Introduction to Continuum Mechanics

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1 Introduction

For many materials the behaviour of large samples can be studied without recourse to the details of atomic level structure. A familiar example is the behaviour of fluids, but we can describe solids, glasses etc by making use of the framework provided by continuum mechanics.

1.1 The Notion of a Continuum

Continuum Mechanics ignores all the fine detail of atomic level structure and assumes that

- the highly discontinuous structure of real materials can be replaced by a smoothed hypothetical continuum.
- every portion of the continuum, however small, exhibits the macroscopic physical properties of the bulk material.

In any branch of continuum mechanics, the field variables (such as density, displacement, velocity) are conceptual constructs. They are taken to be defined at all points of the imagined continuum and their values are calculated via axiomatic rules of procedure.

The continuum model breaks down over distances comparable to interatomic spacing (in solids about 10^{-10} m). Nonetheless the *average* of a field variable over a small but *finite* region is meaningful. Such an average can, in principle, be compared directly to its nominal counterpart found by experiment - which will itself represent an average of a kind taken over a region containing many atoms, because of the physical size of any measuring probe.

For solids: *the continuum model is valid in this sense down to a scale of order* 10^{-8} *m which is the side of a cube containing a million or so atoms.*

Further, when field variables change slowly with position at a microscopic level $\sim 10^{-6}$ m, their averages over such volumes (10^{-20} m ^{3} say) differ insignificantly from their centroidal values. In this case pointwise values can be compared directly to observations.

Within the continuum we take the behaviour to be determined by:

a) conservation of mass

b) linear momentum balance

the rate of change of total linear momentum is equal to the sum of the external forces

c) angular momentum balance

The continuum hypothesis enables us to apply these laws on a local as well as a global scale.

1.2 Deformation and Strain

If we take a solid cube and subject it to some deformation, the most obvious change in external characteristics will be a modification of the shape.

The specification of the deformation is thus a geometrical problem and may be carried out from two different viewpoints:

a) with respect to the undeformed state (Lagrangian)

b) with respect to the deformed state (Eulerian)

Locally, the mapping from the deformed to the undeformed state can be assumed to be linear and described by a differential relation, which is a combination of pure stretch (a rescaling of each coordinate) and a pure rotation.

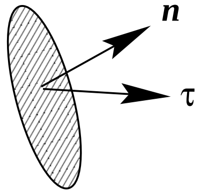
The mechanical effects of the deformation are confined to the stretch and it is convenient to characterise this by a *strain* measure. For example for a wire under load the strain ε would be the relative extension

$$\text{(i. e.) } \varepsilon = \frac{\text{change in length}}{\text{initial length}}$$

The generalisation of this idea requires us to introduce a strain *tensor* at each point of the continuum.

1.3 The stress-field

Within a deformed continuum there will be a force system acting. If we were able to cut the continuum in the neighbourhood of a point, we would find a force acting on a cut surface which would depend on the inclination of the surface and is not necessarily perpendicular to the surface.



This force system can be described by introducing a *stress tensor* at each point whose component describe the loading characteristics.

For a loaded wire, the stress σ would just be the force per unit area.

1.4 Constitutive relations

The specification of the stress and strain states of a body is insufficient to describe its full behaviour, we need in addition to link these two fields.

This is achieved by introducing a *constitutive relation*, which prescribes the response of the continuum to arbitrary loading and thus defines the connection between the stress and strain tensor for the particular material.

At best a mathematical expression provides an approximation to the actual behaviour of the material, but as we shall see we can simulate the behaviour of a wide class of media.

When we come to postulate constitutive relations for a material we shall invoke the *principle of local action* i.e. we shall assume that the forces acting at a point depend on the *local* geometry of deformation and its history and possibly also on the history of the local temperature. This is to exclude "action at a distance" for stress and strain

1.4.1 Solids

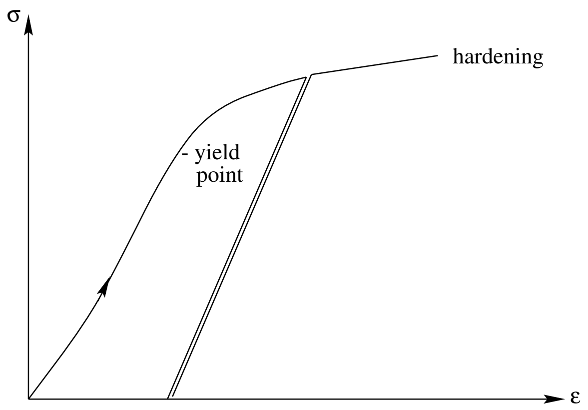
To get an idea of the behaviour of solids, we consider extension of a wire under loading. The tensile stress σ and tensile strain ε are then typically related as If the wire returns to its original configuration when the load is removed, the behaviour is said to be elastic.

