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Entanglement of electron pairs extracted from a many-body system

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Abstract – Entanglement of spins is analyzed for two electrons extracted from a mixed many-electron state by projecting onto the two-electron subspace. The concurrence formulae are expressed in a compact form for states with a well-defined square of the total spin projection. As an example, the thermal entanglement for a qubit pair with an anisotropic Heisenberg and the Dzyaloshinskii-Moriya interactions in an inhomogeneous magnetic field is given analytically. Remarkably, the concurrence of a pair of electrons with antiparallel spins and in a delocalised orbital state is given by the scalar product of the state with its spin-flipped state and not with the time-reversed state.

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Quantum entanglement is considered a key resource for quantum cryptography and quantum computation [1]. Quantifying entanglement [2,3] and identifying maximally entangled states is thus important in the planning of devices. On the other hand, the research of the entanglement and its decoherence can also lead to a better understanding of the foundations of physics, for example of the quantum to classical crossover [4] and of the origins of the thermodynamic laws [5].

Quantum dot arrays, controlled by electrical gating, are, due to their scalability, promising candidates for operational devices [6,7]. The building blocks are coupled few-electron quantum dots enabling full control over individual electrons [8,9]. In a quantum dot the qubit is usually represented by the spin of an electron. Alternatively, charge pseudo-spin entanglement in double quantum dots can also be exploited [10], but is prone to the decoherence due to the Coulomb interaction with the environment.

The description of electrons by the spin degrees of freedom only is a simplification valid when the electrons are localized and the charge fluctuations are negligible. In general, both orbital and spin degrees of freedom are present, but if one is interested in the spin entanglement only, one should trace out the spatial dependence. Typical examples are a recently proposed route to generation of

perfectly entangled electron pairs by the use of acoustic waves in the surface of a GaAs/AlGaAs structure [11] or elastic scattering of electrons in semiconducting carbon nanotube structures with orbital degeneracy [12]. For a special case, where there are precisely two electrons in a pure state on the lattice, the entanglement can be given by simple formulae expressed in terms of the wave function [13]. Such formulae can be applied to the determination and optimization of entanglement generation between static and flying qubits in one-dimensional systems [14,15]. They also enable the analysis of the entanglement between qubit pairs in various double-quantum-dot configurations coupled to external leads [16].

Here we are interested in the entanglement of an electron pair extracted from a many-body state, which can be, for example, an open system of interacting electrons in a solid-state structure of several coupled quantum dots. In particular, we take that the measurement apparatus extracts precisely one electron from each of two non-overlapping regions of the structure —domains A and B . The state of the system is arbitrary and includes fluctuations of electrons between the domains or between the domains and the environment, which introduces spin and charge fluctuations to the subsystem $A \cup B$.

In this letter we express the reduced density matrix of two spin-qubits in terms of projected spin-spin correlators which allows the analysis of entanglement of qubit pairs

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extracted from a general many-electron state. The corresponding concurrence is then given explicitly for systems conserving the square of the total spin projection, which is illustrated by several examples.

First consider two separated electrons, one from domain A and the other from domain B , with spin states labeled by $s = \pm\frac{1}{2}$ and $t = \pm\frac{1}{2}$, respectively. Let the electrons be in a pure state expressed in the standard basis $|\mu\rangle \equiv |st\rangle \in \{|\uparrow\uparrow\rangle, |\uparrow\downarrow\rangle, |\downarrow\uparrow\rangle, |\downarrow\downarrow\rangle\}$ for $\mu \in \{1, 2, 3, 4\}$, as

$$|\psi\rangle = \sum_{\mu} \alpha_{\mu} |\mu\rangle. \quad (1)$$

Because the electrons are in a state completely determined by the spin degrees of freedom only, the entanglement can be quantified with the entanglement of formation or, equivalently, with the concurrence $C = 2|\alpha_1\alpha_4 - \alpha_2\alpha_3|$ [17]. In general, two spins may be a subsystem of a larger system with many degrees of freedom and the subsystem is then described by a reduced 4×4 density matrix ρ . In this case the concurrence is given by the Wootters formula [18],

$$C = \max \left(0, 2\lambda_{max} - \sum_{j=1}^4 \lambda_j \right), \quad (2)$$

where λ_j are the square roots of the eigenvalues of the non-Hermitian matrix $\rho\tilde{\rho}$ among which λ_{max} is the largest, and $\tilde{\rho}$ is the time-reversed density matrix ρ .

