THE PHYSICAL HERITAGE OF SIR W.R. HAMILTON

André Gsponer and Jean-Pierre Hurni

Independent Scientific Research Institute
Box 30, CH-1211 Geneva-12, Switzerland.
e-mail: isri@vtx.ch

Presented at the Conference:

The Mathematical Heritage of
Sir William Rowan Hamilton

commemorating the sesquicentennial of
the invention of quaternions

Trinity College, Dublin, 17th – 20th August, 1993

Abstract

150 years after the discovery of quaternions, Hamilton's conjecture that quaternions are a fundamental language for physics is reevaluated and shown to be essentially correct, provided one admits complex numbers in both classical and quantum physics, and accepts carrying along the intricacies of the relativistic formalism. Examples are given in classical dynamics, electrodynamics, and quantum theory. Lanczos's, Einstein's, and Petiau's generalizations of Dirac's equation are shown to be very naturally formulated with biquaternions. The discussion of spin, isospin, and mass quantization is greatly facilitated. Compared with other formalisms, biquaternions have the advantage of giving compact but at the same time explicit formulas which are directly usable for algebraic or numerical calculations.

Notations

In general scalars, vectors and quaternions are represented by the following types :

- Scalars (elements of $\mathbb{R}$ or $\mathbb{C}$) :
  Preferably lower case Roman or Greek: $a, b, c, \alpha, \beta ...$
- Vectors (elements of $\mathbb{R}^3$ or $\mathbb{C}^3$) :
  Any arrowed character: $\vec{a}, \vec{B}, \vec{V}, \vec{\omega}, \vec{\pi}, ...$
- Quaternions (elements of $\mathbb{H}$ or $\mathbb{B}$) :
  Preferably upper case roman or greek: $A, B, C, \Lambda, \Sigma ...$

Please distinguish :

- B electromagnetic field bivector : $B = [0; \vec{E} + i\vec{B}]$
- $\vec{B}$ magnetic induction vector (arrowed)
- $\mathbb{B}$ algebra of biquaternions ('mathbb' font as for $\mathbb{R}, \mathbb{C}$, and $\mathbb{H}$)

Square brackets are used to emphasize that bracketed quantities (e.g., $[x]$ for a variable, or [ ] for an operand within an expression) are quaternions, or to represent quaternions as $[scalar; vector]$ pairs :

- $[x] = [s, \vec{v}]$

Angle brackets are used to emphasize that bracketed quantities, e.g., $\langle S \rangle$, are scalars, or to restrict quaternions to their scalar part :

- $\langle s + \vec{v} \rangle = s$

Please note the use of the following operators :

- $\overline{(\,)}$ or $(\,)^-$ quaternion conjugation (bar or minus)
- $(\,)^*$ imaginary conjugation (star)
- $(\,)^+$ biconjugation (plus)
- $(\,)^\sim$ order reversal or ordinal conjugation (tilde) : $(AB)^\sim = BA$

- $(\,)^{=}$ transposition (transpose) : $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{=} = \begin{pmatrix} \overline{a} & \overline{c} \\ \overline{b} & \overline{d} \end{pmatrix}$
- $A \circ B$ scalar part of quaternions AB : $A \circ B = \langle AB \rangle = \mathbb{S}[AB]$
- $A \wedge B$ vector part of quaternions AB : $A \wedge B = \mathbb{V}[AB]$
- $\dot{(\,)}$ proper time total derivative (dot) : $\dot{(\,)} = \gamma \frac{d}{dict}(\,)$

With c the velocity of light (we use generally the convention $c = 1$) the most important four-dimensional physical quaternions are :

- four-position : $\mathcal{X} = [ict; \vec{x}]$
- four-gradient : $\nabla = [\frac{\partial}{\partial ict}; \frac{\partial}{\partial \vec{x}}] = [\partial_{ict}, \vec{\nabla}]$
- four-velocity : $\mathcal{U} = \gamma[1; -i\vec{\beta}]$
- energy-momentum : $\mathcal{P} = [E; -ic\vec{p}]$
- generalized momentum : $\Pi = [H; -ic\vec{\pi}]$
- electromagnetic four-potential : $A = [V; -i\vec{A}]$
- charge-current density : $J = [\rho; -i\vec{j}]$

Lorentz transformation quaternions operators are ‘mathcal’ characters :

- Spinor-rotation : $\mathcal{R}(\,) = \exp(\frac{1}{2}\theta\vec{a})[\,] = [\cos(\frac{1}{2}\theta); \sin(\frac{1}{2}\theta)\vec{a}][\,]$
- Spinor-boost : $\mathcal{B}(\,) = \exp(i\frac{y}{2}\vec{b})[\,] = [\cosh(\frac{y}{2}); i\sinh(\frac{y}{2})\vec{b}][\,]$
- Spinor-Lorentz : $\mathcal{L}(\,) = \mathcal{B}\mathcal{R}[\,]$

Quaternionic and quantum mechanical scalar products :

- Angle “bra” and “ket” delimiters are used to emphasize the range of symbols over which a scalar part is calculated :
  $$\langle A|B \rangle = \langle AB \rangle = \mathbb{S}[AB]$$
- Formal quantum mechanical Hilbert spaces’s scalar products are emphasized by using double “bra-ket” symbols, i.e., $\langle\langle ... \rangle\rangle$ instead of $\langle ... \rangle$. E.g.:
  $$\langle\langle \psi | ... | \psi \rangle\rangle \;=\; \iiint \langle d^3V \; \psi^{+}(...)\psi \rangle$$
  where $(\,)^{+}$ is biconjugation, not Hermitian conjugation denoted by $(\,)^{\dagger}$.

THE PHYSICAL HERITAGE OF SIR W.R. HAMILTON

Andre Gsponer and Jean-Pierre Hurni

Abstract

150 years after the discovery of quaternions, Hamilton's conjecture that quaternions are a fundamental language for physics is reevaluated and shown to be essentially correct, provided one admits complex numbers in both classical and quantum physics, and accepts carrying along the intricacies of the relativistic formalism. Examples are shown in classical dynamics, electrodynamics and quantum theory. Lanczos's, Einstein's, and Petiau's generalizations of Dirac's equation are shown to be very naturally formulated with biquaternions. The discussion of spin, isospin, and mass quantization is greatly facilitated. Compared with other formalisms, biquaternions have the advantage of giving compact but at the same time explicit formulas which are directly usable for algebraic or numerical calculations.

Content

1 Introduction

For a contemporary physicist, the name Hamilton is primarily associated with what is known as the "Hamiltonian" formulation of dynamics. Although mathematically equivalent to other formulations such as the Lagrangian formalism, Hamilton's method provides a description of a classical system which has the considerable advantage that the problem can easily be "quantized," i.e., generalized from classical to quantum physics. For this reason, in all text books, the operator H appearing on the right-hand side of the Schrödinger wave equation

$$i\hbar\frac{\partial}{\partial t}\Psi = H\Psi \tag{1}$$

is called the "Hamilton operator" or, simply, the "Hamiltonian."

On its own, this formulation of dynamics discovered by Hamilton in 1834 (which allows problems of optics and problems of mechanics to be worked out interchangeably) is enough to put Hamilton among the greatest physicists of all times, at least at the level of Newton or Maxwell, and not very far from Einstein. At this Conference, however, we celebrate another major discovery of Hamilton: the invention of quaternions in 1843.

But quaternions were not Hamilton's only important discovery in algebra: complex numbers were first. Indeed, in 1835, Hamilton had already found a mathematically appealing and consistent way of interpreting the so-called "imaginary numbers." By considering pairs of ordinary numbers, and defining a suitable multiplication rule, he showed that all operations that could be made with ordinary numbers could also be made with his number doublets.

However, as it often happens in the mysterious process that we call "discovery," Hamilton had a peculiar mental image in his mind when he was thinking of algebra: While he imagined geometry of a science of "space," he conceived algebra as the science of "pure time" [1], and therefore understanding imaginary numbers meant for him coming closer to understanding the essence of time. Thus, when Hamilton was thinking of algebra, his mental image was that of a physicist, an image of somebody whose ambition is to discover the laws of inanimate nature and motion.

Therefore, when after many unsuccessful attempts Hamilton finally succeeded in generalizing complex numbers to quaternions (which require for their representation not just two but four ordinary numbers) he definitely believed to have made a very important discovery. This conviction, however, Hamilton would not include in his scientific writings. But in his correspondence,[1] and in his poetry,[2] Hamilton made it plain that he really thought he had discovered some synthesis of three-dimensional space, the vector-part, and time, the scalar-part of the quaternion.

The astonishing fact is that indeed quaternions do foreshadow “our four-dimensional world, in which space and time are united into a single entity, the space-time world of Einstein’s Relativity” [Ref.2, page 136]. In effect, as science advances, more and more evidence accumulates, showing that essentially all fundamental physics results can easily and comprehensibly be expressed in the language of quaternions. If that is so, then the often-made criticism that Hamilton had “exaggerated views on the importance of quaternions” [Ref.2, page 140] was ill-founded. And therefore, contrary to what is often said, Hamilton was right to have have spent the last twenty-two years of his life studying all possible aspects of quaternions.

However, what Hamilton did not know, and could not have known at his time, is that quaternions would only become really useful in applied and theoretical physics when problems are dealt with in which relativistic and quantum effects play an essential role. In such problems, in effect, it is not so much the "real quaternions" Hamilton was studying that are useful, but the so-called *biquaternions*, which are obtained when every four components of the quadruplet are allowed to become complex numbers. This is not to say that Hamilton’s work on real quaternions was vain. Quite the contrary: most algebraic properties of real quaternions that Hamilton so carefully studied can be carried over to biquaternions.

