



PAPER

Exact analysis of gate noise effects on non-adiabatic transformations of spin-orbit qubits

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Abstract

We considered various types of potential noise in gates controlling non-adiabatic holonomic transformations of spin-qubits in one and two-dimensional systems with the Rashba interaction. It is shown how exact results can be derived for deviations of spin rotation angle and fidelity of the qubit transformation after a completed transformation. Errors in initial values of gate potentials and time-dependent drivings are considered and exact results for white gate noise are derived and analysed in detail. It is demonstrated how the drivings can be tuned to optimise the final fidelity of the transformation and to minimise the variances of qubit transformations.

1. Introduction

The new branch of electronics, spintronics, has been the object of intense activity over the past decade since it promises enhanced performance with smaller power consumption compared with classical electronics [1]. Spintronics has potential for realising the fundamental building blocks of a quantum computer via electron spin qubits. Implementation of such qubits is relatively simple in gated semiconductor devices based on quantum dots and quantum wires [2]. Qubit manipulation may be achieved by rotating the electron's spin by the application of an external magnetic field [3]. However, this is unwieldy and not easily confined and controlled in small regions occupied by qubits. The main challenge is therefore how to accurately manipulate the spin of a single electron without using an external magnetic field.

A possible solution is to make use of the spin-orbit interaction (SOI). In semiconductor heterostructures there are two types of SOI, the Dresselhaus interaction [4] due to bulk inversion asymmetry of a crystal, and the Rashba interaction [5] which is a consequence of structural inversion asymmetry of the confining potential of the two-dimensional (2D) electron gas. In spintronic devices the latter is particularly suitable for qubit manipulation since it can be tuned locally via electrostatic gates. Furthermore, since the Rashba interaction couples electron's spin to its orbital motion, qubit spin rotation can be performed by adiabatic spatial translation of a quantum dot containing a single electron, for a distance of the order of the spin-orbit length [6–11]. In one-dimensional (1D) quantum systems also electric-field-induced resonance can manipulate electron spin [12], tunnel-coupled spin qubits can be driven by ac fields [13], and most recently it was shown that time dependent Rashba interaction in a quantum wire can contribute to the rotation of electron spin [14–16]. Experimentally such systems with the ability of controlling electrons have been realised in InSb [17], InAs [18–20] and Ge [21] quantum wires. Recently this type of qubit manipulation has been generalised to non-adiabatic quantum dot motions due to external time-dependent potentials [22], thus opening up the possibility of much faster spin-qubit transformations.

The simplest non-adiabatic qubit manipulation with exact analytical solution is achieved by translating a qubit in one-dimension in the presence of constant Rashba interaction [22]. A drawback of such manipulation is that after the transformation, the qubit is trapped in a displaced quantum dot. This deficiency can be remedied by applying a time-dependent electric field which produces a time-dependent Rashba coupling [23, 24]. The

qubit manipulation then consists of first displacing the quantum dot, followed by changing the Rashba coupling, then returning the quantum dot to the original spatial position and finally tuning the Rashba coupling to its initial value. Such a system thus represents a 1D spatial motion in a 2D parametric space—spanned by the position of the quantum dot and the strength of the Rashba coupling. For quantum dots with harmonic confining potential the exact analytical solution is known for various quantum phases [25] including non-adiabatic non-Abelian Anandan phase [26] which opens the possibility of qubit holonomic transformations [27, 28] where the non-Abelian phase acquired during one cycle in the parametric space can be studied exactly as is discussed in detail in [25]. However, the transformations are limited to cases of rotations with fixed axis. Most recently this limitation posed by fixed axis of spin rotation was also eliminated in a quantum ring structure where full coverage of the Bloch sphere is possible [29, 30].

Exact solutions for all three methods of qubit manipulation also simplifies the analysis of possible effects of environment which result in decoherence and relaxation of the qubit's state. The stability of holonomic transformations depends on precise driving control which can be uncertain due to various sources of parametric noise [31]. For example, due to fluctuating electric fields, caused by the piezoelectric phonons and conduction electrons in the circuit [9, 32] and due to ionised dopant nuclei in a heterostructure [33] or the hyperfine interaction with the nuclei [34]. In molecular systems with phonon assisted potential barriers phonon-mediated instabilities could introduce noise in the confining potentials [35, 36] and in non-adiabatic qubit transformations the effects are also related to the speed of the moving quantum dot [25]. Electrons could be carried also by surface acoustic waves, where additional noise could be introduced by the electron–electron interaction [37, 38]. In a two-qubit system of coupled quantum dots the tunnelling can cause double occupancy leading to processing errors [39].

In this paper we concentrate on the analysis of errors of such qubit transformations and decoherence. Essential for a correct transformation is the precise application of external electric fields via various top gates. An important consideration in the practical implementation of this scheme is the effect of random fluctuations in both the time-dependent SOI and the QD motion as well as the influence of errors in the initialisation of qubit states. The paper is organised as follows. After the introduction, in section 2 we introduce the model and show exact solutions of the time-dependent Schrödinger equation. In section 3 errors in spin-qubit transformations in one- and 2D parametric space are analysed. Exact expressions are given for white noise in electric potentials and an explicit example is presented. Next in section 4 exact results for fidelity due to the white noise are derived and section 5 is devoted to the summary and conclusion.

2. Model

We consider an electron in a quantum wire confined in a harmonic trap [22, 25]. The centre of the potential (1D quantum dot), $\xi(t)$, can be arbitrarily translated along the wire by means of time dependent external electric fields. Spin–orbit Rashba interaction between electric field and electron spin couples with orbital motion, resulting in the following Hamiltonian

$$H(t) = \frac{p^2}{2m^*}I + \frac{m^*\omega^2}{2}[x - \xi(t)]^2I + \alpha(t)p\mathbf{n} \cdot \boldsymbol{\sigma}, \quad (1)$$

where m^* is the electron effective mass, ω is the frequency of the harmonic trap, $\alpha(t)$ is the strength of SOI, possibly time dependent due to appropriate time dependent external electric fields. Throughout the paper, we use units with $\hbar = 1$. The spin rotation axis \mathbf{n} is fixed and depends on the crystal structure of the quasi-1D material used and the direction of the applied electric field [17]. $\boldsymbol{\sigma}$ and I are Pauli spin matrices and unity operator in spin space, respectively. Exact solution of the time dependent Schrödinger equation corresponding to the Hamiltonian equation (1) is given by [22, 25, 40]