In this paper we consider the electrons in domains A and B as a subsystem of a total system described by a density matrix

$$\rho_{tot} = \sum_n p_n |n\rangle\langle n|. \quad (3)$$

Then a projective measurement is performed by an apparatus which extracts an electron pair: one electron from A and another one from B in such a way that after the projection, the system is in one of the normalized states $\mathcal{P}|n\rangle/\sqrt{\langle n|\mathcal{P}|n\rangle}$, where the projector \mathcal{P} projects onto the subspace where in each of the domains A and B there is exactly one electron. What we are left with is the density matrix

$$\rho_{\mathcal{P}} = \sum_n q_n \frac{\mathcal{P}|n\rangle\langle n|\mathcal{P}}{\langle n|\mathcal{P}|n\rangle} = \frac{\mathcal{P}\rho_{tot}\mathcal{P}}{\text{Tr}\mathcal{P}\rho_{tot}}, \quad (4)$$

where $q_n = P(n|\mathcal{P})$ is the conditional probability that after the measurement the system will be in the projected state $\mathcal{P}|n\rangle$, *i.e.*, that the electrons were extracted from the state $|n\rangle$. This probability is given by Bayes' formula [1], $P(n|\mathcal{P}) = P(\mathcal{P}|n)p_n/\langle\mathcal{P}\rangle$, where $P(\mathcal{P}|n) = \langle n|\mathcal{P}|n\rangle$ is the conditional probability for a single occupancy of each of the domains for a particular state $|n\rangle$ and $\langle\mathcal{P}\rangle = \sum_n p_n \langle n|\mathcal{P}|n\rangle = \text{Tr}\mathcal{P}\rho_{tot}$ is the probability that the apparatus will click, *i.e.*, that the desired two electrons will be extracted.

The projected states $\mathcal{P}|n\rangle$ read

$$\mathcal{P}|n\rangle = \sum_{ij, st} \psi_{n,ij}^{st} c_{is}^{\dagger} c_{jt}^{\dagger} |\Phi_{n,ij}^{st}\rangle, \quad (5)$$

where the operators c_{is}^{\dagger} and c_{jt}^{\dagger} create electrons with spin s at sites $i \in A$ and with spin t at sites $j \in B$, respectively, and, being ordinary electron creation operators, obey the fermionic rule $c_{is} c_{i's'}^{\dagger} + c_{i's'}^{\dagger} c_{is} = \delta_{ii'} \delta_{ss'}$.

The number of sites within the domains is arbitrary. States $|\Phi_{n,ij}^{st}\rangle$ are normalized and represent empty domains A and B with the rest of the system in an arbitrary configuration. In general, these vacuum states may be different for each of the states $|n\rangle$ and also for each particular occupation of pairs of sites (i, j) within the domains. The projector \mathcal{P} removes from $|n\rangle$ all components except those where each of the domains is occupied by precisely one electron and may be written explicitly as

$$\mathcal{P} = \prod_{k=0, k \neq 1}^{N_A} \frac{k - \hat{n}_A}{k - 1} \prod_{k=0, k \neq 1}^{N_B} \frac{k - \hat{n}_B}{k - 1}, \quad (6)$$

where $\hat{n}_{A(B)} = \sum_{l \in A(B), s} c_{ls}^{\dagger} c_{ls}$ is the number operator for domains $A(B)$ and $N_{A(B)}$ is the maximum possible number of electrons in $A(B)$.

Being interested in the spin entanglement we consider the reduced density matrix where only the spin degrees of freedom are retained, $\rho = \sum_{\mu\nu} \rho_{\mu\nu} |\mu\rangle\langle\nu|$, with $\rho_{\mu\nu} \equiv \rho_{(st)(s't')}$ and

$$\rho_{(st)(s't')} = \frac{1}{\langle\mathcal{P}\rangle} \sum_{n, ij} p_n \langle \Phi_{n,ij}^{s't'} | \Phi_{n,ij}^{st} \rangle (\psi_{n,ij}^{s't'})^* \psi_{n,ij}^{st}. \quad (7)$$