But, for the more practical purposes of non-relativistic or non-quantum physics and engineering, it is true that operations with quaternions are not sufficiently flexible. We are therefore in a fortunate position today, that after the simplification of the somewhat cumbersome notations used by Hamilton, we have at our disposal the modern vector notation introduced by J.W. Gibbs. Using this notation, we can now work with quaternions much more easily than with Hamilton’s original notation.[3] In particular, either we separate the quaternion into its scalar and vector parts, separating or mixing freely "scalars" and "vectors" [3], or use it as a whole, especially when we deal with the fundamental aspects of theoretical physics, as in this paper.

[1]“My letter relates to a certain synthesis of the notions of Time and Space, ...” Cited in Ref.1, page 149.

[2]“And how the One of Time, of Space the Three, Might in the Chain of Symbol, girdled be.” Cited in Ref.1, page 192.

[3]When using the modern vector notation with quaternions it is important to keep in mind Gibbs’s redefinition of the sign of the scalar product: $\vec{a}\cdot\vec{b} = -\langle\vec{a}\vec{b}\rangle = -\vec{a}\circ\vec{b} = -\mathbb{S}[\vec{a}\vec{b}]$.

Hence, if we now anticipate what we will develop in the sequel, and jump to our conclusion, we are indeed going to show that while quaternions are not the panacea for solving all possible physical or mathematical problems, they do nevertheless provide an extraordinaryly powerful framework for any problem in which some four-dimensional or quantal aspect of our physical world intervenes.

That this is so, and why it is so, is mysterious. As Wigner stressed in his often quoted article entitled "The Unreasonable Effectiveness of Mathematics in the Natural Sciences" [4], the biggest mystery is, possibly, the fact that once a particularly efficient mathematical scheme has been found for the description of some often crude physical experiment, it turns out that the same mathematical tool can be used to give an amazingly accurate description of a large class of phenomena. This is what we have discovered: once the biquaternion formalism is taken seriously as a language for expressing fundamental physical laws, it so happens that more and more phenomena can be predicted by simply generalizing the accepted results while staying within the frameworks of quaternions.

To do so, one has however to follow a few simple guide-lines. For instance, the time variable "it"[4] should always be written as a pure imaginary number and, consequently, derivation with respect to time should always be written $\frac{d}{dit}$. Hence, the fundamental space-time variable $\mathcal{X}$ and the corresponding four-gradient ∇ will always be written as

$$\mathcal{X} = [ict; \vec{x}] \quad , \tag{2}$$

$$\nabla = [\frac{\partial}{\partial ict}; \frac{\partial}{\partial \vec{x}}] \quad . \tag{3}$$

A second rule is that there should be no *hidden* "i"; in other words, that the imaginary unit "i" should always be explicit, and that imaginary conjugation should always apply to all "i"s. This means that contrary to the convention of some physicists [3,5] one should not use the so-called "Hermitian-" or "Pauli-units," but only the real quaternion units defined by Hamilton, which together with the scalar "1" have the advantage to form a closed four-element group, which is not the case with the "Pauli-units."

If these two rules are followed, one discovers that there is one and only one "i" in physics and in mathematics; that imaginary conjugation can always be given a consistent interpretation (either in classical or quantum physics); and that while "i"[5] is necessary in quantum theory, "i" is also a very useful symbol in classical physics because it often contributes to distinguish quantities which are of a different physical nature.

[4]Or "ict," where c is the velocity of light, to make explicit that the time and space variables have different physical dimensions.

[5]Or, more precisely, a "complex structure" [6].

To conclude this introduction, let us summarize our main point: if quaternions are used consistently in theoretical physics, we get a comprehensive and consistent description of the physical world, with relativistic and quantum effects easily taken into account. In other words, we claim that *Hamilton's conjecture*, the very idea which motivated more then half of his professional life, i.e., the concept that somehow quaternions are a fundamental building block of the physical universe, appears to be essentially correct in the light of contemporary knowledge.

2 Some definitions and properties of quaternion automorphisms

Let us review some definitions and properties of the four basic quaternion linear automorphisms, i.e., the non-trivial involutions of $\mathbb{H}$, the field of quaternions, or $\mathbb{B}$ the algebra of biquaternions.

The first two are *quaternion conjugation*, which reverses the sign of the vector part, and *imaginary conjugation*, which replaces the scalar and vector parts by their imaginary conjugate

$$Q \to \overline{Q} = [s; -\vec{v}] \quad , \tag{4}$$

$$Q \to Q^* = [s^*; \vec{v^*}] \quad . \tag{5}$$

Quite often in practice $\overline{()}$ and $()^*$ are used in combination. Following Hamilton's usage of the prefix <u>bi</u>- we call this third involution *biconjugation*[6] and use for it the symbol $()^+$

$$Q \to Q^+ = [s^*; -\vec{v^*}] = (\overline{Q})^* \quad . \tag{6}$$

In the same spirit we call a complex vector a bivector (rather than a "six-vector"), but we will refrain from using the term "biscalar" suggested by Hamilton for complex numbers.

Using these three involutions we have the following definitions:

- Q is a scalar if $\overline{Q} = Q$
- Q is a vector if $\overline{Q} = -Q$
- Q is real if $Q^* = Q$
- Q is imaginary if $Q^* = -Q$
- Q is bireal if $Q^+ = Q$
- Q is antibireal if $Q^+ = -Q$

[6]Rather than "Hermitian conjugation," symbol $()^\dagger$, as it is often improperly called.

When operating on a quaternion expression, quaternion conjugation reverses the order of the factors. Thus

$$\overline{AB} = \overline{B}\,\overline{A} \quad \text{and} \quad (AB)^+ = B^+A^+ \quad . \tag{7}$$

The last non-trivial involution, *order reversal* (or *ordinal conjugation*), is more subtle and requires some explanations for which it is best to return to Hamilton's "Elements of Quaternions," and more specifically to a note added by C.J. Joly in 1898, at the end of section ten [7, Vol.I, p.162].

Starting from the set of quadruplets of real or complex numbers, the quaternion algebra is obtained by requiring their product to be associative, and the division to be feasible always, except possibly in some singular cases. Then, writing two quadruplets A and B as scalar-vector doublets $[a;\vec{a}]$ and $[b;\vec{b}]$, and using contemporary vector notations, their product has the following explicit form

$$[a;\vec{a}][b;\vec{b}] = [\ \vec{a}\vec{b} + p\ \vec{a}\cdot\vec{b}\ ;\ a\vec{b} + \vec{a}b + q\ \vec{a}\times\vec{b}\ ] \quad . \tag{8}$$

The two constants p and q are related by the equation

$$q^2 + p^3 = 0 \tag{9}$$

which shows that there is some residual arbitrariness when defining the product of two quadruplets. For instance, taking $p = -1$, q can be equal to either $+1$ or -1. On the other hand, taking $p = +1$, q may be $+i$ or $-i$. Thus, the choice $p = +1$ corresponds to the Pauli algebra. But, as we have already said, we will keep Hamilton's choice, $p = -1$, which is also more fundamental because it corresponds to the Euclidian metric in the case of real quaternions, and to Minkowski's metric in the case where the four-dimensional space-time position vector is written as in formula (2). Moreover, with Hamilton's choice, the imaginary conjugate of a product is equal to the product of the imaginary conjugate of the factors: this dispenses of special rules which are sometimes necessary when using "Pauli units."[7]

In short, the arbitrariness in the sign of p is connected with the signature of the metric, and the choice of the metric determines the sign of p and the usage of "i" when defining physically meaningful four-dimensional quantities. In effect, since the square of the norm of a quaternion A is by definition its product by its conjugated quaternion, we have $|A|^2 = A\overline{A} = a^2 + p|\vec{v}|^2$. Therefore, for a given signature, the choice of the sign of p is immaterial because one can always multiply the vector part of all quaternions by "i" in order to get the desired signature.

[7]Such special rules are necessary in the standard "γ-matrices" formulation of Dirac's theory because they are based on the Pauli matrices which contain an algebraic $\sqrt{(-1)}$ which should not be mistaken with the "i" of the complex scalars that multiply the "γ"s.

The arbitrariness in the sign of q is due to the non-commutativity of the quaternion product. Indeed, changing the order of the factors A and B is equivalent to changing the sign of q. The involution associated with the changing of this sign is called "order reversal" (or simply "reversal") and is designated by the symbol $()^{\sim}$. When biquaternions are used to represent physical quantities in space-time, since q is the sign associated with the vector product, there is a close connection between order reversal and space inversion. However, contrary to the case of p, there appears to be no invariant overall criterion to decide for the sign of q.[8] Therefore, in accordance with the principle of relativity, one has to make sure that fundamental physical entities are "order-reversal covariant" (or simply "ordinal covariant"), i.e., that they do not arbitrarily depend on the sign of q.

In this respect a last point is of importance: whereas the problem of signature is common to all formalisms, order reversal is specific to quaternions and Clifford numbers,[9] and therefore something that should be carefully considered when any Clifford algebra is used in physics. For this reason, because reversal was not properly considered — for instance, in defining fundamental quantities as "ordinal invariant" — many authors using quaternions in physics have met with problems. Indeed, as we will see, reversal plays an essential role when writing fundamental equations of physics.

3 Irreducible representations of biquaternions: Spinors and bispinors.

Before going to the physical applications, let us remind (without proof) some elementary theorems concerning the irreducible decompositions of biquaternions.

(I) Any real quaternion $R = R^*$ such that $|R|^2 = 1$ can be written

$$R = \exp(\tfrac{\alpha}{2}\vec{a}) = [\cos(\tfrac{\alpha}{2}); \sin(\tfrac{\alpha}{2})\vec{a}] \tag{10}$$

where α is a real number called the angle and $\vec{a}$ a unit vector called the axis.