$$|\Psi_{ms}(t)\rangle = e^{-i\omega_m t} \mathcal{U}^\dagger(t) |\psi_m(x)\rangle |\chi_s\rangle, \quad (2)$$

$$\mathcal{U}^\dagger(t) = \mathcal{A}_\alpha \mathcal{X}_\xi, \quad (3)$$

$$\mathcal{A}_\alpha = e^{-i[\phi_\alpha(t) + m^* \dot{a}_c(t) a_c(t) / \omega^2] I + \phi_A(t) \mathbf{n} \cdot \boldsymbol{\sigma} / 2} e^{-i \dot{a}_c(t) p \mathbf{n} \cdot \boldsymbol{\sigma} / \omega^2} e^{-i m^* a_c(t) x \mathbf{n} \cdot \boldsymbol{\sigma}}, \quad (4)$$

$$\mathcal{X}_\xi = e^{-i \phi_\xi(t) I} e^{i m^* [x - x_c(t)] \dot{x}_c(t) I} e^{-i x_c(t) p I}. \quad (5)$$

Here $\psi_m(x)$ represents the m th eigenstate of a harmonic oscillator with eigenenergy $\omega_m = (m + 1/2)\omega$ and $|\chi_s\rangle$ is spinor of the electron in the eigenbasis of operator σ_z . Unitary transformations \mathcal{A}_α and \mathcal{X}_ξ transform the system into the ‘moving frame’ of SOI and position, respectively, and therefore $\mathcal{U}^\dagger(t)$ transforms the Hamiltonian equation (1) into a simple time independent harmonic oscillator Hamiltonian. The phase $\phi_\xi(t) = -\int_0^t L_\xi(t') dt'$ is the coordinate action integral, with $L_\xi(t) = m^* \dot{x}_c^2(t) / 2 - m^* \omega^2 [x_c(t) - \xi(t)]^2 / 2$ being the Lagrange function of a driven harmonic oscillator and $x_c(t)$ is the solution to the equation of motion of a classical driven oscillator

$$\ddot{x}_c(t) + \omega^2 x_c(t) = \omega^2 \xi(t). \quad (6)$$

Another phase factor is the SOI action integral phase $\phi_\alpha(t) = -\int_0^t L_\alpha(t') dt'$, where $L_\alpha(t) = m^* \dot{a}_c^2(t)/(2\omega^2) - m^* a_c^2(t)/2 + m^* a_c(t) \alpha(t)$ is the Lagrange function of another driven oscillator, satisfying

$$\ddot{a}_c(t) + \omega^2 a_c(t) = \omega^2 \alpha(t). \quad (7)$$

Spin-qubits are rotated around \mathbf{n} by two ‘dynamical’ terms proportional to operators $a_c(t)x$, $\dot{a}_c(t)p$, and by the angle $\phi_A(t) = -2m^* \int_0^t \dot{a}_c(t') \xi(t') dt'$, the Anandan phase for the case of cyclic motions, as analysed in [25] where it is shown also that the phase in the adiabatic limit reduces to the Wilczek–Zee non-Abelian phase [41]. By the application of an external magnetic field the degeneracy is lifted and the phase reduces to the non-adiabatic Aharonov–Anandan phase [42] in the adiabatic limit simplified further to the ordinary Berry phase [43].

3. Spin-qubit transformations

3.1. 1D parametric space

First we consider a special case of constant SOI, $\alpha(t) = \alpha_0$, which means that the parameter space is 1D. The exact solution, equation (2) is completely determined by the classical response of the oscillator, equation (6), which makes exact analysis of the qubit transformation very simple. For example, if the electron is initially in the m th excited state of $H(0)$, the spin is rotated around \mathbf{n} for angle,

$$\phi(T) = 2m^* \alpha_0 x_c(T), \quad (8)$$

where T is the transformation time [22]. When the driving $\xi(t)$ is chosen to give full spin-flip $\phi(T) = \pi$, the final displacement of the electron is $x_c(T) = \pi/(2m^* \alpha_0)$ and no residual angle oscillations are present. This is fulfilled exactly when the final state of the electron is in the m th eigenstate of $H(T)$, that is when $x_c(T) = \xi(T)$ and $\dot{x}_c(T) = 0$.

In qubit transformations of this type it is essential to control precisely the initial state and driving electric fields. Noise in fields of gate electrodes is reflected in fields which translate the trap potential minimum and is consequently manifested as noise in the initial position $\delta x_c(0)$, initial velocity $\delta \dot{x}_c(0)$ and driving function $\delta \xi(t)$. This produces noise in $x_c(T)$ which further induces noise in $\phi(T)$. Using the exact solution, equation (2) it is straightforward to analyse the noise in the transformation angle ϕ , which is dispersed by some probability distribution, given by a change of variables when the probability density function of x_c is known.

We assume that errors in initial position and velocity may be described by normal distributions with variances $\sigma_{x_0}^2$ and $\sigma_{\dot{x}_0}^2$, respectively,

$$\frac{dP}{dx_c(0)} = \frac{1}{\sqrt{2\pi} \sigma_{x_0}} e^{-x_c(0)^2/2\sigma_{x_0}^2}, \quad \frac{dP}{d\dot{x}_c(0)} = \frac{1}{\sqrt{2\pi} \sigma_{\dot{x}_0}} e^{-\dot{x}_c(0)^2/2\sigma_{\dot{x}_0}^2}. \quad (9)$$

The driving function $\xi(t) = \xi^0(t) + \delta \xi(t)$ consists of ideal driving part without noise, $\xi^0(t)$ with superimposed stochastic part with vanishing mean $\langle \delta \xi(t) \rangle = 0$ and characterised by the time autocorrelation function $\langle \delta \xi(t') \delta \xi(t'') \rangle$ [44]. We consider here coloured noise, in particular the Ornstein–Uhlenbeck process [45–47] with exponential correlations $\langle \delta \xi(t') \delta \xi(t'') \rangle = \frac{\sigma_\xi^2}{2\tau_\xi} e^{-|t' - t''|/\tau_\xi}$ with noise intensity σ_ξ^2 and correlation time τ_ξ .

