}$ is also related to the uniqueness of the nonequilibrium steady state [10]. Here, we focus on this regime and do not delve into the possible scenario of a singular matrix $\mathbf{S}$ which we leave for a future study.

By literally following the procedure introduced in [11], we successfully transform the Liouvillean into its final form:

$$\hat{\mathcal{L}} = -2\sum_{r=1}^{2n}\beta_r\hat{\zeta}'_r\hat{\zeta}_r - \Gamma \tag{17}$$

where

$$\underline{\hat{\zeta}} = \mathbf{P}^T\left(\left(\underline{\hat{a}} - \underline{s}\right) - \mathbf{Z}\left(\underline{\hat{a}}' - \underline{s}'\right)\right), \qquad \underline{\hat{\zeta}}' = \mathbf{P}^{-1}\left(\underline{\hat{a}}' - \underline{s}'\right). \tag{18}$$

The rapidities β_r and the matrix $\mathbf{P}$ are obtained from the diagonalization of the matrix $\mathbf{X}$, namely:

$$\mathbf{X} = \mathbf{P}\boldsymbol{\Delta}\mathbf{P}^{-1}, \qquad \boldsymbol{\Delta} = \mathrm{diag}\left\{\beta_1,\ldots,\beta_{2n}\right\}, \tag{19}$$

and the matrix $\mathbf{Z}$ is a solution of the continuous Lyapunov equation

$$\mathbf{X}^T\mathbf{Z} + \mathbf{Z}\mathbf{X} = \mathbf{Y}. \tag{20}$$

If all the rapidities have a non-negative real part, $\forall\,\mathrm{Re}\,\beta_r \geqslant 0$, one can obtain nonequilibrium steady states $|\mathrm{NESS}^{\underline{s}}\rangle$ and all the corresponding decay modes $|\Theta^{\underline{s}}_{\underline{m}}\rangle$ ($\underline{m} \in \mathbb{Z}^{2n}_+$ is a multi-index) of the Liouvillean for each combination of joint eigenvalues $(\underline{s}^L, \underline{s}^R) \equiv \underline{s}$ of the hermitian operators $(\underline{\sigma}^L, \underline{\sigma}^R)$, such that

$$\left(\hat{\mathcal{L}} + \Gamma\right)|\mathrm{NESS}^{\underline{s}}\rangle = 0, \tag{21}$$

or more precisely

$$\hat{\zeta}_r|\mathrm{NESS}^{\underline{s}}\rangle = 0, \tag{22}$$

and

$$\hat{\mathcal{L}}|\Theta^{\underline{s}}_{\underline{m}}\rangle = \left(\lambda_{\underline{m}} - \Gamma\right)|\Theta^{\underline{s}}_{\underline{m}}\rangle, \tag{23}$$

where

$$|\Theta_{\underline{m}}^{\underline{s}}\rangle = \prod_r \frac{\left(\hat{\zeta}_r'\right)^{m_r}}{\sqrt{m_r!}} |\mathrm{NESS}^{\underline{s}}\rangle, \qquad \lambda_{\underline{m}} = -2\sum_r m_r\beta_r. \tag{24}$$

Note that $|\Theta_{\underline{m}}^{\underline{s}}\rangle$ generally do not constitute an orthogonal basis.

Given an initial condition $|\rho_0\rangle = \sum_{\underline{s}} |\tilde{\rho}_0^{\underline{s}}\rangle \otimes |s^L\rangle\langle s^R|$, with $|\tilde{\rho}_0^{\underline{s}}\rangle$ being the bosonic part, the solution to the time-dependent problem for the density matrix is given by:

$$|\rho(t)\rangle = \sum_{\underline{s},\underline{m}} c_{\underline{m}}^{\underline{s}} e^{\left(\lambda_{\underline{m}} - \Gamma\right)t} |\Theta_{\underline{m}}^{\underline{s}}\rangle \tag{25}$$

where coefficients $c_{\underline{m}}^{\underline{s}}$ solve the set of linear systems for each joint eigenvalue $\underline{s}$:

$$|\tilde{\rho}_0^{\underline{s}}\rangle = \sum_{\underline{m}} c_{\underline{m}}^{\underline{s}} |\tilde{\Theta}_{\underline{m}}^{\underline{s}}\rangle, \tag{26}$$

where $|\tilde{\Theta}_{\underline{m}}^{\underline{s}}\rangle$ is the bosonic part of $|\Theta_{\underline{m}}^{\underline{s}}\rangle$, i.e. $|\Theta_{\underline{m}}^{\underline{s}}\rangle = |\tilde{\Theta}_{\underline{m}}^{\underline{s}}\rangle \otimes |s^L\rangle\langle s^R|$.