(II) Any bireal quaternion $B = B^+$ such that $|B|^2 = 1$ can be written

$$B = \exp(i\tfrac{y}{2}\vec{b}) = [\cosh(\tfrac{y}{2}); i\sinh(\tfrac{y}{2})\vec{b}] \tag{11}$$

[8] This observation relates to the experimental fact that while time flows in only one direction, space is not oriented.

[9] "Order reversal" is related (but in general not identical) to "reversion", one of the three basic involutions defined on any Clifford algebra.

where y is a real number called the rapidity and $\vec{b}$ a unit vector called the boost direction.[10]

(III) Any biquaternion Q with non-zero norm, i.e., $|Q|^2 \neq 0$, can be written

$$Q = cRB \tag{12}$$

where c is a complex number, R a normed real quaternion, and B a normed bireal quaternion.

(IV) Any biquaternion S with zero norm, i.e., $|S|^2 = 0$, in which case S is called *singular*[11], can be written as a product of three factors

$$Q = rR\sigma \tag{13}$$

where r is a real number, R a normed real quaternion and σ a primitive nullquat

$$\sigma = \tfrac{1}{2}[1; i\vec{\nu}] \tag{14}$$

where $\vec{\nu}$ is a real unit vector and σ has the property of being an *idempotent*, i.e., $\sigma^2 = \sigma$.

(V) Multiplying a nullquat from one side by any biquaternion does not change its primitive nullquat.

Since two real parameters are needed to fix the direction of a unit vector, we see from (10) and (11) that three parameters are necessary to represent a normed real or bireal quaternion. Similarly, a general biquaternion (12) requires eight parameters, while six suffice for a nullquat (13).

Theorem (V) provides the basis for defining spinors in the quaternion formalism. In effect, for a given primitive nullquat σ, the left- (or right-) ideal forms a four parameter group that is isomorph to the spin $\frac{1}{2}$ spinor group [9]. As a consequence, we have the *spin* $\frac{1}{2}$ *decomposition theorem* which establishes the link between Dirac's bispinors and biquaternions:

(VI) Relative to a given primary nullquat σ, any normed biquaternion Q can be written as a *bispinor*, i.e., as the sum of two conjugated *spinors*

$$Q = R_1\sigma + R_2\overline{\sigma} \tag{15}$$

where R_1, R_2, are two real quaternions.

[10] A bireal biquaternion is called a *minquat* by Synge [8].

[11] A singular biquaternion is called a *nullquat* by Synge [8].

4 Relativity and quaternionic tensor calculus

Usually, when special relativity is introduced, one does a lot of algebra in order to work out the somewhat complicated formulas of Lorentz transformations. Even when quaternions are used for this purpose, the proof of the equivalence of the quaternion formulas with the usual ones is rather complicated [10]. In the present paper, our intention is to develop the fundamental concepts and present the main results without giving the details of the proofs.

We start therefore from the fundamental ideas of relativity, covariance, and tensor calculus which are that all observers are equivalent for writing the physical laws, and that all meaningful physical quantities should have well defined transformation properties when going from one observer to another one.[12] Hence, if Hamilton's conjecture is correct, i.e., if indeed biquaternions can be used as elementary building blocs of theoretical physics, any meaningful physical quantity should be writable as a simple explicit quaternion expression which should have the same form in all *referentials*.[13] In other words, the components which in ordinary tensor calculus are represented by symbols such as $t_{ij...}^{kl...}$, where the various indices show how the physical quantity varies in a change of referential, should be replaced by quaternionic monomials $QRST...$ where the indices are replaced by some convention making the *variance*[14] of each factor in the monomial explicit.

Let us take, for example, the general Poincaré transformation law. This is a change of referential which corresponds to the affine function

$$Q' = AQB + C \tag{16}$$

where (a priori) A, B and C are any kind of quaternion expressions. If A and B are functions of the four-position vector $\mathcal{X}$, and the translation term C is zero, we have a local Lorentz transformation, and if A and B are independent of $\mathcal{X}$ a global Lorentz transformation.

The most basic tensor quantity is obviously the *four-vector* such as, for example, the space-time vector $\mathcal{X}$ given in formula (2). Four-vectors have the following variance

$$V' = \mathcal{L}V\mathcal{L}^+ \tag{17}$$

where $\mathcal{L}$ is restricted, by Einstein-Minkowski's condition

$$\overline{V}V = \overline{V'}V' = \text{invariant scalar}. \tag{18}$$

[12]For an enlightening introduction to tensor calculus and its relation to quaternions see [11]. See also Ref.2 page 140.

[13]I.e., "reference frames" (gallicism).

[14]I.e., "transformation law" (gallicism).

Using (17) we see that this condition implies $\overline{\mathcal{L}}\mathcal{L} = \mathcal{L}^*\mathcal{L}^+ = 1$. Thus, the quaternion $\mathcal{L}$ which represents the most general Lorentz transformation, is simply a biquaternion of unit norm. By theorem IV, such a transformation can therefore be decomposed into a product where $\mathcal{R}$ is a rotation and $\mathcal{B}$ a boost. To find the explicit form of the boost, we apply (17) to the velocity four-vector $\mathcal{U}$. Then, transforming from the rest-frame (in which $\mathcal{U} = 1$) to a moving frame, we find

$$\mathcal{U}' = \mathcal{B}\mathcal{B}^+ = \mathcal{B}\mathcal{B} = \gamma[1; -i\vec{\beta}] \tag{20}$$

where $\vec{\beta} = \vec{v}/c$ is the relative velocity of the moving frame and $\gamma = (1-\beta^2)^{-1/2}$ the Lorentz factor. We see therefore that the Lorentz boost is a kind of quaternionic square-root of the four-velocity.

But, the four-vector is not the most simple non-trivial covariant quantity. Hence, for a *spinor*, there are four possible transformation laws (or eight, if one takes order reversal into account)

$$S_1' = \mathcal{L}S_1 \quad , \quad S_2' = S_2\overline{\mathcal{L}} \quad , \quad S_3' = \mathcal{L}^*S_3 \quad , \quad S_4' = S_4\mathcal{L}^+ \quad . \tag{21}$$

In fact, as it is immediately seen, these four types of variances are the counterparts to the four basic variances of tensor/spinor calculus: contra- or co-variance, dotted or undotted indices. But, here, the last three can be deduced from the first by means of the automorphisms $\overline{()}$, $()^*$, and $()^+$. This leads directly to the general idea of quaternionic tensor calculus: any time some covariant four-dimensional quantity is introduced, the only possible new variance it may obtain is the result of operating with one of the three basic involutions, possibly combined with reversal $()^\sim$. Multiplying these quantities, and alternating the variances by making use of quaternion conjugation, one obtains more complicated tensors. For example,

$$\vec{T} = \mathbb{V}[\overline{V_1}V_2] \tag{22}$$

is a *six-vector*[15] which has the variance $\vec{T}' = \mathcal{L}^*\vec{T}\mathcal{L}^+$. In fact, since $\mathcal{L}^*\mathcal{L}^+ = \mathcal{L}\overline{\mathcal{L}} = 1$, the scalar part $\langle\overline{V_1}V_2\rangle$ of $\overline{V_1}V_2$ is an invariant, while its vector part $\vec{T} = \overline{V_1} \wedge V_2$ is a complex vector such as, for example, the electromagnetic field bivector $B = \vec{E} + i\vec{B}$.

Of course, when changing referential, besides the proper transformations such as (16), there are also the so-called improper ones which involve space- or time-reversal, complex conjugation, and order-reversal. These can be taken care of in the quaternion formalism, using in particular the four basic involutions, so that, in the spirit of relativity, covariance can also be insured with respect to them. For

[15] Similarly to the concept of *four-vector*, the concept of *six-vector* refers to a variance, not just to the fact that such objects are necessarily bivectors which have six real components.

example, in the case of tensors constructed by multiplying four-vectors of odd parities, a tensor that is order-reversal invariant will also be of odd parity.

In summary, as long as one remains within four-dimensional space-time and works with biquaternions, it is possible to achieve the same power and flexibility as with ordinary tensor/spinor calculus without having to manipulate explicitly a large number of indices. This is possible in the realms of both special and general relativity [12].

5 Classical dynamics and Hamilton's principle

Classical mechanics is a domain in which Hamilton himself found many brilliant applications of real quaternions. Just think, for example, of his very elegant and general resolution of the Kepler problem, in which the quaternion formalism leads directly to the conservation of angular momentum and of the misnamed "Runge-Lenz" vector.[16]

Here, however, we will consider the classical dynamics of a system of point particles, *without* specifying a priori whether it is a relativistic or non-relativistic problem. The fundamental concept is then the "Hamiltonian," which is a scalar function of time, position $\vec{x}$, and canonical conjugate momentum $\vec{\pi}$. This Hamiltonian function H can be merged together with the canonical momentum into one bireal quaternion that we call the *four-Hamiltonian*

$$\Pi = [H(t,\vec{x},\vec{\pi}); -i\vec{\pi}(t,\vec{x})] \quad . \tag{23}$$

For example, for a system point particles in a time independent external field, Π is the sum

$$\Pi = \sum \mathcal{P}_n + q_n A(\mathcal{X}_n) \tag{24}$$

where $\mathcal{P}_n = [E_n; -ic\vec{p}_n]$ is the energy-momentum of each particle, q_n their electric charge, and $A = [V; -i\vec{A}]$, the external electromagnetic four-potential.

The equations of dynamics can then be expressed in a number of equivalent forms, the simplest one being possibly the *action postulate* which states that the four-Hamiltonian derives from an invariant scalar function, the action $S(t,\vec{x})$:

$$\nabla S = i\Pi \quad . \tag{25}$$

[16]See Ref.7, Vol II, art. 419, pages 298–299. The angular momentum and Runge-Lenz vectors are denoted by β and ϵ, respectively.