A general solution of equation (6) $x_c(t)$ is given by

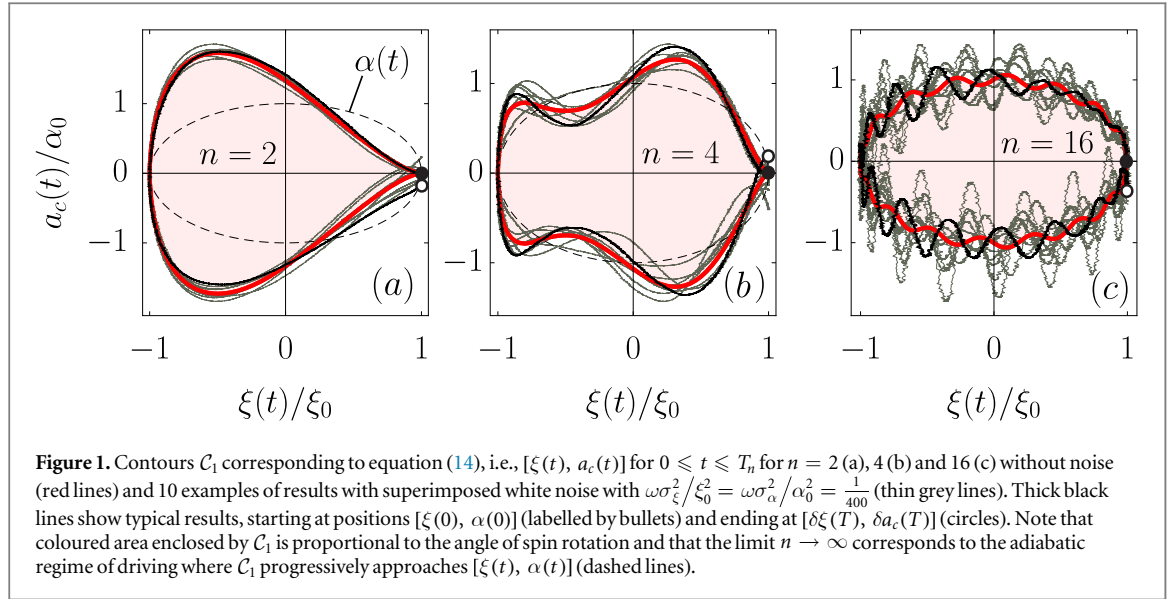
$$x_c(t) = x_c(0) \cos \omega t + \frac{\dot{x}_c(0)}{\omega} \sin \omega t + \omega \int_0^t \sin[\omega(t - t')] \xi(t') dt', \quad (10)$$

where all three terms are stochastic, independent and normally distributed variables. Their sum is also normally distributed with variance equal to the sum of variances of all variables [48]. The variance of the first two terms is obtained by the change of variables formula while the variance corresponding to the third term is evaluated as equal-times autocorrelation function [49],

$$\sigma_x^2(t) = \omega^2 \lim_{\Delta t \rightarrow 0} \left\langle \int_0^t \sin[\omega(t - t')] \delta \xi(t') dt' \int_0^{t+\Delta t} \sin[\omega(t - t'')] \delta \xi(t'') dt'' \right\rangle. \quad (11)$$

For the Ornstein–Uhlenbeck noise considered here the integrals can be evaluated exactly and the final result is that the angle of spin is distributed normally with the time dependent variance

$$\sigma_\phi^2(t) = (2m^* \alpha_0)^2 \left(\sigma_{x_0}^2 \cos^2 \omega t + \sigma_{\dot{x}_0}^2 \frac{\sin^2 \omega t}{\omega^2} + \sigma_x^2(t) \right), \quad (12)$$



with $\sigma_x^2(t) = \frac{1}{4}\omega\sigma_\xi^2(2\omega t - \sin 2\omega t) + \mathcal{O}(\tau_\xi)$, where only the short correlation time ($\tau_\xi \rightarrow 0$) contributions—corresponding to the white noise in driving—are explicitly shown here. The first two terms in equation (12) are limited by the precision of the initial conditions while the third contribution, related to the noise in driving, diverges at large ωt , since the Lorentzian noise power spectrum $\sigma_\xi^2/[1 + (2\pi f\tau_\xi)^2]$ considered here consists of different driving frequencies including the resonant value $\omega = 2\pi f$, resulting in the asymptotic response $\sigma_x^2(t) \propto t$ —similar to the 1D random walk problem [45]. In order to keep the noise in the final results low, fast, non-adiabatic transformations are therefore favourable.

3.2. 2D parametric space

Although the 1D spin transformation scheme can be implemented in a controllable manner and also the driving noise level optimised by a suitable driving, an important drawback is the fact that after the transformation is completed the electron is spatially shifted from its initial position. This problem is resolved if the position of the quantum dot and the Rashba interaction are both time dependent, thus spanning a 2D parameter space. As demonstrated in [25], the quantum dot can, for example, be first spatially shifted with some initial Rashba coupling value α_1 and then displaced back to the starting position, while keeping the Rashba coupling fixed at different value α_2 and finally setting the Rashba coupling back to its initial value α_1 . This transformation depends only on the area of the loop in the 2D parameter space. In particular, when the system is driven by a cyclic evolution, that is $\xi(t+T) = \xi(t)$, $\alpha(t+T) = \alpha(t)$, $x_c(t+T) = x_c(t)$, $\dot{x}_c(t+T) = \dot{x}_c(t)$, $a_c(t+T) = a_c(t)$ and $\dot{a}_c(t+T) = \dot{a}_c(t)$, the angle of the spin rotation around direction \mathbf{n} is given by

$$\phi(T) = \phi_A(T) = -2m^* \int_0^T \dot{a}_c(t)\xi(t) dt = 2m^* \oint_{\mathcal{C}_1} a_c[\xi] d\xi, \quad (13)$$

where $a_c[\xi]$ represents the contour \mathcal{C}_1 in 2D parametric space $[\xi(t), a_c(t)]$ for $0 \leq t \leq T$ thus the spin rotation angle is simply given by the area enclosed by \mathcal{C}_1 .