This formula is useful if the wave functions are known. However, in some cases it is possible to determine various correlation functions of the system without an explicit knowledge of the wave functions. Then it is advantageous to express the density matrix in terms of spin correlators¹

$$\rho = \frac{1}{\langle\mathcal{P}\rangle} \begin{pmatrix} \langle P_A^{\uparrow} P_B^{\uparrow} \rangle & \langle P_A^{\uparrow} S_B^{-} \rangle & \langle S_A^{-} P_B^{\uparrow} \rangle & \langle S_A^{-} S_B^{-} \rangle \\ \langle P_A^{\uparrow} S_B^{+} \rangle & \langle P_A^{\uparrow} P_B^{\downarrow} \rangle & \langle S_A^{-} S_B^{+} \rangle & \langle S_A^{-} P_B^{\downarrow} \rangle \\ \langle S_A^{+} P_B^{\uparrow} \rangle & \langle S_A^{+} S_B^{-} \rangle & \langle P_A^{\downarrow} P_B^{\uparrow} \rangle & \langle P_A^{\downarrow} S_B^{-} \rangle \\ \langle S_A^{+} S_B^{+} \rangle & \langle S_A^{+} P_B^{\downarrow} \rangle & \langle P_A^{\downarrow} S_B^{+} \rangle & \langle P_A^{\downarrow} P_B^{\downarrow} \rangle \end{pmatrix},$$

where $\langle\mathcal{P}\rangle = \sum_{st} \langle P_A^s P_B^t \rangle$ is the probability that in the subsystem $A \cup B$ there will be precisely two electrons, one in each of the domains. The correlators are expressed as the expectation values of projected operators in the sense $\langle\mathcal{O}\rangle \equiv \sum_n p_n \langle n|\mathcal{P}\mathcal{O}\mathcal{P}|n\rangle$, where \mathcal{O} consists of A - B pairs of operators

$$S_{A(B)}^{x,y,z} = \frac{1}{2} \sum_{l \in A(B), ss'} \sigma_{ss'}^{x,y,z} c_{ls}^{\dagger} c_{ls'},$$

$$P_{A(B)}^s = \sum_{l \in A(B)} \hat{n}_{l,s} (1 - \hat{n}_{l,-s}).$$

¹The application of correlators to the present case of indistinguishable fermions is a straightforward generalization of the approach applied in the case of spin systems [19].

Here $\sigma_{ss'}^{x,y,z}$ are the Pauli matrices, $S_{A(B)}^{\pm} = S_{A(B)}^x \pm iS_{A(B)}^y$ and $\hat{n}_{l,s} = c_{ls}^{\dagger}c_{ls}$ is the electron number operator.

The evaluation of the correlators is simplified if each of the domains A and B consists of one site only, in which case $\mathcal{P}\mathcal{O}\mathcal{P} = \mathcal{O}$, *i.e.*, the states corresponding to empty or doubly occupied sites are projected away by the operator \mathcal{O} . A prototype example is a Hubbard dimer, *i.e.*, two electrons on two sites and described by the Hubbard model, as studied by Zanardi [20]. Note, however, that the entanglement measures introduced for fermionic systems where multiple occupancy is retained [20–24] are different from the entanglement of formation studied here.

The concurrence for the domains A and B is determined from the projected density matrix by the Wootters formula, eq. (2). In general, λ_j can be computed numerically, but in some cases, due to symmetry the density matrix simplifies and analytic evaluation is possible. Such symmetries were exploited in various coupled spin systems on a lattice with translational and parity invariance [19,25–27].

In the present case of interacting electrons in coupled-quantum-dot structures, translational and parity invariance is an exception. Still, an analogous simplification is possible in a special case when the density operator commutes with the square of the total spin projection for $A \cup B$, $S^z = S_A^z + S_B^z$. For such *biaxial* systems [28] with

$$[\rho, (S^z)^2] = 0, \quad (8)$$

ρ is a block matrix: $\rho_{12} = \rho_{13} = \rho_{24} = \rho_{34} = 0$ or, equivalently, $\langle S_{A(B)}^{x,y} \rangle = 0$ and $\langle S_{A(B)}^z S_{B(A)}^{x,y} \rangle = 0$.