3. Example

Let us consider a 1-dimensional system ($n = 1$) with two Lindblad operators, $L_1 = a + z_1\sigma^z$ and $L_2 = z_2\sigma^z$, and a Hamiltonian $H = \omega a^\dagger a + \alpha a\sigma^z + \bar{\alpha} a^\dagger \sigma^z$, where $\omega \in \mathbb{R}$ and $z_1, z_2, \alpha \in \mathbb{C}$. Here, σ^z is the Pauli matrix for the spin degree of freedom, and terms with α represent, for example, the coupling of the spin to an inhomogeneous magnetic field for the real part of α and the Rashba coupling for the imaginary part [25–27].

Thus, in accordance with the previous section, we have $\mathbf{H} = \omega$, $\mathbf{\Omega} = \alpha$, $\mathbf{M} = 1$, $\mathbf{W} = |z_1|^2 + |z_2|^2$, $\mathbf{E} = \bar{z}_1$, and $\mathbf{K} = \mathbf{N} = \mathbf{L} = \mathbf{F} = 0$. Subsequently, employing our established formalism, we proceed to compute:

$$\mathbf{X} = \frac{1}{2}\begin{pmatrix} 1 + i\omega & 0 \\ 0 & 1 - i\omega \end{pmatrix}, \quad \mathbf{Y} = 0, \quad \mathbf{S} = -\begin{pmatrix} 0 & \mathbf{X} \\ \mathbf{X} & 0 \end{pmatrix}, \tag{27}$$
$$\mathbf{G}^T = -\begin{pmatrix} -\bar{z}_1 - i\alpha & z_1 - i\bar{\alpha} & -z_1 - i\bar{\alpha} & 0 \\ \bar{z}_1 + i\alpha & -z_1 + i\bar{\alpha} & 0 & -\bar{z}_1 + i\alpha \end{pmatrix}.$$

As $\omega \in \mathbb{R}$, the matrices $\mathbf{X}$ and $\mathbf{S}$ are non-singular, implying a non-degenerate NESS.

Given that $\mathbf{X} = \mathbf{\Delta}$ is already diagonal ($\mathbf{P} = I$), the rapidities are $\beta_{1,2} = (1 \pm \omega)/2$. Moreover, since $\mathbf{Y} = 0$, we also find that $\mathbf{Z} = 0$, and consequently, from equation (18), we conclude that $(\hat{\underline{\zeta}}, \hat{\underline{\zeta}}')^T = \hat{\underline{b}} - \underline{d}$. Solving the linear system (15) for $\underline{d}$ we get

$$\underline{d}^T = \left(-\eta_+\sigma^L, -\bar{\eta}_+\sigma^R, \bar{\eta}_-\left(\sigma^R - \sigma^L\right), \eta_-\left(\sigma^L - \sigma^R\right)\right) \tag{28}$$

where $\eta_\pm = (z_1 \pm i\bar{\alpha})/(1 \pm i\omega)$, and for brevity we omitted the superscript z in σ^z. Finally, for the asymptotic decoherence, we obtain

$$\Gamma = \left(\sigma^R - \sigma^L\right)\left(\gamma\sigma^L - \bar{\gamma}\sigma^R\right), \quad \gamma = \eta_+\bar{\eta}_-(1 + i\omega) - |z_1|^2 - |z_2|^2. \tag{29}$$

For each sector $\underline{s} = (s^L, s^R) \in \{-1, 1\}^2$, we can compute the NESS which satisfies the constraint (22):

$$|\mathrm{NESS}^{\underline{s}}\rangle = |\alpha^L\rangle\langle\alpha^R| \otimes |s^L\rangle\langle s^R|, \tag{30}$$

where $|\alpha^{L,R}\rangle$ with $\alpha^{L,R} = -\eta_+ s^{L,R}$ are coherent states, i.e. $|\alpha\rangle = e^{-|\alpha|^2/2}\sum_l \frac{\alpha^l}{\sqrt{l!}}|l\rangle$, and $|s^L\rangle\langle s^R|$ represents the spin degree of freedom. The form of $|\mathrm{NESS}^{\underline{s}}\rangle$ indicates the generation of entanglement between the bosonic and spin degrees of freedom (as further discussed in section 3.1).