Additionally to coordinate noises in 1D spin transformations, in 2D spin transformations are also normally distributed noise in initial SOI response $a_c(0)$, initial time derivative of the response $\dot{a}_c(0)$ and stochastic noise in SOI driving function $\alpha(t) = \alpha^0(t) + \delta\alpha(t)$, where $\alpha^0(t)$ is ideal noiseless driving. SOI noise $\delta\alpha(t)$ is similar to the previous case of spatial driving and is again of the Ornstein–Uhlenbeck type with autocorrelation function $\langle \delta\alpha(t')\delta\alpha(t'') \rangle$, noise intensity σ_α^2 and correlation time τ_α . One should note that although the driving parameters $\xi(t)$ and $\alpha(t)$ are directly subjected to the noise, error in qubit transformation is related to the noise in $\xi(t)$ and noise in the response function $a_c(t)$ as seen from equation (13).

As an example of a 2D qubit rotation we present the family of transformations with sinusoidal driving parametrised as

$$\xi(t) = \xi_0 \cos(2\pi t/T_n), \quad \alpha(t) = \alpha_0 \sin(2\pi t/T_n), \quad (14)$$

with transformation times $T_n = nT_1$, where $T_1 = 2\pi/\omega$ is the period of the confining potential and $n > 1$. In figure 1 are shown some paths in the parametric space $[\xi(t), a_c(t)]$ during the 2D transformation $0 \leq t \leq T_n$. Thin lines represent ten different results with white noise. Time dependent variance $\sigma_a^2(t)$ is manifested as a spread of these curves around the ideal closed line \mathcal{C}_1 (red). Thick black lines represent typical case starting at positions $[\xi(0), \alpha(0)]$ (bullets) and ending at $[\delta\xi(T), \delta a_c(T)]$ (circles). It should be noted that accumulated

errors in $a_c(T) = \delta a_c(T)$, with variance σ_a^2 , are much larger than the corresponding $\delta\xi(T)$ which has only an instant white noise contribution. Finally, it should be noted also that the transformation angle ϕ —proportional to the coloured area enclosed by C_1 —is due to oscillations of individual noisy curves around the ideal value relatively less prone to the noise.

The fact that, in comparison to 1D spin transformations, these transformations include noise in 2D parameter space, where transformations need to be periodic, already indicates that fundamental differences might arise. One such difference is in the effect of noise of initial conditions. As seen from equation (10), eventual non-zero initial conditions directly contribute to response with cosine and sine time-dependency. Spin transformations in 2D fulfil the condition of periodic response $x_c(t + T) = x_c(t)$ and $a_c(t + T) = a_c(t)$ and have transformation times T a multiple of the oscillator period $2\pi/\omega$. The errors in initial conditions then only translate the curve C_1 in the parametric space but do not change the area enclosed by the contours and thus the angle of rotation is not affected.

The only relevant source of gate noise is thus the noise in driving functions $\xi(t)$ and $\alpha(t)$, which induce noise in the appropriate variables which are for the case of white noise all independent, stochastic and normally distributed with the corresponding variances [50]. As before, only short correlation times are considered with approximation, $\tau_\xi, \tau_\alpha \rightarrow 0$, leading to the time-dependent variances, $\sigma_x^2(t)$ as in equation (12),

$$\sigma_x^2(t) = \frac{1}{4}\omega^3\sigma_\xi^2[2\omega t + \sin(2\omega t)], \quad (15)$$

$$\sigma_a^2(t) = \frac{\sigma_\alpha^2}{\sigma_\xi^2}\sigma_x^2(t) \quad \text{and} \quad \sigma_{\dot{a}}^2(t) = \frac{\sigma_\alpha^2}{\sigma_\xi^2}\sigma_{\dot{x}}^2(t), \quad (16)$$

corresponding to $x_c(t)$, $\dot{x}_c(t)$, $a_c(t)$ and $\dot{a}_c(t)$, respectively.

Induced noise in the angle of spin qubit rotation ϕ is given by

$$\delta\phi = -2m^* \int_0^T \dot{a}_c^0(t) \delta\xi(t) dt - 2m^* \int_0^T \delta\dot{a}_c(t) \xi^0(t) dt - 2m^* \int_0^T \delta\dot{a}_c(t) \delta\xi(t) dt, \quad (17)$$

where $\dot{a}_c(t) = \dot{a}_c^0(t) + \delta\dot{a}_c(t)$, $\dot{a}_c^0(t)$ being response without noise. $\delta\phi$ is the sum of three independent and normally distributed stochastic processes which lead to the variance $\sigma_\phi^2 = \sigma_I^2 + \sigma_{II}^2 + \sigma_{III}^2$, where σ_I^2 is the variance of the first, σ_{II}^2 of the second and σ_{III}^2 of the third term [48]. These terms are evaluated directly from the phase average, for example $\sigma_I^2 = 4m^{*2} \langle \int_0^T \dot{a}_c^0(t') \delta\xi(t') dt' \int_0^T \dot{a}_c^0(t'') \delta\xi(t'') dt'' \rangle$. For arbitrary driving these terms are equal to

$$\sigma_I^2 = (2m^*\sigma_\xi)^2 \int_0^T \dot{a}_c^0(t)^2 dt = (2m^*\sigma_\xi)^2 \oint_{C_a} \dot{a}_c^0[a_c^0] da_c^0, \quad (18)$$

$$\begin{aligned} \sigma_{II}^2 &= m^{*2}\omega^3\sigma_\alpha^2 \int_0^T \xi^0(t') \int_0^T \{ \sin[\omega(2\min(t', t'') - t' - t'')] \\ &+ \sin[\omega(t' + t'')] + 2\omega \min(t', t'') \cos[\omega(t' - t'')] \} \xi^0(t'') dt'' dt', \end{aligned} \quad (19)$$

$$\sigma_{III}^2 = \frac{1}{4}(m^*\omega\sigma_\xi\sigma_\alpha)^2[(2\omega T)^2 + 2\sin^2(\omega T)]. \quad (20)$$