Since S is a scalar, operating on both sides with $\overline{\nabla}$ and taking the vector part we find

$$\overline{\nabla} \wedge \Pi = 0 \quad . \tag{26}$$

This is, written in quaternions, the condition for $d\Pi$ to be a total differential, or, equivalently, for Π to be an exact one-form, i.e.,

$$\oint \langle \overline{d\mathcal{X}} \Pi \rangle = 0 \tag{27}$$

which, for any given two fixed point $\mathcal{X}_1$ and $\mathcal{X}_2$, is the same as *Hamilton's principle*

$$\delta \int_{\mathcal{X}_1}^{\mathcal{X}_2} \langle \overline{d\mathcal{X}} \Pi \rangle = 0 \quad . \tag{28}$$

Moreover, for a system in which H does not depend explicitly on time, (27) is also fully equivalent to *Hamilton's equations* of motion:

$$\dot{\vec{x}} = \frac{d}{d\vec{\pi}} H \quad , \quad \dot{\vec{\pi}} = -\frac{d}{d\vec{x}} H \quad . \tag{29}$$

The sequence of transformations we have gone through may look like a succession of trivialities. This, in fact, is not the case. Had we not put the "i" in front of $\vec{\pi}$ in (23), we would not have been able to work out these results. Moreover, while all expressions are formally covariant, they are the same whether we assume the kinematical expression for $\mathcal{P}$ to be relativistic or not. Hence, although we have done nothing more than rewriting well known results, we see that complex quaternions provide a compact and convenient framework for writing the equations of Hamiltonian dynamics, and that the resulting expressions are automatically in relativistic covariant form. It is the same with Maxwell's equations: as will be recalled in the next section, writing them down in compact quaternionic form requires the use <u>bi</u>quaternions.

To show what happens if we introduce relativity suppose for example that we apply (25) to a single particle. The fourth component, i.e., time or energy, is no longer an independent variable. For instance, we have now the relativistic identity $|\mathcal{P}|^2 = m^2$ and (25) can be rewritten as

$$(i\overline{\nabla} S + e\overline{A})(i\nabla S + eA) = m^2 \quad . \tag{30}$$

This is *Hamilton-Jacobi*'s equation for a particle in an electromagnetic field.

6 Maxwell's equations and quaternionic analyticity

After the casting of Lorentz transformations into quaternion form, one of the first modern applications of biquaternions was the rewriting of Maxwell's equations in 1911 by Conway [13], and in 1912 by Silberstein [14], as

$$\overline{\nabla} A = B \quad , \quad \nabla B = -4\pi J \tag{31}$$

where $A = [V, -i\vec{A}]$ is the electromagnetic potential and $J = [\rho, -i\vec{j}]$ the source current density. This very compact form allows many calculations to be done very effectively. In particular, as shown by Silberstein in 1913, the energy-momentum tensor of the electromagnetic field, i.e., Maxwell's stress-energy tensor, has a very simple explicit quaternionic form [15]

$$4\pi T(\,) = \tfrac{1}{2} B^{\sim}[\,]B = -\tfrac{1}{2} B^{+}[\,]B \tag{32}$$

where the free space $[\,]$ corresponds to the position of a quaternionic argument.[17] When this tensor is used to calculate the flow of energy and momentum through some given hypersurface, the result is automatically covariant and there is no "4/3 problem" as with the obnoxious Poynting vector [16]. Moreover, using (31) to calculate its divergence, one immediately obtains Lorentz's force equation

$$\dot{\mathcal{P}} = T(\nabla) = -\tfrac{1}{2}(JB + B^{\sim}J) = -F(J) \tag{33}$$

which shows that the quaternion form of the electromagnetic field *tensor* is $F(\,) = \frac{1}{2}([\,]B + B^{\sim}[\,])$, a physical object which should not be confused with the electromagnetic field *bivector* $B = \vec{E} + i\vec{B}$, or its reverse $B^{\sim} = \vec{E} - i\vec{B}$, a non-trivial distinction first made in 1955 by Kilmister [17].

A most interesting idea suggested by Maxwell's equations in quaternion form was developed by Lanczos [18] in his PhD thesis of 1919. In effect, Maxwell's second equation in vacuum, $\nabla B = 0$, is the direct generalization of the Cauchy-Riemann analyticity condition from two to four dimensions. It is therefore natural to envisage classical electrodynamics as a biquaternionic field theory in which point singularities are interpreted as electrons [19]. In this case the field at some point $\mathcal{X}$ is calculated by means of the appropriate generalization of Cauchy's formula in which the integration contour becomes an hypersurface $\Sigma(\mathcal{Y})$ surrounding the point

$$B(\mathcal{X}) = \frac{-1}{2\pi^2} \iiint \frac{\overline{\mathcal{R}}}{|\mathcal{R}|^4} \, d^3\Sigma \, B(\mathcal{Y}) \tag{34}$$

[17] This convention, due to Hamilton and promoted by Conway and Synge [8], generalizes Dirac's "bra–ket" notation to biquaternions. Its value stems from the speed of calculation which derives from the simplicity of the composition rule: $a[\,]a' \odot b[\,]b' = ab[\,]b'a'$. Moreover, it provides a clear distinction between "numbers" (or "vectors") Q, and "fonctions" (or "operators") $Q[\,]$.

where $\mathcal{R} = \mathcal{Y} - \mathcal{X}$ and $|\mathcal{R}|^2 = \overline{\mathcal{R}}\mathcal{R}$. This generalization of complex analysis has been extensively studied by Fueter in the case of real quaternions [20] and more recently extended to biquaternions and higher dimensional Clifford algebras [21]. This formalism can now very efficiently be applied to standard problems, such as the calculation of retarded potential and fields [22].

7 Spinors in kinematics and classical electrodynamics

Spinors are increasingly often used in classical physics and relativity [23]. However, possibly the first significant use of spinors in classical physics was made in 1941 by Paul Weiss [24], the particularly brilliant first PhD student of P.A.M. Dirac. By "significant" we mean that Weiss's applications of quaternions was not just rewriting an otherwise known result in quaternion form. In fact, Weiss gave an independent interpretation and derivation of an important physical law: the Lorentz-Dirac equation.

Weiss's starting point was the fact that the quaternion formalism provides explicit formulas which are difficult to obtain by the ordinary methods of analysis. For instance, in kinematics, taking the square root of the four-velocity as in (20) is the same as making the spinor decomposition of the four-velocity. An explicit formula for the four-acceleration is then obtained by taking the total proper-time derivative on both sides

$$\dot{\mathcal{Z}} = \mathcal{U} = \mathcal{B}\mathcal{B}^+ \quad , \quad \ddot{\mathcal{Z}} = \dot{\mathcal{U}} = \mathcal{B}\vec{\mathcal{A}}\mathcal{B}^+ \tag{35}$$

where, as shown by Weiss, the invariant real vector $\vec{\mathcal{A}}$ is the acceleration in the rest-frame. Similarly, since null-four-vectors can explicitly be formulated with biquaternions, one has explicit formulas for the light-cone and retarded coordinates

$$\mathcal{X} - \mathcal{Z} = 2is\mathcal{B}\overline{\sigma}\mathcal{B}^+ \tag{36}$$

where s is the invariant retarded distance from the position of the charge $\mathcal{Z}$ to the space-time point $\mathcal{X}$, and σ an idempotent such as (14) with $\vec{\nu}$ pointing from $\mathcal{Z}$ to $\mathcal{X}$.

In his paper, Weiss does not speak of spinors. On the contrary, he makes it clear that his decomposition has nothing to do with Dirac's *bispinors*. But what he does is exactly the kind of spinor decomposition we use today, e.g., in general relativity.

Weiss's application is to show that in this formalism the flow of energy and momentum through a hypersurface surrounding a point charge in arbitrary motion can be calculated exactly using Silberstein's form of the Maxwell tensor (32). He then proceeds to find the world-lines for which the energy-momentum flow is stationary, and discovers that the resulting equation of motion is nothing but the Lorentz-Dirac equation [25]

$$mc^2\dot{\mathcal{U}} = \tfrac{2}{3}ie^2(\dot{\mathcal{U}}\dot{\overline{\mathcal{U}}}\mathcal{U} + \ddot{\mathcal{U}}) - \tfrac{1}{2}e(\mathcal{U}B + B^{\sim}\mathcal{U}) \quad . \tag{37}$$

8 Lanczos's generalization of Dirac's equation: Spin and isospin.

About one year after Dirac discovered his relativistic wave-equation for spin $\frac{1}{2}$ particles, Lanczos [26] published a series of three articles in which he showed how to derive Dirac's equation from the more fundamental coupled biquaternion system[18]

$$\overline{\nabla}A = mB \quad , \quad \nabla B = mA \quad . \tag{38}$$

Obviously, Lanczos was inspired by his previous work with quaternions [18]. Indeed, comparing with (31), it is clear that (38) can be seen as Maxwell's equations with feedback, and that, following Lanczos [27], this feedback can be interpreted as a distinctive feature of massive particles. However, there is a problem. In Dirac's equation, the wave function is a four-component bispinor, while A and B are biquaternions with four complex components each. This is the "doubling" problem that puzzled Lanczos a lot, as well as others who later tried to cast Dirac's equation in quaternion form [28].