3.1. Initial condition and the expectation value of spin

In this section, we present the calculation for the time evolution of the expectation value of spin, $\langle\sigma^\mu(t)\rangle = \mathrm{tr}\,(\sigma^\mu\rho(t))$ $(\mu = x, y, z)$, given an initial condition.

Utilizing equations (26) and (30), the determination of coefficients $c^{\underline{s}}_{\underline{m}}$ involves solving the system of linear equations:

$$|\tilde{\rho}^{\underline{s}}_0\rangle = \sum_{\underline{m}} c^{\underline{s}}_{\underline{m}} \frac{\left(\hat{\zeta}_1'\right)^{m_1}}{\sqrt{m_1!}} \frac{\left(\hat{\zeta}_2'\right)^{m_2}}{\sqrt{m_2!}} |\alpha^L\rangle\langle\alpha^R|, \quad \forall \underline{s}, \tag{31}$$

where $\hat{\zeta}_1' = \hat{a}^{\dagger L} - \hat{a}^{\dagger R} + \bar{\eta}_-(s^L - s^R)$ and $\hat{\zeta}_2' = \hat{a}^R - \hat{a}^L + \eta_-(s^R - s^L)$. We examine a scenario where the system is initially in the product state $|\rho_0\rangle = |0\rangle\langle 0| \otimes \frac{1}{2}(I + \sigma_0\sigma^x)$, i.e. with the bosonic part in the ground state and spin pointing in the x direction where $\sigma_0 = \langle\sigma^x(t=0)\rangle$. Although the above linear system for $c^{\underline{s}}_{\underline{m}}$ can be solved numerically for arbitrary values of $\eta_\pm$ and $\underline{m}$, we will proceed with an analytical treatment for the slowest decay modes, i.e. $m_{1,2} \leqslant 1$, in the regime $|\eta_\pm| \ll 1$.

To obtain the coefficients $c^{\underline{s}}_{\underline{m}}$, we perform a Taylor expansion of the expression (31) up to quadratic terms in $\eta_\pm$. This expansion results in a 4×4 system of equations to solve for each $\underline{s} = (s^L, s^R)$:

$$\begin{pmatrix} \frac{1}{2}\sigma_0^{\tilde{s}} \\ 0 \\ 0 \\ 0 \end{pmatrix} = e^{-|\eta|^2} \begin{pmatrix} 1 & \eta' s^R & \bar{\eta}' s^L & -1 + |\eta'|^2 s^L s^R \\ -\bar{\eta} s^R & 1 - \bar{\eta}\eta' & -\bar{\eta}\bar{\eta}' s^L s^R & \bar{\eta} s^R + \bar{\eta}' s^L \\ -\eta s^L & -\eta\eta' s^L s^R & 1 - \eta\bar{\eta}' & \eta s^L + \eta' s^R \\ |\eta|^2 s^L s^R & -\eta s^L & -\bar{\eta} s^R & 1 - \tilde{\eta}^2 \end{pmatrix} \begin{pmatrix} c^{\underline{s}}_{(0,0)} \\ c^{\underline{s}}_{(0,1)} \\ c^{\underline{s}}_{(1,0)} \\ c^{\underline{s}}_{(1,1)} \end{pmatrix}, \tag{32}$$

where $\tilde{s} = (1 - s^L s^R)/2$, $\eta = \eta_+$, $\eta' = \eta_- - (\eta_- - \eta_+) s^L s^R$ and $\tilde{\eta}^2 = 2\mathrm{Re}(\eta\bar{\eta}') + |\eta|^2 s^L s^R$. The solution for the coefficients is given by:

$$\begin{pmatrix} c^{\underline{s}}_{(0,0)} \\ c^{\underline{s}}_{(0,1)} \\ c^{\underline{s}}_{(1,0)} \\ c^{\underline{s}}_{(1,1)} \end{pmatrix} = \frac{\mathcal{M}\sigma_0^{\tilde{s}}}{2} \begin{pmatrix} 1 + |\eta|^2 s^L s^R + 2\mathrm{Re}\,(\eta\bar{\eta}')\left(-1 + |\eta|^2 s^L s^R\right) + 4|\eta\bar{\eta}'|^2 \\ \bar{\eta}\left(1 - \eta\bar{\eta}' - (\eta\bar{\eta}')^2 + |\eta\bar{\eta}'|^2\right) s^R \\ \eta\left(1 - \bar{\eta}\eta' - (\bar{\eta}\eta')^2 + |\eta\bar{\eta}'|^2\right) s^L \\ |\eta|^2\left(1 + 2\mathrm{Re}\,(\eta\bar{\eta}')\right) s^L s^R \end{pmatrix}$$