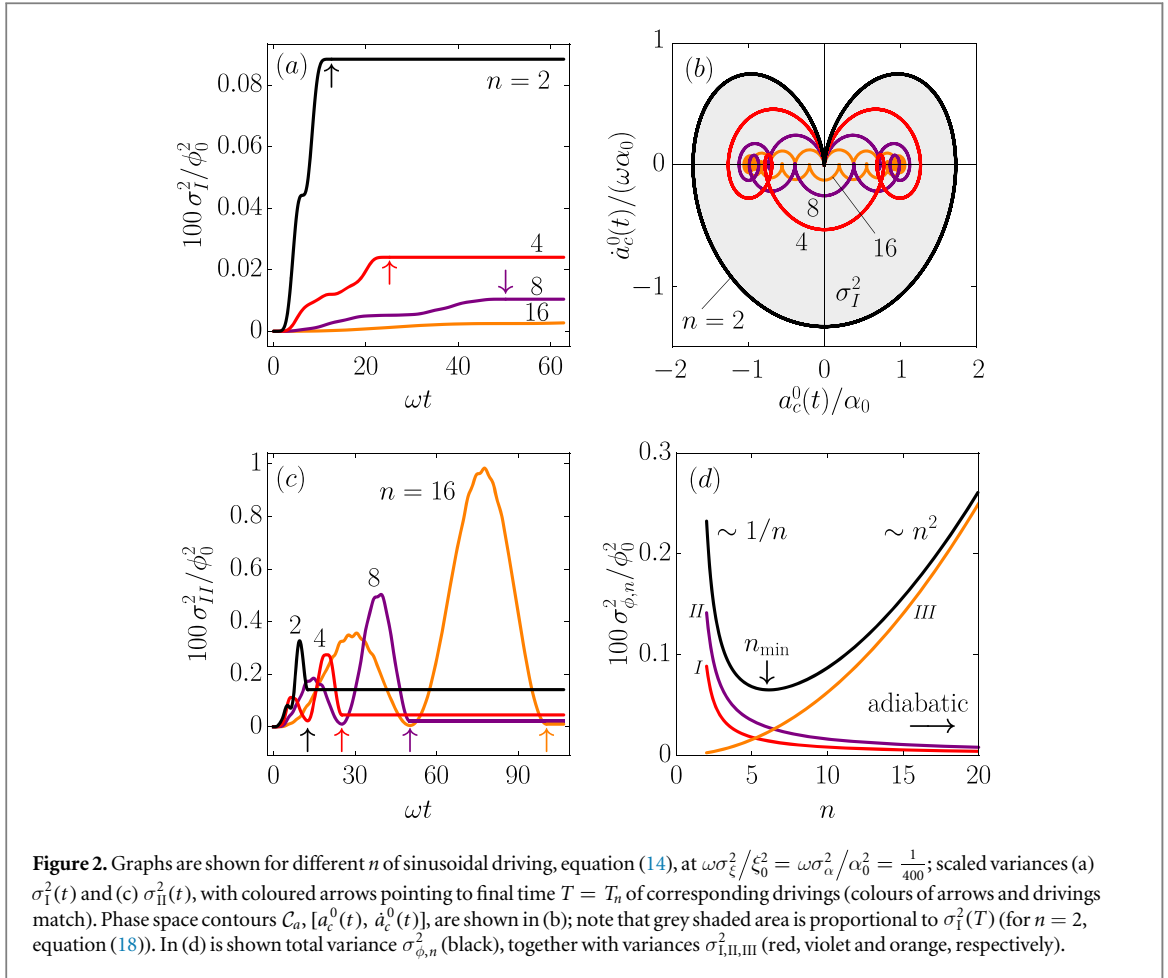
The first contribution σ_I^2 is proportional to the intensity of the ξ -noise and to the action integral associated with the SOI response, which vanishes in the adiabatic limit of the Rashba-driving. The second term, σ_{II}^2 , originates in the α -noise and is non-trivially related to the time dependence of the spatial driving function $\xi^0(t)$. As shown later in an example, this term can be made small by appropriate choice of driving. The last contribution to the angle variance, σ_{III}^2 , is of higher order in position and Rashba driving noise intensities and thus negligible for fast, non-adiabatic qubit transformations while quadratically increasing for large $\omega T \gg 1$ adiabatic-like spin transformations. Variance σ_ϕ^2 of noise in the angle of qubit rotation is therefore due to terms I and II enhanced for fast non-adiabatic drivings whereas it increases for large driving times due to the term III. This sets the condition for minimal total induced noise.

In order to elucidate this point we investigate angle variance for the circular driving scheme given by equation (14). At completion of the transformation at time T_n the variance is given by

$$\frac{\sigma_{\phi,n}^2}{\phi_0^2} = \frac{n(1+n^2)}{\pi(n^2-1)^2} \frac{\omega\sigma_\xi^2}{\xi_0^2} + \frac{2n^3}{\pi(n^2-1)^2} \frac{\omega\sigma_\alpha^2}{\alpha_0^2} + n^2 \left(\frac{\omega\sigma_\xi\sigma_\alpha}{\xi_0\alpha_0} \right)^2, \quad (21)$$

where $\phi_0 = 2\pi m^* \xi_0 \alpha_0$ is the qubit rotation angle equation (13) in the noiseless and $n \rightarrow \infty$ limit.

In figure 2(a) is presented σ_I as a function of time for various driving times for $n = 2, 4, 8$ and 16 with corresponding contours $[a_c^0(t), \dot{a}_c^0(t)]$ in figure 2(b). Note that at larger n the area within particular contour C_a is progressively smaller—as expected in the adiabatic limit. Figure 2(c) shows that the contribution σ_{II} exhibits oscillations with time, but at final times the level of noise is minimum, decreasing with increasing n . It should be noted that the noise can not be avoided in any limit—the first two contributions to σ_ϕ^2 are larger at short



transformation times, i.e., both decrease as $\propto 1/n$ for large $n \gg 1$ while the third term σ_{III}^2 increases as $\propto n^2$, figure 2(d), therefore there exists some minimum variance $\sigma_{\phi,n_{\min}}$ at optimal driving time for n_{\min} .