The eigenvalues λ_j^2 of $\rho\tilde{\rho}$ (which, again, is a block matrix) follow trivially from two decoupled blocks corresponding to subspaces with parallel, $(S^z)^2 = 1$ and $\{\mu = 1, \nu = 4\}$, or antiparallel spins, $(S^z)^2 = 0$ and $\{\mu = 2, \nu = 3\}$. The matrix elements $\rho_{\mu\nu}$ are interrelated [29], $|\rho_{\mu\nu}| \leq \sqrt{\rho_{\mu\mu}\rho_{\nu\nu}}$, which leads to $\lambda_j = \sqrt{\rho_{\mu\mu}\rho_{\nu\nu}} \pm |\rho_{\mu\nu}|$. The concurrence is then determined by

$$C = \max(0, C_{\uparrow\downarrow}, C_{\parallel}) / \sum_{st} \langle P_A^s P_B^t \rangle, \quad (9)$$

$$C_{\uparrow\downarrow} = 2|\langle S_A^+ S_B^- \rangle| - 2\sqrt{\langle P_A^{\uparrow} P_B^{\uparrow} \rangle \langle P_A^{\downarrow} P_B^{\downarrow} \rangle},$$

$$C_{\parallel} = 2|\langle S_A^+ S_B^+ \rangle| - 2\sqrt{\langle P_A^{\uparrow} P_B^{\downarrow} \rangle \langle P_A^{\downarrow} P_B^{\uparrow} \rangle},$$

which represents a generalisation of the result derived for the case of precisely two delocalised electrons in a pure state [13].

For *axially* symmetric systems, *i.e.*, conserving the total spin projection, $[\rho, S^z] = 0$, the formula simplifies because $\langle S_A^+ S_B^+ \rangle = 0$ and

$$C = \max(0, C_{\uparrow\downarrow}) / \langle \mathcal{P} \rangle. \quad (10)$$

For $SU(2)$ spin symmetric case, $[\rho, \mathbf{S}_A + \mathbf{S}_B] = 0$, the concurrence is completely determined by a single spin invariant and $C = 2 \max(0, -\langle \mathbf{S}_A \cdot \mathbf{S}_B \rangle / \langle \mathcal{P} \rangle - \frac{1}{4})$.

In practice, several specific cases are of interest. Let us first consider a special case of the total system being in a pure state $|m\rangle$ containing only two electrons, *i.e.*, $p_n = \delta_{nm}$ and $|\Phi_{m,ij}^{st}\rangle = |0\rangle$. We assume the electrons are in a state with the amplitudes $\psi_{m,ij}^{\uparrow\downarrow} = \alpha_2\varphi_{ij}$, $\psi_{m,ij}^{\downarrow\uparrow} = \alpha_3\varphi_{ij}$ and $\psi_{m,ij}^{\uparrow\uparrow} = \alpha_1\chi_{ij}$, $\psi_{m,ij}^{\downarrow\downarrow} = \alpha_4\chi_{ij}$, where φ_{ij} and χ_{ij} are normalized. Then, if $\varphi_{ij} = \chi_{ij}$, the concurrence is given by $C = 2|\alpha_1\alpha_4 - \alpha_2\alpha_3| / \langle \mathcal{P} \rangle$ with $\langle \mathcal{P} \rangle = \sum_{\mu} |\alpha_{\mu}|^2$, which is the pure spin-subsystem result, renormalized due to the projection. If $\sum_{ij} \varphi_{ij}^* \chi_{ij} = 0$ the concurrence is $C = 2|\alpha_1\alpha_4 - \alpha_2\alpha_3| / \langle \mathcal{P} \rangle$. Additionally, if the state $|m\rangle$ is an eigenstate of S^z , the concurrence simplifies further to $C = 2|\alpha_2\alpha_3| / \langle \mathcal{P} \rangle$.

An interesting case in point is a pure state with a zero-spin projection, $S^z|m\rangle = 0$, and $\langle \mathcal{P} \rangle = 1$. Such states are important, for example, in the realization of entangled flying qubit pairs, when two initially unentangled electrons approach each other and the interaction conserves S^z . The concurrence is given by $2|\langle m|S_A^+ S_B^-|m\rangle|$, but can also be expressed as

$$C = \sqrt{\langle \mathcal{F} \rangle^2 + 4\langle m|\mathbf{S}_A \times \mathbf{S}_B|m\rangle^2}. \quad (11)$$

Here $\langle \mathcal{F} \rangle = \langle m|m_{\text{flip}}\rangle$ is the scalar product of the state $|m\rangle$ with its spin-flipped state $|m_{\text{flip}}\rangle = \mathcal{F}|m\rangle$ where the spin-flip operator $\mathcal{F} = S_A^- S_B^+ + S_A^+ S_B^-$ reverses the spins in $A \cup B$.