$$\approx \frac{\sigma_0^{\tilde{s}}}{2} \begin{pmatrix} 1 - \left(2\mathrm{Re}\,(\eta_+\bar{\eta}_-) - |\eta_+|^2\right)\left(1 - s^L s^R\right) \\ \bar{\eta}_+ s^R \\ \eta_+ s^L \\ |\eta_+|^2 s^L s^R \end{pmatrix}, \tag{33}$$

where $\mathcal{M} = e^{|\eta|^2}/(1 + |\eta\bar{\eta}'|^2(3 + 2\mathrm{Re}(\eta\bar{\eta}')))$.

After computing the coefficients, it becomes evident that the dynamics produce entanglement between the bosonic and spin degrees of freedom, as suggested earlier.

One approach to observe this is by explicitly expressing the asymptotic behavior, $t\to\infty$: $|\rho(t)\rangle \asymp e^{-\Gamma t}\sum_{\underline{s}} c^{\underline{s}}_{(0,0)}|\mathrm{NESS}^{\underline{s}}\rangle \approx e^{-\Gamma t}\left(\frac{1+\tilde{\sigma}_0}{2}|a_+\rangle\langle a_+| + \frac{1-\tilde{\sigma}_0}{2}|a_-\rangle\langle a_-|\right)$, where $\tilde{\sigma}_0 = \sigma_0\left(1-4\mathrm{Re}(\eta_+\bar{\eta}_-)+2|\eta_+|^2\right)$ and $|a_\pm\rangle = \frac{1}{\sqrt{2}}\left(|-\eta_+\rangle\otimes|\uparrow\rangle \pm |+\eta_+\rangle\otimes|\downarrow\rangle\right)$, which shows that the state is entangled. However, note that the coherent states $|\pm\eta_+\rangle$ are not orthogonal.

To determine the time evolution of $\langle\sigma^x(t)\rangle$, it is necessary to compute the expectation values of σ^x for the slowest decay modes. Following the notation in [11], i.e. $(A|\rho\rangle = \mathrm{tr}(A\rho)$, we have:

$$
\begin{aligned}
(\sigma^x|\Theta^{\underline{s}}_{(0,0)}\rangle &= \delta_{(-s^L,s^R)}e^{-2|\eta|^2},\\
(\sigma^x|\Theta^{\underline{s}}_{(0,1)}\rangle &= \delta_{(-s^L,s^R)}e^{-2|\eta|^2}(\eta+\eta')s^R,\\
(\sigma^x|\Theta^{\underline{s}}_{(1,0)}\rangle &= \delta_{(-s^L,s^R)}e^{-2|\eta|^2}(\bar{\eta}+\bar{\eta}')s^L,
\end{aligned}
\tag{34}
$$

where δ is the Kronecker delta. The final result in the limit $|\eta_\pm|\to 0$ for the slowest decay modes $(\lambda_{(0,0)}=0, \lambda_{(0,1)}=\bar{\lambda}_{(1,0)}=-1+i\omega)$ is then:

$$
\langle\sigma^x(t)\rangle = \sum_{\underline{s},\underline{m}} c^{\underline{s}}_{\underline{m}} e^{(\lambda_{\underline{m}}-\Gamma)t}(\sigma^x|\Theta^{\underline{s}}_{\underline{m}}\rangle \approx \sigma_0\left(1-4\mathrm{Re}\left(\eta_+\bar{\eta}_-(1-e^{-(1+i\omega)t})\right)\right)e^{-\Gamma t},
\tag{35}
$$

where equation (29) evaluates to

$$
\begin{aligned}
\Gamma &= -4\mathrm{Re}\,(\gamma)\\
&= 4\left(\omega\,\mathrm{Im}\,(\eta_+\bar{\eta}_-) - \mathrm{Re}\,(\eta_+\bar{\eta}_-) + |z_1|^2 + |z_2|^2\right)\\
&\geqslant 4\left(\omega|z_1| - |\alpha|\right)^2/\left(1+\omega^2\right) + 4|z_2|^2 \geqslant 0
\end{aligned}
\tag{36}
$$

where the first inequality is a consequence of the Cauchy-Schwarz inequality. The real and imaginary parts of $\eta_+\bar{\eta}_-$ equal to