For larger $n \gg 1$ such optimal regime can readily be evaluated,

$$n_{\min} = \left(\frac{\xi_0^2}{\pi\omega\sigma_\xi^2} + \frac{\alpha_0^2}{2\pi\omega\sigma_\alpha^2} \right)^{\frac{1}{3}}, \quad (22)$$

$$\frac{\sigma_{\phi,n_{\min}}^2}{\phi_0^2} = 3n_{\min}^2 \left(\frac{\omega\sigma_\xi\sigma_\alpha}{\xi_0\alpha_0} \right)^2 \propto T_{n_{\min}}^2, \quad (23)$$

therefore in order to minimise the variance the parameters should be chosen such that the driving time $T_{n_{\min}}$ is minimal. The variance is for $T < T_{n_{\min}}$ limited by the extreme non-adiabatic value at $n = 2$,

$$\frac{\sigma_{\phi,n}^2}{\phi_0^2} < \frac{\omega\sigma_\xi^2}{\xi_0^2} + \frac{\omega\sigma_\alpha^2}{\alpha_0^2}, \quad (24)$$

which is qualitatively correct also for other types of driving with the contour C_1 approximately bounded by the area $\xi_0\alpha_0$, with noise intensities σ_ξ and σ_α .

4. Fidelity

A fundamental property of adiabatic quantum phase is its invariance to changes in time-dependent Hamiltonian parameters, the actual phase being given by the area enclosed by the path in the parametric space. For fast, non-adiabatic holonomic transformations considered here, the phase is given by the area, in combined space of driving and response parameters, which only in the adiabatic limit is independent of time. More importantly, for non-adiabatic qubit manipulations the electron state is determined by the time-dependent Hamiltonian during the evolution and will in general be a superposition of excited states, becoming the ground state when the transformation is complete. As shown in [22, 25, 29] such motions in parametric space can easily be performed if the driving functions are appropriately chosen.

In the previous section the analysis of spin-rotation angle variance demonstrated that due to gate noise in the driving functions, spin transformations are not completely faithful and that additional fine tuning is required in order to minimise these noise effects. Moreover, for the present case of non-adiabatic qubit transformations, an important additional question is relevant: how well does the final state of the electron relax to the desired final state energy manifold after the transformation if the driving function is not ideal as in the presence of noise?

In order to answer this question we consider the qubit wave function $|\Psi_{0\frac{1}{2}}(t)\rangle$, equation (2), at $t = 0$ in the ground state of the harmonic quantum dot with $m = 0$ and spin $\frac{1}{2}$. We observe the relaxation to the ground state, spanned by the basis of the time dependent Hamiltonian equation (1) at time t [22],

$$|\widetilde{\Psi}_{0s}\rangle = e^{-im^*[x-\xi(t)]\alpha(t)\mathbf{n}\cdot\sigma}|\psi_0[x-\xi(t)]|\chi_s\rangle. \quad (25)$$

As the appropriate measure of the relaxation accuracy we define fidelity $F = \langle\Psi_{0\frac{1}{2}}(t)|P_0|\Psi_{0\frac{1}{2}}(t)\rangle$, where

$P_0 = \sum_s|\widetilde{\Psi}_{0s}\rangle\langle\widetilde{\Psi}_{0s}|$ is the projector onto the ground state manifold. We choose \mathbf{n} perpendicular to the z -axis and a lengthy but straightforward derivation yields the expression

$$F = \frac{1}{2}(e^{-E_+} + e^{-E_-}), \quad (26)$$

$$E_{\pm} = \frac{m^*}{2\omega} \{ [\omega(x_c(t) - \xi(t)) \pm \dot{a}_c(t)/\omega]^2 + [\dot{x}_c(t) \mp (a_c(t) - \alpha(t))]^2 \}, \quad (27)$$

where E_{\pm} resembles normalised classical energy with additional terms for spin-orbit coupling and is equal to the normalised classical energy if the spin-orbit driving is constant [22]. Fidelity for 1D driving is obtained as a limit of 2D case when one of the drivings is constant, for example $a_c(t) = \alpha_0$, $\dot{a}_c(t) = 0$ thus $\sigma_{\alpha} = \sigma_{a_c} = \sigma_{\dot{a}_c} = 0$. The expression for fidelity then simplifies to $F = e^{-E}$, where $E = E_+ = E_-$ is now equal to the normalised classical energy of harmonic oscillator.

Ideal qubit transformations, $F = 1$, are achieved by applying ideal drivings, where the energies E_{\pm} vanish at final time $t = T$, i.e., when $x_c = \xi$, $a_c = \alpha$, $\dot{x}_c = 0$, and $\dot{a}_c = 0$. However, the presence of noise in spin-orbit and spatial driving terms makes fidelity a random quantity, described by a probability density function $\frac{dP(F)}{dF}$. It can be calculated from the probability density for variables $E_{\pm} = f_{\pm}(x_c, \dot{x}_c, a_c, \dot{a}_c)$ which are functions of independent random variables, normally distributed and with variances equations (15) and (16). The probability density functions for E_{\pm} can be calculated using the formula

$$\frac{dP(E_{\pm})}{dE_{\pm}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dP(x_c)}{dx_c} \frac{dP(\dot{x}_c)}{d\dot{x}_c} \frac{dP(a_c)}{da_c} \frac{dP(\dot{a}_c)}{d\dot{a}_c} \times \delta[E_{\pm} - f_{\pm}(x_c, \dot{x}_c, a_c, \dot{a}_c)] dx_c d\dot{x}_c da_c d\dot{a}_c. \quad (28)$$

The result is obtained by first calculating the characteristic function, followed by the inverse Fourier transform yielding distributions with the same functional form for variables E_+ and E_- ,

$$\frac{dP(E_{\pm})}{dE_{\pm}} = 2\sigma_1^{-1}\sigma_2^{-1}I_0[(\sigma_1^{-2} - \sigma_2^{-2})E_{\pm}]e^{-(\sigma_1^{-2} + \sigma_2^{-2})E_{\pm}}, \quad (29)$$

with

$$\sigma_1^2(t) = \left(\frac{2m^*}{\omega}\right)(\omega^2\sigma_x^2(t) + \sigma_a^2(t)/\omega^2), \quad (30)$$

$$\sigma_2^2(t) = \left(\frac{2m^*}{\omega}\right)(\sigma_{\dot{x}}^2(t) + \sigma_a^2(t)), \quad (31)$$

where $I_0(z)$ is the modified Bessel function of the first kind.