The first step towards a Dirac bispinor is to postulate that A and B have spinor variances, i.e., that $A' = \mathcal{L}A$ and $B' = \mathcal{L}^*B$, which leaves the possibility of making a gauge transformation, i.e., a right-multiplication by some arbitrary biquaternion G. Then, to get Dirac's spin $\frac{1}{2}$ field, Lanczos had to make the superposition

$$D = A\sigma + B^*\overline{\sigma} \quad . \tag{39}$$

Here σ is an idempotent such as (14) with, for definitiveness, $\vec{\nu} = e_3$, the third quaternion unit. Comparing with (15), we see that σ has the effect of projecting out half of A, which added to another half of the complex conjugated of B, gives

[18] We write m for $mc/\hbar$ taking $c = \hbar = 1$ for simplicity. Note that Lanczos could have taken the reverse of (38) as his fundamental equation: $A\overline{\nabla} = mB^{\sim} \quad , \quad B^{\sim}\nabla = mA \quad . \qquad (38^{\sim})$

a Lorentz covariant superposition that obeys the wave equation

$$\overline{\nabla} D = m D^* i e_3 \quad . \tag{40}$$

This equation, to be called the *Dirac-Lanczos equation*, is precisely equivalent to Dirac's equation. It will be rediscovered by many people, in particular by Gürsey [29] and Hestenes [30]. While equivalent to other possible forms, (40) has the considerable didactic advantage of making "spin" explicit. Indeed, the vector on the right-hand side shows that Dirac's equation singles out an arbitrary but unique direction in ordinary space: the *spin* quantization axis.

Using this equation, it is easy to construct and study the various covariant quantities which are important in quantum electrodynamics. For example, the conserved probability current is $J = DD^+$, and Tetrode's energy-moment tensor is

$$T(\,) = \big(\langle [\,] \overline{\nabla} \rangle D\big) i e_3 D^+ - D i e_3 \big(D^+ \langle \overline{\nabla} [\,] \rangle\big) - \langle \overline{A} [\,] \rangle D D^+ \quad . \tag{41}$$

However, the superposition $(39)$is not the only one leading to a spin $\frac{1}{2}$ field obeying equation (40). As shown by Gürsey in 1957, if (39) represents a *proton*, the *neutron* is then [31,32]

$$N = (A\overline{\sigma} - B^* \sigma) i e_1 \quad . \tag{42}$$

Hence, Lanczos's doubling is nothing but *isospin*. Gürsey's articles had a tremendous impact [27] and inspired ideas like chiral symmetry and the sigma model [32]. Indeed, "internal" symmetries such as isospin are explicit and trivial in Lanczos's double equation (38), while only space-time symmetries are explicit in Dirac's traditional 4×4 matrix formulation or the biquaternionic formulation (40). Unfortunately, except in his PhD dissertation, Gürsey made no reference to Lanczos's work, and Lanczos never learned that he had anticipated isospin in 1929 already!

Since the fundamental fields are A and B, while D and N are the physically observed ones, it is of interest to find the most general gauge transformations on A and B which are compatible with the superpositions (39) and (42). In fact, these transformations form a group that was discovered in another context by Nishijima [33] and which has the following explicit representation

$$G_N = \sigma \exp(i\alpha) + \overline{\sigma} \exp(i\beta) \quad . \tag{43}$$

By direct calculation, one finds that while A and B transform under G_N, D transforms as $\exp(-e_3\alpha)$ and N as $\exp(-e_3\beta)$, respectively, so that the system (38) describes two particles of equal mass but different electric charges, such as the proton and the neutron. Hence, by just trying to write Dirac's equation in quaternions, one is automatically led to discover the existence of *isospin*, a fundamental feature that indeed is found in nature.

9 Proca's equations and the absence of magnetic monopoles

When we wrote Maxwell's equation (31) we made the implicit assumptions $A = A^{\sim}$, i.e., that A was a ordinal invariant fundamental four-vector. If we try now to put a mass term on the right of the second Maxwell equation, and therefore introduce a "feedback" to get the wave equation for a massive spin 1 field, we find that Maxwell's equations have necessarily to be generalized to the following form

$$\overline{\nabla} \wedge A = B \quad , \tag{44$'$}$$

$$\tfrac{1}{2}(\nabla B + B^{\sim}\nabla) = m^2 A \quad . \tag{44$''$}$$

This is, written in biquaternions, the correct spin 1 wave equation discovered in 1936 by Proca [34]. As with Dirac's equation it is easy to write in quaternions the conserved current and the energy-momentum tensor

$$2J = A^{+}B + B^{+}A + (...)^{\sim} \quad , \tag{45}$$

$$8\pi T(\,) = B^{+}[\,]B + m^2 A^{+}\overline{[\,]}A + (...)^{\sim} \tag{46}$$

where $(...)^{\sim}$ means that the expression has to be completed by adding the reverse of the part on the left. Hence, the current and the energy-momentum are bireal and ordinal invariant four-vectors, as it should be.

Now, just as we derived Dirac's equation from Lanczos's equation (38) by making the superposition (39), Proca's equation $(44'')$ can also be derived from (38) by adding the second Lanczos equation to its reverse equation. If this is so, what then is the meaning of the equation obtained by subtracting Lanczos's second equation from its reverse, assuming that the potential A is ordinal invariant

$$\nabla B - B^{\sim}\nabla = 0 \quad ? \tag{47}$$

Obviously, this is just the part of Maxwell's equation which specifies that there are *no* magnetic monopoles! Hence, if *Lanczos's* system (38) is taken as the *fundamental equation* from which Dirac's and Proca's equations are derived, Maxwell's equation is obtained by taking the $m^2 = 0$ limit in (44), and (47) insures the absence of magnetic monopoles.

However, if we would have assumed that $A = -A^{\sim}$ instead of $A = +A^{\sim}$, we would have found another fully covariant field equation, only differing from Proca's by the fact that a minus-sign would replace the plus-sign in $(44'')$: in fact, the correct equation for a massive pseudo-vector particle. Therefore, as shown by Gürsey in his PhD thesis [29], the wave-equations of all scalar and vector particles, and of all pseudo-scalar and pseudo-vector particles, are just degenerated cases of Lanczos's fundamental equation (38).

10 Einstein-Mayer: electron-neutrino doublets in 1933!

When he wrote his 1929 papers on Dirac's equation, Lanczos was with Einstein in Berlin. In 1933, Einstein and Mayer (using semi-vectors, a formalism allied to quaternions) derived a spin $\frac{1}{2}$ field equation (in fact, a generalized form of Lanczos's equation) predicting that particles would come in doublets of different masses [35,36]. The idea was that the most general Lagrangian for quaternionic fields, to be called the Einstein-Mayer-Lanczos (EML) Lagrangian, should have the form

$$L = \mathbb{S}\Big[A^+\overline{\nabla}A + B^+\nabla B - (A^+BE^+ + B^+AE) + (...)^+\Big] \quad . \tag{48}$$

where $E \in \mathbb{B}$. The field equations are then

$$\overline{\nabla}A = BE^+ \quad , \quad \nabla B = AE \tag{49}$$

which reduce to (38) when $E = m$. In the general case, the second order equations for A or B become eigenvalue equations for the *mass* (the m factor appearing in the argument of $\exp im(Et - \vec{p}\cdot\vec{x})$ of plane wave solutions. This generalization is obtained by the substitutions $Am \rightarrow AE$ and $Bm \rightarrow Bm$ in the Lagrangian leading to (38). Therefore, mass-generation is linked to a maximally parity violating field.

There are two basic conserved currents: the probability current J, and the barycharge current K

$$J = AA^+ + \overline{BB^+} \quad , \quad K = AEA^+ + \overline{BEB^+} \quad . \tag{50}$$

Keeping E constant, J is invariant in any non-abelian unitary $SU(2)\otimes U(1)$ gauge transformation of A or B.[19] On the other hand, K is only invariant for abelian gauge transformations which also commute with E, i.e., elements of the general Nishijima group (43) such that E and σ commute.

Of special interest are the cases in which E is also a global gauge field. The first such gauge is when E is idempotent. One solution of (49) is then massive and the other one massless: an electron-neutrino doublet! The most general local gauge transformations compatible with (50) are then elements of the unitary Nishijima group $U_N(1,\mathbb{C})$ combined with *one* non-abelian gauge transformation which operates on either E and A, or E and B, exclusively. This leads directly to the Standard model of electro-weak interactions [37].

[19] In biquaternions, these transformations have the form $G(\,) = [\,]e^{i\phi}\exp(\frac{1}{2}\theta\vec{a})$.

The second fundamental case is when E is real: $E = E^*$. The eigenvalue equation is degenerate and the two masses are equal. As shown by Gürsey [32], equation (49) describes a nucleonic field, non-locally coupled to a pseudoscalar sigma-pion field.

Again, as in the examples shown in the previous sections, Hamilton's conjecture seems to be realized. Einstein-Mayer's generalization (49) lifts the mass degeneracy of Lanczos's original equation (38) and leads to electron-neutrino doublets and weak interaction on one hand, and to proton-neutron doublets and strong interaction in the form of the well known charge-independent pion-nucleon theory, on the other hand [27].

11 Petiau waves and the mass spectrum of elementary particles

One of the central problems of contemporary physics is the question of the origin of the mass of the elementary particles. As we have seen, by replacing the mass term in Lanczos's equation by some biquaternionic parameter, Einstein has been able to show that elementary particles come in doublets of different masses. In fact, in 1930 already, Lanczos wondered whether a theory in which the mass term is replaced by a variable would not simultaneously solve the problem of mass quantization and that of infinities in field theory [38].

Well, as nobody after Einstein and Mayer seemed to have taken Lanczos's suggestion seriously, one had to wait until 1965 (about the time when Gell-Mann and Zweig proposed the idea of quarks) for somebody to reinvent the concept. This year, in complete independence from mainstream research, the French physicist Gerard Petiau wrote a system of equations which may precisely give a solution to the problem of the mass of the electrons and quarks [39].