$$
\mathrm{Re}\,(\eta_+\bar{\eta}_-) = \left(\left(1-\omega^2\right)\left(|z_1|^2-|\alpha|^2\right) + 4\omega\,\mathrm{Re}\,(z_1\alpha)\right)/\left(1+\omega^2\right)^2,
\tag{37}
$$

$$
\mathrm{Im}\,(\eta_+\bar{\eta}_-) = \left(-2\omega\left(|z_1|^2-|\alpha|^2\right) + 2\left(1-\omega^2\right)\mathrm{Re}\,(z_1\alpha)\right)/\left(1+\omega^2\right)^2.
\tag{38}
$$

Taking the logarithm of expression (35) and approximating it for $|\eta_\pm|\ll 1$, yields

$$
\log\left(\langle\sigma^x(t)\rangle/\sigma_0\right) = -4\mathrm{Re}\left(\eta_+\bar{\eta}_-\left(1-e^{-(1+i\omega)t}\right)\right) - \Gamma t.
\tag{39}
$$

Surprisingly, this expression holds for every $\eta_\pm$, regardless of their magnitude (see section 3.3), however, comprehending this generalization surpasses the current treatment and remains an open question. Also, it is essential to emphasize that equations (35) and (39) are valid only for the specific initial condition $|\tilde{\rho}^{\underline{s}}_0\rangle \sim |0\rangle\langle 0|$. For different initial conditions, such as $|\tilde{\rho}^{\underline{s}}_0\rangle \sim |n\rangle\langle n|$ with $n\neq 0$, one must include higher orders of $\underline{m}$ in the computation of the coefficients $c^{\underline{s}}_{\underline{m}}$ to obtain the correct result.

Introducing an argument ϕ of the complex number $\eta_+\bar{\eta}_-$, i.e. $\eta_+\bar{\eta}_- = |\eta_+\bar{\eta}_-|e^{i\phi}$, allows us to rewrite the resulting expression (39) as

$$
\begin{aligned}
\log\left(\langle\sigma^x(t)\rangle/\sigma_0\right) &= -4|\eta_+\bar{\eta}_-|\left(\cos(\phi) - e^{-t}\cos(\omega t-\phi)\right) - \Gamma t\\
&= -4\mathrm{Re}\,(\eta_+\bar{\eta}_-)\left(1-\frac{\cos(\omega t-\phi)}{\cos(\phi)}e^{-t}\right) - \Gamma t.
\end{aligned}
\tag{40}
$$

Clearly, for large t, the solution exhibits exponential decay with an asymptotic decay rate Γ.

Regarding the behavior of the quantities present in the final expression, namely $\mathrm{Re}(\eta_+\bar{\eta}_-)$, ϕ, and Γ, with respect to ω, refer to appendix.

3.2. Limiting cases

Here, we explore several limiting cases of the expression (40) with respect to t and ω.

As suggested above, in the limit $t\to\infty$, the expectation value of spin decays exponentially with time:

$$\log\left(\langle\sigma^x(t)\rangle/\sigma_0\right)\asymp -4\mathrm{Re}\left(\eta_+\bar{\eta}_-\right)-\Gamma t. \tag{41}$$

On the other hand, in the limit $t\to 0$, equation (39) reduces to

$$\log\left(\langle\sigma^x(t)\rangle/\sigma_0\right)=-4\left(|z_1|^2+|z_2|^2\right)t+2\left(|z_1|^2-|\alpha|^2\right)t^2 \tag{42}$$

where we retained terms up to the second order in t. Interestingly, up to the quadratic order in t, the time evolution of the expectation value of spin does not depend on ω. Moreover, by restricting ourselves solely to the linear order, the time evolution becomes independent of α as well, effectively governed by an effective Liouvillean $\hat{\mathcal{L}}_{\text{eff}}$, where $H_{\text{eff}}=0$ (or $H_{\text{eff}}=\omega a^\dagger a$) and $L_{\text{eff}}=\sqrt{|z_1|^2+|z_2|^2}\sigma^z$.

Next, we investigate the limits in ω. To obtain simpler results, we also consider α to be proportional to ω.