Distributions for $e^{-E_{\pm}}$ are calculated by using a simple change of variables formula. Since the fidelity is a sum of two dependent random variables, its probability distribution is calculated from the joint probability distribution function for those two variables, which in general cannot be evaluated analytically. However, one can examine $\frac{dP}{dF}$ exactly in two convenient limiting cases. The first is the case when the noise in one of the driving variables, for example $\alpha(t)$, is much weaker than the other, i.e., $\sigma_{\alpha} \ll \sigma_{\xi}$. In this case, the noise properties are essentially those of a 1D problem and the exact expression for probability density function of fidelity is

$$\frac{dP(F)}{dF} = 2\sigma_1^{-1}\sigma_2^{-1}I_0[(\sigma_1^{-2} - \sigma_2^{-2})\ln F]F^{\sigma_F^{-2}-1}, \quad \sigma_F^{-2} = \sigma_1^{-2} + \sigma_2^{-2}, \quad (32)$$

which for $\sigma_{1,2} \rightarrow 0$ leads to $\frac{dP}{dF} \propto F^{\sigma_F^{-2}}$.

The second limiting case is when $\sigma_x^2(t) = \sigma_a^2(t)/\omega^4$ and $\sigma_{\dot{x}}^2(t) = \sigma_a^2(t)$, which is satisfied for $t = T_n$ if the coordinate and the SOI driving noise intensities are equal, i.e., $\sigma_{\alpha} = \omega\sigma_{\xi}$. In this case E_+ and E_- are independent random variables [48] and $\frac{dP}{dF}$ can be calculated as the convolution of probability distributions for e^{-E_+} and e^{-E_-} .

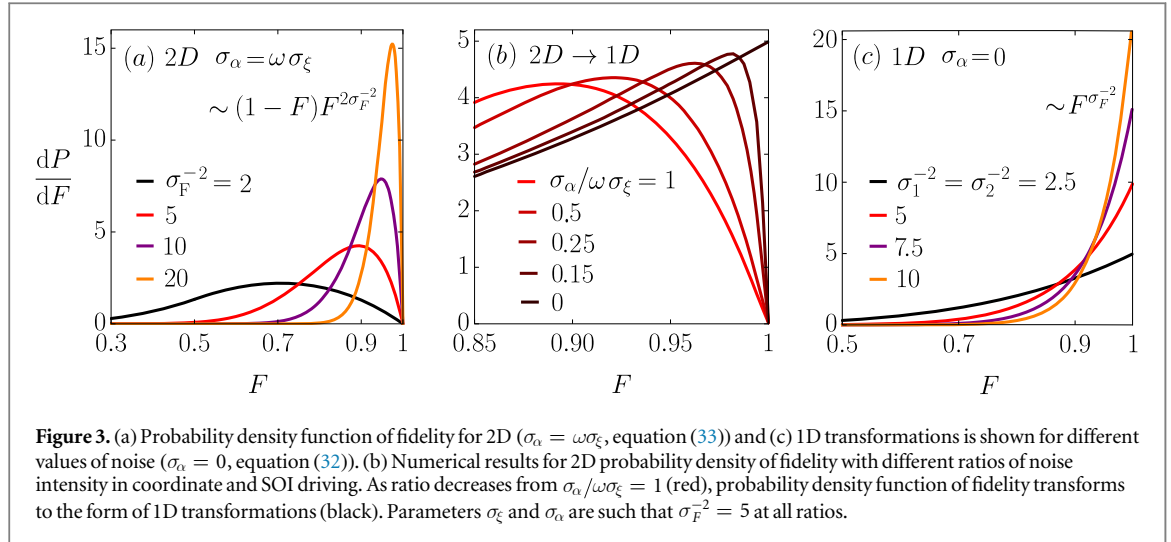


Figure 3. (a) Probability density function of fidelity for 2D ($\sigma_\alpha = \omega\sigma_\xi$, equation (33)) and (c) 1D transformations is shown for different values of noise ($\sigma_\alpha = 0$, equation (32)). (b) Numerical results for 2D probability density of fidelity with different ratios of noise intensity in coordinate and SOI driving. As ratio decreases from $\sigma_\alpha/\omega\sigma_\xi = 1$ (red), probability density function of fidelity transforms to the form of 1D transformations (black). Parameters σ_ξ and σ_α are such that $\sigma_F^2 = 5$ at all ratios.

The exact result for $F \geq \frac{1}{2}$ is

$$\frac{dP(F)}{dF} = 2\sigma_F^{-4} \left[B\left(\frac{1}{2F}, \sigma_F^{-2}, \sigma_F^{-2}\right) - B\left(1 - \frac{1}{2F}, \sigma_F^{-2}, \sigma_F^{-2}\right) \right] (2F)^{2\sigma_F^{-2}-1}, \quad (33)$$

where $B(x, a, b)$ is the incomplete beta function. For $F < \frac{1}{2}$ the probability distribution is given by $\frac{dP}{dF} = 2\sigma_F^{-4} B(\sigma_F^{-2}, \sigma_F^{-2})(2F)^{2\sigma_F^{-2}-1}$, where $B(a, b)$ is the beta function.

In practice the most relevant regime is $\sigma_F \rightarrow 0$ for which the probability distribution equation (33) simplifies to $\frac{dP}{dF} \propto (1-F)F^{2\sigma_F^{-2}}$. At first glance the surprising result is that the probability distribution for $F \rightarrow 1$ tends to zero, in other words, it is not possible to exactly achieve a flawless spin-flip transformation with 2D driving. This is quite different from transformations with 1D driving, where the probability for flawless transformation at $F = 1$ is maximum. For intuitive interpretation one can compare the noise in 1D and 2D transformations to Brownian motion in 1D and 2D parameter space and the fidelity to the probability of finding a particle after some time near the starting point [47, 51]. In 1D the particle always returns to the starting point while in 2D the particle returns *almost surely* with maximum probability at the annulus near the starting point.

In figure 3(a) is shown the fidelity probability density function equation (33) for different σ_F . It is clear that although the probability for driving flawlessly is zero, the position of the maximum of $\frac{dP}{dF}$ is for small σ_F very close to 1,

$$F_{\max} = 1 - \frac{1}{2}\sigma_F^2, \quad (34)$$

the width at half maximum is

$$\sigma_{2D} = \frac{e}{2}\sigma_F^2, \quad (35)$$

and linear growth of probability from zero to maximum value indicates that when the noise is low enough, the transformation is with very high probability achieved with almost zero error. Note that the fidelity is less sensitive to the noise than the transformation angle studied in section 3 as indicated by the quadratic dependence of the shift in fidelity distribution maximum away from $F = 1$. Hence $\delta F \sim \sigma_F^2$ is of higher order in the noise intensities than the width of the transformation angle, $\sigma_\phi \sim \sigma_\xi$ (or σ_α).