For a special case $\langle \mathbf{S}_A \times \mathbf{S}_B \rangle = 0$, the expression (11) resembles the result for a general pure spin state, eq. (1), with the concurrence given by $C = |\langle m|\tilde{m}\rangle|$, where $|\tilde{m}\rangle = \mathcal{T}|m\rangle$ is the time-reverse of $|m\rangle$ [18]. The time reversal operator is given by $\exp[-i\pi(S_A^y + S_B^y)]\mathcal{K}$, where \mathcal{K} is the complex conjugation operator [30], which for the present case of two electrons with antiparallel spins gives $\mathcal{T} = i\mathcal{F}\mathcal{K}$. For a special case of a pure spin state, where $\psi_{m,ij}^{\uparrow\downarrow} \propto \psi_{m,ij}^{\downarrow\uparrow}$ and $\rho^2 = \rho$, the conjugation has no effect and the concurrence takes the customary form $C = 2|\alpha_2\alpha_3| = 2|\alpha_2^*\alpha_3|$. In general, however, if each of the domains consists of at least two sites, the overlap of the state $|m\rangle$ with the spin-flipped state $|m_{\text{flip}}\rangle$ is different from its overlap with the time-reversed state $|\tilde{m}\rangle$ because the amplitudes in $|\tilde{m}\rangle$ are complex conjugated while in $|m_{\text{flip}}\rangle$ they are not. To be specific, let two electrons be in the state

$$|\Psi\rangle = \frac{1}{2}(c_{1\uparrow}^{\dagger}c_{3\downarrow}^{\dagger} + c_{1\downarrow}^{\dagger}c_{3\uparrow}^{\dagger} + ic_{2\uparrow}^{\dagger}c_{4\downarrow}^{\dagger} + ic_{2\downarrow}^{\dagger}c_{4\uparrow}^{\dagger})|0\rangle, \quad (12)$$

where the sites 1, 2 and 3, 4 represent the domains A and B , respectively. The correct expression for the concurrence is $C = |\langle \Psi|\Psi_{\text{flip}}\rangle| = 1$ while the scalar product with the time-reversed state is $\langle \Psi|\tilde{\Psi}\rangle = 0$. Thus, in general the entanglement of electron pairs is not related to the scalar product of a state with its time reverse but to the spin-flipped state only. This apparent disagreement with the pure spin result is no paradox; it should simply be a warning and a demonstration that the electron pair with a general orbital extend can not be described by a pure spin state.

Finally, we present an example where the biaxial symmetry $[\rho, (S^z)^2] = 0$ arises naturally in a system in thermal equilibrium. Consider a pair of electrons in domains A and B representing a weakly coupled-double-quantum-dot structure where charge fluctuations between A and B and with the rest of the system are negligible (*i.e.*, $\langle \mathcal{P} \rangle = 1$). Besides the Coulomb interaction, the spin-orbit interaction may also be present and a general effective Hamiltonian can contain the anisotropic Heisenberg exchange interaction, the Dzyaloshinskii-Moriya term [31] and the coupling to an inhomogeneous external magnetic field,

$$\begin{aligned}
H = & J_x S_A^x S_B^x + J_y S_A^y S_B^y + J_z S_A^z S_B^z \\
& + \mathbf{B}_A \cdot \mathbf{S}_A + \mathbf{B}_B \cdot \mathbf{S}_B \\
& + \mathbf{D} \cdot (\mathbf{S}_A \times \mathbf{S}_B) + \mathbf{S}_A \cdot \mathbf{T} \mathbf{S}_B.
\end{aligned} \quad (13)$$

We assume the magnetic field and \mathbf{D} are parallel to the z -axis and a symmetric tensor \mathbf{T} is of the form

$$\mathbf{T} = \begin{pmatrix} 0 & t & 0 \\ t & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