In the limit $\omega\to 0$ we get:

$$\eta_+\bar{\eta}_-\to|z_1|^2-|\alpha|^2+2i\mathrm{Re}\left(z_1\alpha\right)\xrightarrow{\alpha\propto\omega}|z_1|^2, \tag{43}$$

$$\phi\to\arctan\left(\frac{2\mathrm{Re}\left(z_1\alpha\right)}{|z_1|^2-|\alpha|^2}\right)\xrightarrow{\alpha\propto\omega}0, \tag{44}$$

$$\Gamma\to 4\left(|\alpha|^2+|z_2|^2\right)\xrightarrow{\alpha\propto\omega}4|z_2|^2, \tag{45}$$

and thus

$$\log\left(\langle\sigma^x(t)\rangle/\sigma_0\right)\xrightarrow{\alpha\propto\omega}-4|z_1|^2\left(1-e^{-t}\right)-4|z_2|^2t. \tag{46}$$

Clearly, the last limit $\alpha\propto\omega\to 0$ implies $H\to 0$. Taking the limit $t\to 0$ recovers equation (42).

If we take the limit $\omega\to\infty$ while considering $\alpha\propto\omega$, we observe:

$$\eta_+\bar{\eta}_-\xrightarrow{\alpha\propto\omega}\left(|\alpha|/\omega\right)^2, \tag{47}$$

$$\phi\xrightarrow{\alpha\propto\omega}0, \tag{48}$$

$$\Gamma\xrightarrow{\alpha\propto\omega}4\Big(|z_1|^2+|z_2|^2+\left(|\alpha|/\omega\right)^2-2\mathrm{Re}\left(z_1\alpha/\omega\right)\Big), \tag{49}$$

and

$$\log\left(\langle\sigma^x(t)\rangle/\sigma_0\right)\xrightarrow{\alpha\propto\omega}-4\frac{|\alpha|^2}{\omega^2}\left(1-e^{-t}\cos\left(\omega t\right)\right)-\Gamma t. \tag{50}$$

If α does not scale with ω, we simply obtain $\log\left(\langle\sigma^x(t)\rangle/\sigma_0\right)=-4(|z_1|^2+|z_2|^2)t$, which corresponds to the linear term in equation (42). Notably, this expression remains valid even for large t.

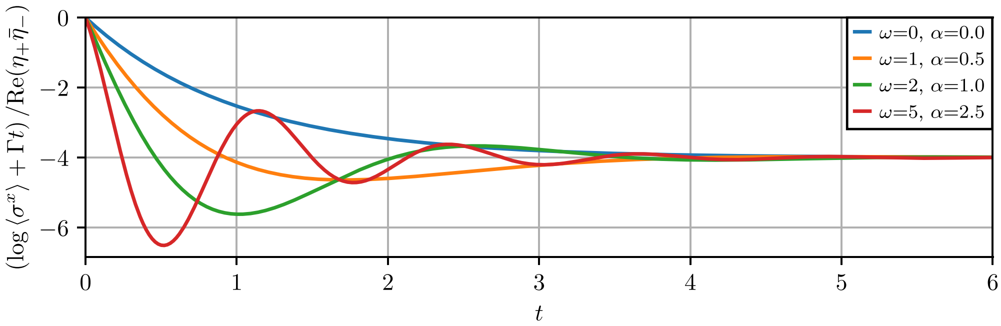

Figure 1. Time evolutions of the expectation value $\langle\sigma^x(t)\rangle$ for different values of ω and $\alpha = \omega/2$ ($z_1 = 1$, $z_2 = 0$). The presented solutions were obtained using analytical expression (39) and are graphically indistinguishable from numerical results obtained via *QuTiP*.

3.3. Comparison with the numerical results

To validate our findings, we utilized the *QuTiP* Python library [28, 29], which allows for the computation of the time evolution of the density matrix following the Lindblad master equation.

As the parameter z_2 contributes trivially to decoherence, we disregarded it in our calculations by setting it to 0. We simplify the presentation by displaying outcomes for various ω ranging from 0 to 5, with $\alpha = \omega/2$ and $z_1 = 1$, which is depicted in figure 1. Note that in the presented solutions we subtracted the asymptotic decoherence $-\Gamma t$ which dominates for large t.

The numerical computations (*QuTiP*) for the time dependence of $\langle\sigma^x(t)\rangle$ align with the conjectured analytical expression (39) within numerical precision, $\mathcal{O}(10^{-6})$. This consistency holds even for parameter choices far from the limit $\eta_\pm \to 0$, such as $z_1 = 1$ and $\omega = 0$.