Although Petiau was thinking in very general terms, considering complicated couplings between particles of various intrinsic spin, it is easy with quaternions to write his fundamental equation in the case of spin $\frac{1}{2}$ particles [35]. It just amounts, in the spirit of Lanczos's feedback idea, to adding a third equation to Einstein's system (49) in order to close it:

$$\begin{aligned} \overline{\nabla} A &= BC \quad , \\ \nabla B &= AC \quad , \\ \nabla C &= A\overline{B} \quad . \end{aligned} \tag{51}$$

Here A and B are the usual Lanczos spin $\frac{1}{2}$ fields, and the scalar C an additional Einstein-Mayer field of spin 0. Because the system is now closed, it becomes non-linear, and the solutions are much more constrained than with any usual linear type of wave equations.

For instance, the single-periodic *de Broglie waves* that quantum mechanics associates with a particle become double-periodic *Petiau waves* [40]. Instead of being linear combinations of $\sin(z)$ and $\cos(z)$ functions, these waves are superpositions of elliptic functions $\mathrm{sn(z,k)}$, $\mathrm{cn(z,k)}$, etc. A very appealing feature of Petiau waves is that their dependence on the modulus interpolates between pure de Broglie waves (for $k = 0$) and pure solitonic waves (for $k = 1$): a beautiful realization of the wave/particle duality of quantum mechanics. Moreover, both the amplitudes and the *proper mass* (the μ factor appearing in the argument of $\mathrm{sn}(\mu(Et - \vec{p} \cdot \vec{x}), k)$, for example) will be quantized.

The most interesting thing, however, is happening when, in order to quantize the system, the Hamiltonian function is constructed. Taking, for example, A as the fundamental field Petiau showed that, in terms of the first integrals, the Hamiltonian has the very simple form [39]

$$H = C_0 \mu^4 k^2 \tag{52}$$

where k is the modulus of the elliptic function, μ the proper mass, and C_0 some constant. The exciting thing is that the Hamiltonian, and thus the total energy in the field (i.e., for a single particle, the *effective mass*) scales with the fourth power of μ.

In effect, in 1979, Barut discovered a very good empirical formula for the mass of the leptons [41]. Assuming that a quantized self-energy of magnitude $\frac{3}{2}\alpha^{-1} M_e c^2 N^4$, where $N = 0, 1, 2, ...$, is a new quantum number, be added to the rest-mass of a electron to get the next heavy lepton in the chain e, μ, τ, ..., Barut got the following expression (where $\alpha = 1/137$)

$$M(N) = M_e\Big(1 + \tfrac{3}{2}\alpha^{-1} \sum_{n=0}^{n=N} n^4\Big) \quad . \tag{53}$$

The agreement with the data of this rather simple formula is surprisingly good, the discrepancy being of order 10^{-3} for μ and 10^{-3} for τ, respectively [42]. In order to get the masses of the quarks [43], it is enough to take for the mass of the lightest quark $M_u = M_e/7.47$. Again, as can be seen in **Table 1** the agreement between the theoretical quark masses and the "observed" masses is quite good, especially for the three heavy quarks.

Since we have just seen (52) that the energy of a Petiau field is scaling with the fourth power, we are inclined to think that there might indeed be a fundamental link between such non-linear fields and the theory of the mass of quark and electrons. If this is so, what about the factor 7.47 ?

There are two non-trivial exceptional cases for elliptic functions: the harmonic case, $k = \sin(\frac{\pi}{4})$, and the equianharmonic case, $k = \sin(\frac{\pi}{12})$. It is very plausible to associate the former with leptons, and the latter with quarks. Indeed, in either case, the corresponding elliptic functions exhibit several unique symmetry and scaling properties, which come from the fact that in the complex plane their poles form a modular aggregate with $\frac{\pi}{2}$ or $\frac{\pi}{3}$ symmetries. Since according to (52) the masse is proportional to k^2, the electron to quark mass ratio is then equal to $[\sin(\frac{\pi}{4})/\sin(\frac{\pi}{12})]^2 \approx 7.47$.

But this is now very close to pure speculation, and in any case on the frontier of contemporary research [43]. Nevertheless, it is interesting to see how far, just following Hamilton's conjecture, one can go in the direction of a unified picture of fundamental physics.

Table 1

N	electron masses			quark masses		
		Barut's formula	Data		Barut's formula	Data
0	e	0.511	0.511	u	0.068	0 – 8
1	μ	105.55	105.66	d	14.1	5 – 15
2	τ	1786.1	1784.1	s	239	100 – 300
3		10294	?	c	1378	1300 –1500
4		37184	?	b	4978	4700 – 5300
5				t	13766	?
6					31989	?

Comparison of electron and quark masses in MeV/c^2 *calculated with Barut's formula* (53) *to measured masses from Ref. [42]. (Note added in 1996: The observation of a sixth quark of mass in the range of 160'000 to 190'000 MeV/c*^{2} *has been reported at the begining of 1995.)*

12 Quaternions and quantum mechanics

Some of those who have been following us on this upwards trail, starting from Hamilton's principle and ending with a possible solution to the problem of quark masses, might be surprised that just a short section is dedicated to quaternions and quantum mechanics. To these we say — in the spirit of Hamilton's particle-wave duality — that everything we have done can easily be recast in the jargon of "wave mechanics," so that, in this perspective, we have been doing quantum mechanics all along.

In effect, the quantum predicate is no so much in the field equations we have been discussing in this paper, than in the interpretation of the field's amplitudes (the "wave functions": complex number in the non-relativistic case, biquaternions in the relativistic one). As Feynman clearly stated in a review of the principles of quantum mechanics: "It has been found that all processes so far observed can be understood in terms of the following prescription: To every *process* there corresponds an *amplitude*; with proper normalization the probability of the process is equal to the absolute square of this amplitude" [Ref.44, page 1].

Take, for example, the Dirac-Lanczos equation (40), rewritten here in the case where there is an external electromagnetic field A

$$\overline{\nabla} D = (mD^* + e\overline{A}D)ie_3 \quad . \tag{54}$$

Comparing with (1), we see that the Hamiltonian is the following operator

$$H(\,) = -\vec{\nabla}[\,] - m[\,]^* ie_3 - e\overline{A}[\,]ie_3 \quad . \tag{55}$$

Then, following Feynman's prescription, we have to normalize the amplitude D. Since the probability current is the conserved four-vector DD^+, a suitable norm is

$$\langle\!\langle D|\mathbf{1}|D\rangle\!\rangle \;=\; \iiint d^3V \; \langle D^+[\mathbf{1}]D\rangle \;=1 \tag{56}$$

where the dummy operator "**1**" may be replaced by the operator corresponding to the physical quantity whose expectation value is to be calculated. Hence once the above prescription has been accepted as a postulate there is not much mystery left in quantum theory, and it is straightforward, at least in principle, to give the quantum interpretation of the field equations presented in the previous sections.[20]

At this point, it is worth mentioning that in the past decade a kind of a silent revolution has been occurring with respect to quantum theory. Increasingly, quantum and classical theories are seen as part of the same theory: Barut shows that wave mechanics can be formulated without $\hbar$ [45], Lamb suggests that Newton could have invented wave mechanics [46], and several major investigations show that the whole apparatus of the so-called "second quantization" of fields is redundant [47]. Indeed, as is trivially shown in the way Lanczos's equation (38) generalizes

[20]It is remarkable that it is the use of complex conjugation in expressions such as the Hermitian product (56) that distinguishes quantum theory from classical physics.

Maxwell's equation [31], Maxwell's theory can be interpreted as a quantum theory without $\hbar$. For instance, calculating by means of Silberstein's energy-momentum tensor (32) the electromagnetic field's energy-momentum density, or the Lorentz's force density (33), is the same as applying the quantum rule (56). Moreover, the Hamiltonian is simply the operator $H() = \frac{1}{2}(\vec{\nabla}[\,] - [\,]\vec{\nabla})$ just like in Good's quantum interpretation of Maxwell's theory [48].

What, is then the main contribution of biquaternions to quantum theory? Possibly, the clear disentanglement of "i" and "$\hbar$," the two elements which have been traditionally associated with quantum mechanics. Indeed, looking at the Schrödinger equation (1), these two elements appear together as a combined factor. In Lanczos equation [38], however, "i" appears in the scalar part of the four-gradient ∇, and $\hbar$ in combination with the mass on the right-hand side had we not used the convention $\hbar = 1$. Thus, if Hamilton's conjecture is true, "i" is definitely associated with "time" (i.e., Hamilton's intuitive conception of imaginary numbers) et "$\hbar$" is associated with "mass" or, more precisely, with the particle aspect of waves, i.e., lumps of energy localized in space [49,43].

It remains, in conclusion, to stress that the power of Hamilton's conjecture seems not to have shown its limits yet. By this we allude to the numerous investigations in "quaternionic quantum mechanics" which have occurred since the birth of wave mechanics.

Indeed, not to mention the work of Lanczos [26,38], quaternionic and other more general algebraic generalization of quantum mechanics have been actively studied since 1928 already [50]. The best known sequel to this work is possibly what Jordan [51] started in 1932 and which culminated in the famous article [52] in which the first "Jordan algebra" was described. To give another example of the breadth of research in these early days, we mention that the theory of operators in quaternionic Hilbert spaces was the subject of a PhD thesis in 1935, and that the name "Wachs space" was proposed for such spaces [53].

A new impetus was given in the 1960's, mainly after the work of the group around Finkelstein and Jauch [54] at CERN, followed by others [55], up to the synthesis soon to be published by Adler [56]. All these developments contemplate the possibility that the complex numbers of contemporary quantum theory may have to be replaced by quaternions or biquaternions in some more fundamental theory. However, it may well be that Nature is satisfied with complex number as the fundamental scalar field, and that in this respect a single and commutative "i" is enough and playing some essential role that the current experimental situation seems to favor [57,58].