Similar arguments apply to the qubit transformations with 1D driving, where the position of the maximum of $\frac{dP}{dF}$ is equal to 1 and full width at half maximum σ_{1D} is given by

$$\sigma_{1D} = \ln 2 \sigma_F^2. \quad (36)$$

Examples of numerically generated results for general σ_ξ and σ_α [52] are shown in figure 3(b), where in the limit $\sigma_\alpha/\sigma_\xi \rightarrow 0$ the probability distribution gradually transforms from 2D to 1D form and in figure 3(c) such 1D results are presented for various $\sigma_1 = \sigma_2$.

For the example of sinusoidal driving, equation (14) considered in the previous section, the lowest noise in the angle of spin rotation is achieved for driving times $T_{n_{\min}} = n_{\min} T_1$, equation (22). The corresponding probability distribution of fidelity is equal to equation (33), with

$$\sigma_F^2 = \frac{1}{2} m^* (\omega^2 \sigma_\xi^2 + \sigma_\alpha^2) \omega T_{n_{\min}}, \quad (37)$$

therefore regarding the minimisation of both variances, for rotation angle and for fidelity, faster transformations with lower $T_{n_{\min}}$ are favourable.

5. Summary and conclusion

Recent theoretical analysis has revealed that holonomic spin manipulation in linear systems [22, 25] or on an appropriate ring support [29, 30] is feasible from adiabatic to strong non-adiabatic regime of driving. The first prerequisite here is the ability to control the position of the electron $\xi(t)$ and the second is controllable manipulation of the Rashba coupling, regarding the time dependent strength $\alpha(t)$ and also the choice of preferred direction \mathbf{n} . For slow, adiabatic qubit manipulation these requirements lead to an arbitrary transformation, simply determined by the area in the space of driving parameters $[\xi, \alpha]$. During the process of the transformation the electron remains permanently in the same spatial state, the ground state for example, and only spin properties change.

Fast, non-adiabatic spin manipulation is far more challenging since the time-dependence of driving functions have to be appropriately tuned. Unlike the adiabatic regime, the transformation angle of spin is given by the combined space of both the driving function $\xi(t)$ and the SOI response $a_c(t)$ to the driving function $\alpha(t)$. In addition to correct transformation of the spin direction, one has also to take care that the electron state has not left the starting energy manifold at the final time. For example, starting from the ground state the electron should, after performing one cycle with time-dependent Hamiltonian, return to the ground state, although during the cycle the state of the electron may be a superposition of excited eigenstates of the moving potentials. As shown in [22, 25, 29] such drivings are feasible to perform. As long as the approximation of the harmonic potential is justified, the formalism yields exact time dependent wave functions with simple tuning of driving functions in order to achieve desired qubit transformations. However, noise in driving functions will always be present because of unavoidable gate noise, which means that qubit transformation will always deviate from the ideal one.

In this paper we examined in detail the influence on qubit transformations of various imperfections in driving. The formalism allows analytical treatment of arbitrary driving, therefore we concentrated on the exact analysis of the influence of small deviations from ideal qubit manipulation. In particular, for 1D manipulation we show how errors in initial conditions give rise to variance in the transformation angle. It is shown how one can analyse the effects of a general coloured noise to the transformation angle and, as an example, we show the result for Ornstein–Uhlenbeck noise in the limit of short correlation times (white noise) although the formalism can be applied to other types of noise defined by their autocorrelation functions.

The results valid for 1D parametric space are generalised to more involved analysis of the transformation angle for the case of 2D spin manipulation with time dependence of both quantum dot position and SOI. The first result here is that, due to periodicity, holonomic manipulation is completely insensitive to the initial conditions, since the qubit rotation angle is given solely by the area in parameter space for which errors in the starting point are irrelevant. Therefore the only source of errors here is the noise in driving functions. As in the 1D case exact results can be derived for a broad class of coloured noise with given autocorrelation functions and appropriate formulae are given explicitly. As a typical example, considered in detail, is the case of circular driving in the space of parameters for which exact analytical formulae are given and analysed for white noise. It is argued that these particular results are qualitatively valid in general, providing similar size of the contour in parametric space and similar noise intensities. In particular, for non-adiabatic manipulations, errors increase due to the detuning of sensitivity drivings in the presence of the noise, while in the adiabatic limit of driving the accumulation of errors is similar to random walk process. In general we expect some optimum regime between non-adiabatic and adiabatic driving and we also show that minimal variance can be achieved by suitable tuning.

As discussed above, for non-adiabatic regimes a non-trivial point of issue is the ability of the system to return to the ground state after an arbitrary time-dependent driving. For that reason our analysis was focused on fidelity—the overlap of the actual wave function with the desired ideal. For the white noise limit of coloured noise explicit formulae are derived. For the 1D case and general time-dependent variances of response functions the result is given explicitly. For 2D, exact analytical results are derived for symmetric noise in position and spin-orbit driving functions. For more general cases some examples are calculated numerically and shown to demonstrate smooth transition between two limiting cases, totally symmetric 2D and asymmetric 1D.

We conclude with an interesting observation that the noise effects on fidelity have a structure similar to probability density in random walk problems. In 1D random walk the particle always returns to the origin and similarly the fidelity probability distribution $\sim F^{\sigma_F^2}$ for the 1D parametric case which exhibits a maximum at $F = 1$, i.e., although the qubit motion is influenced by the random noise, the wave function still returns with

maximum probability to the ground state. In a 2D random walk the classic result [51] is that the particle after some elapsed time returns to the origin *a.s.* but with maximum probability at annulus displaced from the origin. Similarly in our case the maximum of the fidelity probability distribution $\sim(1 - F)F^{2\sigma_F^2}$ is slightly shifted away from $F = 1$. Finally, we show that errors in fidelity occur at a higher order in noise intensity compared with errors in qubit rotation angle.

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