4. Conclusion

In conclusion, our study delved into the dynamics of a quantum system that exhibits quadratic behavior in bosonic degrees of freedom and linearity in additional commuting (classical) degrees of freedom, employing the formalism of third quantization. In particular, we succeeded in diagonalizing the Liouville superoperator and obtaining nonequilibrium steady states and the corresponding decay modes for various combinations of eigenvalues of the mutually commuting Hermitian operators.

A special case involves a harmonic oscillator coupled through Rashba interaction to the spin degree of freedom, which we solved exactly in the limit $\eta_\pm \to 0$. Our analysis reveals a transient process in the expectation value of spin, ultimately converging to an exponential decay with a rate Γ, highlighting the system's long-term relaxation behavior. Additionally, we explored various limiting scenarios, recovering well-known time dependency for the Lindbladian $L \sim \sigma^z$ in two distinct limits: $t \to 0$ and $\omega \to \infty$. Although we have numerically verified our solutions, a complete analytic proof of generality remains an open question.

The results presented provide valuable contributions to understanding the dynamics of open quantum systems and provide a basis for further studies of related physical phenomena in which spin degrees of freedom are coupled with bosonic ones. For example, these findings could be significant for understanding decoherence in spin systems coupled to a bosonic thermal bath and confined in quadratic potential traps [27]. Such studies could provide important insights into the behavior of quantum systems under realistic experimental conditions and pave the way for advances in quantum technology and quantum information processing.

Data availability statement

No new data were created or analysed in this study.

Acknowledgments

L M and A R acknowledge support from the Slovenian Research and Innovation Agency (ARIS) under Contract No. P1-0044; A R was also supported by Grant J2-2514. T P was supported by Programme P1-0402 and Grants N1-0219 and N1-0233 (ARIS).

Appendix. Dependence of $\mathrm{Re}(\eta_+\bar{\eta}_-)$, ϕ, and Γ on ω

Figure A1 illustrates the behavior of $\mathrm{Re}(\eta_+\bar{\eta}_-)$, ϕ, and Γ as functions of ω. We set $z_1 = 1$ and follow a similar approach as in section 3.2, assuming α to be proportional to ω, specifically $\alpha = \omega\xi$ ($\xi \in \mathbb{C}$). For simplicity, we omit z_2 as it only trivially affects Γ. Note that when $\xi = 1$, we obtain $\Gamma = 0$, or more generally, $\Gamma = 0$ holds for $\alpha = \omega\bar{z}_1$ and $z_2 = 0$.

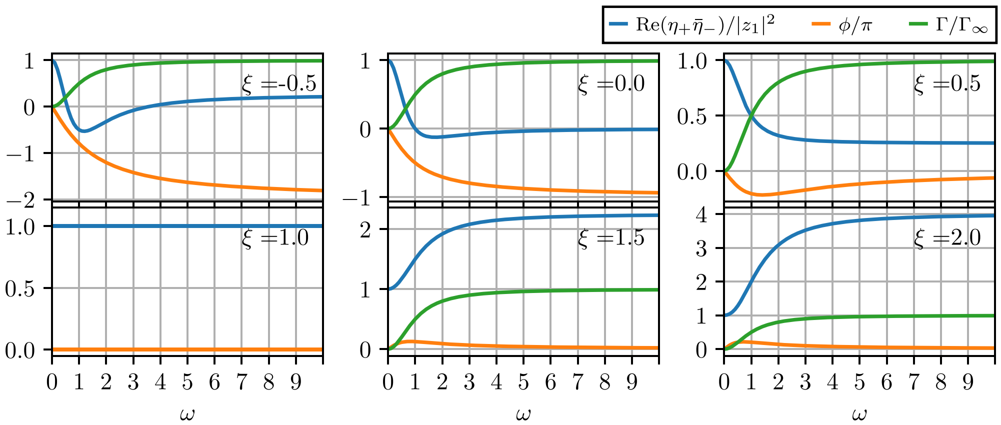

Figure A1. Various examples of the dependence of $\mathrm{Re}(\eta_+\bar{\eta}_-)$, ϕ, and Γ on ω for different choices of ξ ($z_1 = 1$, $z_2 = 0$, $\alpha = \omega\xi$).