13 Conclusion

The physics is in the mathematical *structure*, not in the *formalism*: What are then the advantages of using a formalism such as Hamilton's biquaternions?

- Biquaternion are *whole symbols*, i.e., they compound between one and eight real numbers which belong to a single (or a few related) tensor quantity(ies) so that many formulas written in biquaternions are simpler than their standard matrix, tensor, or higher rank Clifford numbers counterparts. In general, they enable to dispense of at least one level of tensor indices, and quite often to reduce a few indices tensor to a single entity.

- Biquaternions are *expressive*, i.e., being elements of the simplest non-trivial Clifford algebra they provide neat and explict formulas in final form, which are therefore directly amenable to symbolic or numerical calculation, with pencil and paper, or with a computer.

- Biquaternion formulas are *suggestive*, i.e., they often indicate the correct way of generalizing a result, or how to relate seemingly independent results.

- Biquaternions provide a *unifying formalism*, e.g., they enable a fully consistent use of complex numbers in both classical and quantum physics; they lead to expressions that are very similar in both Galilean and Lorentzian relativity; they are very effective in formulating the physics of the current "Standard model" of fundamental interactions [27,35]; etc.

However, some words of caution are in order: Since biquaternions are whole symbols it is important to take care of the problems specifically associated with such symbols. For instance, in order that biquaternion expression have a well defined tensor character they have to be constructed from elementary biquaternions that have such a character. Moreover, special care is required because of noncommutativity and of the need for biquaternion expression to be *ordinal covariant.* For example, the truly correct form of Maxwell's equations is not the Conway-Silberstein expression (31), but the gauge and ordinal invariant system

$$\overline{\nabla} \wedge A = B \quad , \tag{57'}$$

$$\tfrac{1}{2}(\nabla B + B^{\sim}\nabla) = -4\pi J \tag{57''}$$

which like Proca's equations (44) does not require the supplementary conditions $\langle \overline{\nabla} A \rangle = 0, A = A^{\sim}, J = J^{\sim}$, i.e., the Lorentz gauge and the requirements that A and J are ordinal invariant biquaternions. It is is precisely because such problems were not properly understood and cured that biquaternions failed, at the beginning of the twentieth century, to become a widespread language for physics.

14 Acknowledgments

We wish to thank Professor Brendan Goldsmith, Principal Organizer of the Conference commemorating the sesquicentennial of the invention of quaternions, for inviting us to give a lecture on the Physical heritage of Sir W.R. Hamilton, although the audience was primarily composed of mathematicians. We also wish to thank Professor James R. McConnell for his encouragement and suggestion to present a more technical account of our findings at the Cornelius Lanczos International Centenary Conference. And, finally, we are indebted to Mrs Ann Goldsmith, Librarian at the Dublin Institute of Advanced Studies, for her kind assistance in helping us retrieving some rather old but nevertheless important quaternion references [59].

References

[1] Most biographies of Hamilton give some details on his conception of algebra as the science of pure time. See e.g., O'DONNELL S. (1983). William Rowan Hamilton — Portrait of a Prodigy. Bool Press. Dublin. See also [2].

[2] LANCZOS C. (1967). William Rowan Hamilton — An appreciation. Amer. Sci., **55**, 129–143. Reprinted **in** DAVIS W.R. *et al.*, eds., Cornelius Lanczos Collected Published Papers With Commentaries (North Carolina State University, Raleigh, 1998) Vol. IV pages 2-1859 to 2-1873.

[3] GOUGH W. (1990). Mixing scalars and vectors — an elegant view of physics. Eur. J. Phys. **11**, 326–333.

[4] WIGNER E.P. (1960). The unreasonable effectiveness of mathematics in the natural sciences. Comm. Pure and Appl. Math. **13**, 1–14.

[5] GURSEY F. (1956), Contribution to the quaternion formalism in special relativity, Rev. Fac. Sci. Istanbul **A 20** 149–171. Gürsey uses both "quaternion units" $\{1, e_n\}$ and "Hermitian units" $\{1, I_n\}$ with $I_n = ie_n$. See also [24].

[6] E.C.G. Stückelberg et al., Helv. Phys. Acta. **33**, 727–752 (1960); **34**, 621–628 (1961). See also section 2.6 of Ref.56, pages 47–49.

[7] HAMILTON W.R. (1866). Elements of Quaternions. Second edition 1899-1901 enlarged by C.J. Joly. Reprinted in 1969 by Chelsea Publishing, New York.

[8] SYNGE J.L. (1972). Quaternions, Lorentz transformations and the Conway-Dirac-Eddington matrices. Comm. Dublin Inst. for Adv. Studies. Series A. No. 21.

[9] BLATON J. (1935). Quaternionen, semivektoren and spinoren. Z. f. Phys. **95**, 337–354.

[10] See, for example, chapter IX of LANCZOS C. (1986). The Variational Principles of Mechanics. Dover Publ. New York.

For two other good introductions to the use of quaternions in special relativity, see [5] and [8].

[11] LANCZOS C. (1958) Tensor calculus. **in** CONDON E.U. and ODISHAW H., eds., Handbook of Physics. (McGraw-Hill, New York) Part I., Ch. 10, I-140 to 1-151. Reprinted **in** DAVIS W.R. *et al.*, eds., Cornelius Lanczos Collected Published Papers With Commentaries (North Carolina State University, Raleigh, 1998) Vol. IV pages 2-1752 to 2-1763.

[12] RASTALL P. (1960). Quaternions in relativity. Rev. Mod. Phys. **36**, 820-832.

See also [29] in which Gürsey generalizes Lanczos's fundamental equation to curved space-time.

[13] CONWAY A.W. (1911). On the applications of quaternions to some recent developments of electrical theory. Proc. Roy. Irish Acad. **29**, 1–9.

[14] SILBERSTEIN L. (1912). Quaternionic form of relativity. Phil. Mag. **23**, 790–809.

CONWAY A.W. (1912). The quaternion form of relativity. Phil. Mag. **24**, 208.

[15] SILBERSTEIN L. (1913). Second memoir on quaternionic relativity. Phil. Mag. **25**, 135–144.

[16] JACKSON J.D. (1975). Classical Electrodynamics. Wiley. New York.

[17] KILMISTER C.W. (1955). The application of certain linear quaternion functions to tensor analysis. Proc. Roy. Irish Acad. **57**, 37-99.

[18] LANCZOS C. (1919). Die Funktionentheoretischen Beziehungen der Maxwellschen Aethergleichungen — Ein Beitrag zur Relativitäts- und Elektronentheorie. Verlagsbuchhandlung Josef Németh. Budapest. Reprinted **in** DAVIS W.R. *et al.*, eds., Cornelius Lanczos Collected Published Papers

With Commentaries (North Carolina State University, Raleigh, 1998) Vol. VI pages A-1 to A-82.

[19] GSPONER A. and HURNI J.P. (1994). Lanczos's functional theory of electrodynamics. Published **in** DAVIS W.R. *et al.*, eds., Cornelius Lanczos Collected Published Papers With Commentaries (North Carolina State University, Raleigh, 1998) Vol. I pages 2-15 to 2-23.

[20] FUETER R. (1932). Analytische Funktionen einer Quaternionen- variablen. Comm. Math. Helv. **4**, 9–20.

MOISIL C.R. (1931). Sur les quaternions monogŁnes. Bull. Sci. Math. **55**, 168–174.

For a bibliography of Fueter's work, see: HAEFELI H.G. (1947). Hypercomplex Differentiale. Comm. Math. Helv. **20**, 382–420.

For an elementary introduction to quaternion calculus, see: DEAVOURS C.A. (1973). The quaternion calculus. Am. Math. Monthly **80**, 995–1008.

For a more professional introduction, see: SUDBERRY A. (1979). Quaternionic analysis. Math. Proc. Camb. Phil. Soc. **85**, (199–225.

[21] RYAN J. (1982). Complexified Clifford analysis. Complex Variables **1**, 119–149.

BRACKX F., DELANGHE R. and SOMMEN F. (1982). Clifford Analysis. Pitman Books. London. 307 pp.

SOUCEK V. (1983). Complex-quaternionic analysis applied to spin-$\frac{1}{2}$ massless fields. Complex Variables **1**, 327–346.

[22] IMAEDA K. (1976). A new formulation of classical electrodynamics. Nuovo Cim. **32B**, 138–162.

[23] PENROSE R. (1963). Null hypersurface initial data for classical fields of arbitrary spin and for general relativity. Reprinted in 1980 Gen. Rel. and Grav. **12**, 225–264.

GURSEY F. (1957). Relativistic Kinematics of a Classical Point Particle in Spinor Form. Nuovo Cim. **5**, 784–809.

PROCA A. (1955). Particules de trŁs grande vitesse en mécanique spinorielle. Nuovo Cim. **2**, 962–971.

[24] WEISS P. (1941). On some applications of quaternions to restricted relativity and classical radiation theory. Proc. R. Ir. Acad. **46**, 129–168.

Despite the internal consistency of this article, we recommend the readers not to use Weiss's "Hermitian" units, but Hamilton's real quaternion units.

[25] DIRAC P.A.M. (1938). Classical theory of radiating electrons. Proc. Roy. Soc. **A167**, 148–169.

[26] LANCZOS C. (1929). Die tensoranalytischen Beziehungen der Diracschen Gleichung. Z. f. Phys. **57**, 447–473, 474–483, 484–493. Reprinted and translated **in** DAVIS W.R. *et al.*, eds., Cornelius Lanczos Collected Published Papers With Commentaries (North Carolina State University, Raleigh, 1998) Vol. III pages 2-1133 to 2-1225.