In the basis $\{|\mu\rangle\}$ the Hamiltonian takes the form

$$H = \begin{pmatrix} h_{11} & 0 & 0 & h_{14} \\ 0 & h_{22} & h_{23} & 0 \\ 0 & h_{23}^* & h_{33} & 0 \\ h_{14}^* & 0 & 0 & h_{44} \end{pmatrix}, \quad (14)$$

which is the most general form of a Hamiltonian commuting with $(S^z)^2$. The corresponding density matrix describing thermal equilibrium is given by $\rho = \exp(-\beta H) / \text{Tr}[\exp(-\beta H)]$, where β is the inverse temperature. The matrix elements of the Hamiltonian equation (14) are related to eq. (13) as $h_{11(44)} = J_z/4 \pm (B_A + B_B)/2$, $h_{14} = (J_x - J_y)/4 + it/2$, $h_{22(33)} = -J_z/4 \pm (B_A - B_B)/2$, $h_{23} = (J_x + J_y)/4 + iD/2$ with $\mathbf{B}_{A(B)} = B_{A(B)} \hat{\mathbf{z}}$ and $\mathbf{D} = D \hat{\mathbf{z}}$.

The problem decouples into two 2×2 subsystems and the resulting two pairs of eigenenergies are given analytically,

$$\epsilon_{\{\mu\nu\}} = (h_{\mu\mu} + h_{\nu\nu})/2 \pm x_{\{\mu\nu\}},$$

$$x_{\{\mu\nu\}} = [(h_{\mu\mu} - h_{\nu\nu})^2/4 + |h_{\mu\nu}|^2]^{1/2},$$

where the subscript $\{\mu\nu\}$ denotes $\{14\}$ and $\{23\}$ for subspaces with $(S^z)^2 = 1$ and $(S^z)^2 = 0$, respectively. With the corresponding sets of eigenvectors, the density matrix elements are easily expressed,

$$\begin{aligned}
|\rho_{\mu\nu}| &= \frac{|h_{\mu\nu}| \sinh \beta x_{\{\mu\nu\}}}{Z x_{\{\mu\nu\}}} e^{-\beta(h_{\mu\mu} + h_{\nu\nu})/2}, \\
\rho_{\mu\mu} \rho_{\nu\nu} &= |\rho_{\mu\nu}|^2 + Z^{-2} e^{-\beta(h_{\mu\mu} + h_{\nu\nu})},
\end{aligned}$$

with

$$\begin{aligned}
Z = & 2e^{-\beta(h_{11} + h_{44})/2} \cosh \beta x_{\{14\}} \\
& + 2e^{-\beta(h_{22} + h_{33})/2} \cosh \beta x_{\{23\}}.
\end{aligned}$$

The concurrence is given by eq. (9), $C = 2 \max(0, |\rho_{23}| - \sqrt{\rho_{11}\rho_{44}}, |\rho_{14}| - \sqrt{\rho_{22}\rho_{33}})$.

Some particular cases of this problem, *i.e.*, an isotropic Heisenberg model in a magnetic field and spin Hamiltonians with the Dzyaloshinskii-Moriya interaction [32–34], were analyzed before. For $J_x = J_y$ and $t = 0$ the qubit pair is axially symmetric and the concurrence simplifies to eq. (10) with

$$C_{\uparrow\downarrow} = \left(|h_{23}| \frac{\sinh \beta x_{\{23\}}}{x_{\{23\}}} e^{-\beta J_z/4} - e^{\beta J_z/4} \right) / Z. \quad (15)$$

In conclusion, we analyzed the spin entanglement of electron pairs extracted from a system of electrons in a mixed state by projecting it onto the two-electron subspace. The entanglement is quantified by the concurrence obtained from the reduced density matrix which is expressed in terms of projected spin-spin correlators for the measurement domains and normalized by the probability that in each of the two measurement domains there is precisely one electron. The formalism is appropriate for the analysis of the entanglement of formation for the domains of open fermionic systems allowing charge fluctuations of the subsystems, as is, *e.g.*, a system of coupled quantum dots attached to external leads.

Simplified expressions are derived for systems with a good square of spin projection. The result for the most general case of the corresponding two-qubit system is given analytically, which generalizes particular known cases. As an example, the thermal state of a double quantum dot with anisotropic Heisenberg and Dzyaloshinskii-Moriya interactions and in an inhomogeneous magnetic field is considered. The concurrence is presented in a simple closed form.

Also considered is an electron pair in a pure, but orbitally delocalized state. An example, relevant to the analysis of solid-state realizations of flying and static qubits, demonstrates that the concurrence is given as the overlap of the state with its spin-flipped state—but not complex conjugated—therefore not by its time reverse as is the case for pure spin states.

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