ORCID iD

Luka Medic https://orcid.org/0009-0008-8177-7685

References

[1] Schütz G M 2001 1 - Exactly solvable models for many-body systems far from equilibrium *Phase Transitions and Critical Phenomena* (Academic)
[2] Prosen T 2015 Matrix product solutions of boundary driven quantum chains *J. Phys. A: Math. Theor.* **48** 373001
[3] Landi G T, Poletti D and Schaller G 2022 Nonequilibrium boundary-driven quantum systems: models, methods and properties *Rev. Mod. Phys.* **94** 045006
[4] Breuer H-P and Petruccione F 2007 *The Theory of Open Quantum Systems* (Oxford University Press)
[5] Lindblad G 1976 On the generators of quantum dynamical semigroups *Commun. Math. Phys.* **48** 119–30
[6] Gorini V, Kossakowski A and Sudarshan E C G 2008 Completely positive dynamical semigroups of N-level systems *J. Math. Phys.* **17** 821–5
[7] Kraus B, Büchler H P, Diehl S, Kantian A, Micheli A and Zoller P 2008 Preparation of entangled states by quantum Markov processes *Phys. Rev.* A **78** 042307
[8] Diehl S, Micheli A, Kantian A, Kraus B, Büchler H P and Zoller P 2008 Quantum states and phases in driven open quantum systems with cold atoms *Nat. Phys.* **4** 878–83
[9] Prosen T 2008 Third quantization: a general method to solve master equations for quadratic open Fermi systems *New J. Phys.* **10** 043026
[10] Prosen T 2010 Spectral theorem for the Lindblad equation for quadratic open fermionic systems *J. Stat. Mech.* P07020
[11] Prosen T and Seligman T H 2010 Quantization over boson operator spaces *J. Phys. A: Math. Theor.* **43** 392004
[12] Prosen T, Martignon L and Seligman T H 2015 Observables and density matrices embedded in dual Hilbert spaces *Phys. Scr.* **90** 074036
[13] Dzhioev A A and Kosov D S 2011 Super-fermion representation of quantum kinetic equations for the electron transport problem *J. Chem. Phys.* **134** 044121
[14] Guo C and Poletti D 2017 Solutions for bosonic and fermionic dissipative quadratic open systems *Phys. Rev.* A **95** 052107
[15] Thomas B and Yikang Z 2022 Solving quasi-free and quadratic Lindblad master equations for open fermionic and bosonic systems *J. Stat. Mech.* 113101
[16] Zhang Y and Barthel T 2022 Criticality and phase classification for quadratic open quantum many-body systems *Phys. Rev. Lett.* **129** 120401
[17] Eisler V 2011 Crossover between ballistic and diffusive transport: the quantum exclusion process *J. Stat. Mech.* P06007
[18] Temme K, Wolf M M and Verstraete F 2012 Stochastic exclusion processes versus coherent transport *New J. Phys.* **14** 075004
[19] Eisert J and Prosen T 2010 Noise-driven quantum criticality (arXiv:1012.5013)
[20] Höning M, Moos M and Fleischhauer M 2012 Critical exponents of steady-state phase transitions in fermionic lattice models *Phys. Rev.* A **86** 013606
[21] Prosen T and Pižorn I 2008 Quantum phase transition in a far-from-equilibrium steady state of an *XY* spin chain *Phys. Rev. Lett.* **101** 105701
[22] Prosen T and Žunkovič B 2010 Exact solution of Markovian master equations for quadratic Fermi systems: thermal baths, open XY spin chains and non-equilibrium phase transition *New J. Phys.* **12** 025016
[23] Braak D 2011 Integrability of the Rabi model *Phys. Rev. Lett.* **107** 100401
[24] Skrypnyk T 2009 Integrability and superintegrability of the generalized n-level many-mode Jaynes–Cummings and Dicke models *J. Math. Phys.* **50** 103523
[25] Čadež T, Jefferson J H and Ramšak A 2013 A non-adiabatically driven electron in a quantum wire with spin–orbit interaction *New J. Phys.* **15** 013029
[26] Čadež T, Jefferson J H and Ramšak A 2014 Exact nonadiabatic holonomic transformations of spin-orbit qubits *Phys. Rev. Lett.* **112** 150402

[27] Donvil B, Ulčakar L, Rejec T and Ramšak A 2020 Thermal effects on a nonadiabatic spin-flip protocol of spin-orbit qubits *Phys. Rev.* B **101** 205427
[28] Johansson J R, Nation P D and Nori F 2012 QuTIP: an open-source Python framework for the dynamics of open quantum systems *Comput. Phys. Commun.* **183** 1760–72
[29] Johansson J R, Nation P D and Nori F 2013 QuTIP 2: a Python framework for the dynamics of open quantum systems *Comput. Phys. Commun.* **184** 1234–40