[27] GSPONER A. and HURNI J.P. (1994). Lanczos's equation to replace Dirac's equation? **in** BROWN J.D. *et al.*, eds., Proc. of the Cornelius Lanczos Int. Centenary Conf. December 12–17, 1993, Raleigh, NC, USA. (SIAM, Philadelphia, 1994) 509–512. There are a number of typographical errors in this paper. Please ask the authors for a corrected version.

[28] EDMONDS J.D. Jr. (1976). A relativistic "higher spin" quaternion wave equation giving a variation on the Pauli equation. Found. Phys. **6**, 33–36; (1977 ibid **7**, 835–859; 1978 ibid **8**, 439–444.

GOUGH W. (1989). A quaternion expression for the quantum mechanical probability and current densities. Eur. J. Phys. **10**, 188–193.

See also reference [60].

[29] GURSEY F. (1950). Applications of Quaternions to Field Equations. PhD thesis. University of London. 204 pp.

In his PhD thesis, Gürsey reviews the quaternionic theories of LANCZOS [Ref. 26], and others, in particular:

CONWAY A.W. (1937). Quaternion Treatment of the relativistic Wave Equation. Proc. Roy. Soc. **A162**, 145–154.

[30] HESTENES D. (1967). Real spinor fields. J. Math. Phys. **8**, 798–808, 809–812.

Following Hestenes, there is nowadays a renewed interest in the Clifford formulation of Dirac's theory, a framework that was first independently developed by Sauter, Proca and Juvet:

SAUTER F. (1930). Lösung der Diracschen Gleichungen ohne Spezialisierung der Diracschen Operatoren. Z. f. Phys. **63**, 803–814; ibid, **64**, 295–303.

PROCA A. (1930). Sur l'équation de Dirac. J. Phys. Radium **1**, 235–248.

JUVET G. (1930). Operateurs de Dirac et équations de Maxwell. Comm. Math. Helv. **2** 225–235.

[31] GURSEY F. (1958). Relation of charge independence and baryon conservation to Pauli's transformation. Nuovo Cim. **7**, 411–415.

[32] GURSEY F. (1960). On the symmetries of strong and weak interactions. Nuovo Cim. **16**, 230–240.

GELL-MANN M. and LEVY M. (1960). The axial vector current in beta decay. Nuovo Cim. **16**, 705–725.

[33] NISHIJIMA K. (1957). On the theory of leptons. Nuovo Cim. **5**, 1349–1354.

[34] PROCA A. (1936). Sur la théorie ondulatoire des électrons positifs et négatifs. J. Phys. Radium **7**, 347–353.

CONWAY A.W. (1945). Quaternion and matrices. Proc. Roy. Irish Acad. **A50**, 98–103.

[35] GSPONER A. and HURNI J.P. (1994). Lanczos-Einstein-Petiau: From Dirac's equation to non-linear wave mechanics. Published **in** DAVIS W.R. *et al.*, eds., Cornelius Lanczos Collected Published Papers With Commentaries (North Carolina State University, Raleigh, 1998) Vol. III pages 2-1248 to 2-1277.

[36] EINSTEIN A. and MAYER W. (1933). Die Diracgleichung für Semivektoren. Proc. Roy. Acad. Amsterdam **36**, 497–516, 615–619. For the idea of replacing the mass by a biquaternionic parameter, see also EDMONDS, Ref. [28].

[37] BJORKEN J.D. (1979). Neutral-current results without gauges theories. Phys. Rev. **D19**, 335–346.

[38] LANCZOS C. (1930). Dirac's wellenmechanische Theorie des Elektrons und ihre feldtheoretische Ausgestaltung. Physik. Zeitschr. **31**, 120–130. Reprinted and translated **in** DAVIS W.R. *et al.*, eds., Cornelius Lanczos Collected Published Papers With Commentaries (North Carolina State University, Raleigh, 1998) Vol. III pages 2-1226 to 2-1247.

[39] PETIAU G. (1965). Sur les théories quantiques des champs associés à des modŁles simples d'équations d'ondes non linéaires. Nuovo Cim. **40**, 84–101. For a synthesis of Petiau's work on this subject, see: PETIAU

G. (1982). Sur la représentation par des systŁmes différentiels du premier ordre de modéles corpusculaires. Ann. Inst. Henri Poincaré **36**, 89–125.

[40] PETIAU (1958). Sur une généralisation non linéaire de la mécanique ondulatoire et sur les propriétés des fonctions d'ondes correspondantes. Nuovo Cim. **9**, 542–568.

[41] BARUT A.O. (1979). Lepton mass formula. Phys. Rev. Lett. **42**, 1251.

[42] PARTICLE DATA GROUP (1992). Review of Particle Properties. Phys. Rev. **D45**, II-4. According to this reference, page II-4, note [g], "the estimates of d and u masses are not without controversy ... the u-quark could be essentially massless." See also GASSER J. and LEUTWYLER H. (1982). Quark Masses. Phys. Rep. **87**, 77–169.

[43] GSPONER A. and HURNI J.P. (1994). A non-linear field theory for the mass of the electrons and quarks. Published **in** Hadronic Journal **19** (1966) 367-373. After this paper was published we learned from G. Rosen that H. TERAZAWA also noted that the n^4 proportionality could be extended to quarks on page 1769 of Prog. Theor. Phys. **64** (1980) 1763–1771.

[44] FEYNMAN R.P. (1962). The Theory of Fundamental Processes. Benjamin. Reading, MA, USA.

[45] BARUT A.O. (1992). Formulation of wave mechanics without the Planck constant $\hbar$. Phys. Lett. **A171**, 1–2

[46] LAMB W.E. Jr. (1994). Suppose Newton had invented wave mechanics. Am. J. Phys. **62**, 201–206.

[47] SCHWINGER J. (1969, 1973, 1988). Particles, Sources and Fields. 3 volumes. Addison-Wesley. Reading, MA.

BARUT A.O. (1990). Foundations of self-field quantum electrodynamics. **in** BARUT A.O. Ed. New Frontiers in Quantum electrodynamics and Quantum Optics. Plenum Press. New York.

[48] GOOD R.H. (1957). Particle aspect of the electromagnetic field equations. Phys. Rev. **105**, 1914–1919.

[49] BARUT A.O. (1990). $E = \hbar\omega$. Phys. Lett. **A143**, 349–352.

[50] GURSEY F. (1983). Quaternionic and octonionic structures in physics Proc. of the 1st. Int. Meeting on the Hist. of Scientific Ideas. Barcelona. 29, 559–591.

[51] JORDAN P. (1933). Über Verallgemeinerungsmöglichkeiten des Formalismus Quantenmechanik. Nachr. Ges. Wiss. Göttingen **39**, 209–217.

[52] JORDAN P., VON NEUMANN J. and WIGNER E. (1934). On an algebraic generalization of the quantum mechanical formalism. Ann. Math. **35**, 29–64.

[53] TEICHMULLER O. (1935). Operatoren in Wachschen Raum. J. für Math. **174**, 73–124

[54] FINKELSTEIN D., JAUCH J.M. and SPEISER D. (1959). Notes on quaternion quantum mechanics. CERN report 59-7. Published **in** HOOKER C.A., ed., Logico-Algebraic Approach to Quantum Mechanics. (1979, Reidel, Dordrecht). Vol. II 367–421.

[55] In particular:

RUELLE D. (1958). Représentation du spin isobarique des particules à interaction forte. Nucl. Phys. **7**, 443–450.

KANENO T. (1960). On a possible generalization of quantum mechanics. Progr. Theor. Phys. **23**, 17–31.

ADLER S.L. (1980). Quaternion chromodynamics as a theory of composite quarks and leptons. Phys. Rev. **D21**, 2903–2915.

HORWITZ L.P. and BIEDENHARN L.C. (1984). Quaternion quantum mechanics: Second quantization and gauge fields. Ann. Phys. **157**, 432–488.

NASH C.G. and JOSHI G.C. (1987). Composite systems in quaternionic quantum mechanics. J. Math. Phys. **28**, 2883–2885.

ROTELLI P. (1989). The Dirac equation on the quaternion field. Mod. Phys. Lett. **A4**, 933–40.

SHARMA C.S. and ALMEDIA D.F. (1990). Additives isometries on a quaternionic Hilbert space. J. Math. Phys. **31**, 1035–1041.

NASH C.G. and JOSHI G.C. (1992). Quaternionic quantum mechanics is consistent with complex quantum mechanics. Int. J. Theor. Phys. **3**, 965–981

MARCHIAFAVA S. and REMBIELINSKI J (1992). Quantum quaternions. J. Math. Phys. **33**, 171–173.

[56] ADLER S.L. (1995). Quaternionic Quantum Mechanics. Oxford University Press. London.

[57] PERES A. (1979). Proposed test for complex versus quaternion quantum theory. Phys. Rev. Lett. **42**, 683–686.

Note, in particular, Ref. 18 cited therein.

[58] DAVIES A.J. and McKELLAR B.H. (1992). Observability of quaternionic quantum mechanics. Phys. Rev. **A46**, 3671–3675.

[59] In the course of this research we have compiled an essentially complete bibliography on the applications of quaternions in "modern" physics, i.e., starting with relativity. GSPONER A. and HURNI J.P. (1993). Quaternion bibliography 1893–1993. Unpublished report ISRI-93-13.

Additional references:

[60] GSPONER A. and HURNI J.-P. (2001). Comment on Formulating and Generalizing Dirac's, Proca's, and Maxwell's Equations with Biquaternions or Clifford Numbers. Found. Phys. Lett. **14**, 77–85.

[61] GSPONER A (2002). On the "equivalence" of the Maxwell and Dirac equations. Int. J. Theor. Phys. **41**, 689–694.