a) Elasticity

i) linear elasticity $\sigma = E\varepsilon$ - usually valid for small strains.

ii) non-linear elasticity $\sigma = f(\varepsilon)$ - important for rubber like materials

b) Plasticity



Once the yield point is exceeded permanent deformation occurs and there is no unique stress-strain curve, but a unique $d\sigma - d\varepsilon$ relation. Due to microscopic processes the yield stress rises with σ - work hardening.

c) Viscoelasticity (rate-dependent behaviour)

Materials may creep and show slow long-term deformation e.g plastics and metals at elevated temperatures. Such behaviour also seems to be appropriate to the Earth e.g. the slow uplift of Fennoscandia in response to the removal of the loading of the glacial ice sheets.

Simple models of viscoelasticity

i) Maxwell model

$$\dot{\sigma} = E(\dot{\varepsilon} + \varepsilon/\tau),$$

this allows for instantaneous elasticity and represents a crude description of a fluid. Integrating the constitutive relation using e.g. Laplace transform methods we find

$$\sigma(t) = E[\varepsilon(t) + \int^{t} \mathrm{d}t' \varepsilon(t') \exp\{-(t-t')/\tau\}$$

ii) Voigt model

$$\sigma = E(\dot{\varepsilon} + \varepsilon/\tau)$$

which displays long term elasticity.

More complex models can be generated but all have the same characteristic of depending on the time history of deformation.

1.4.2 Fluids

The simplest constitutive equation encountered in continuum mechanics are those for an ideal fluid, where the pressure field is isotropic and depends on density and temperature

$$\sigma = -P(\rho, \mathrm{T})I$$

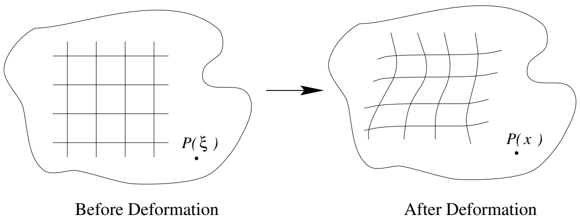
where ρ is the density, T is absolute temperature. If the fluid is incompressible ρ is a constant.

The next level of complication is to allow the pressure to depend on the flow of the fluid. The simplest such form includes a linear dependence on strain rate - a Newtonian viscous fluid

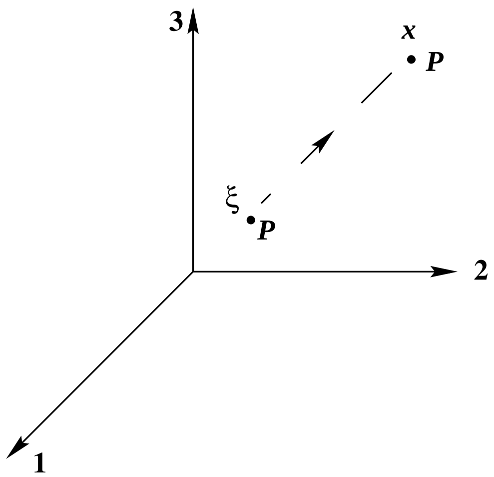
$$\sigma = -P(\rho, \mathrm{T})I + \mu\dot{\varepsilon}$$

2 Arbitrary deformation

After an arbitrary deformation of a material continuum, the amounts of compression (or expansion) and distortion of material vary with position throughout the continuum.



We need to look at the deformation on a local basis and so examine the geometrical aspects in the neighbourhood of a point P. We consider any two configurations of a material continuum. One of these is then taken as a *reference state* relative to which the deformation in the other is assessed.

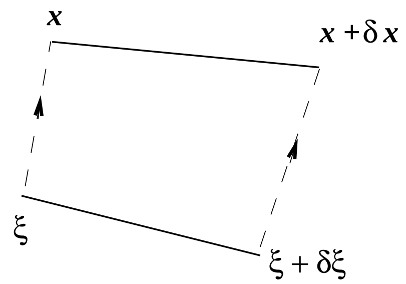


We take a set of rectangular background axes and use these to specify the coordinates of a material point P:
i) in the reference state $\boldsymbol{\xi} \equiv (\xi_1, \xi_2, \xi_3)$
ii) in the deformed state $\mathbf{x} \equiv (x_1, x_2, x_3)$
The deformation is specified by knowing

$$\mathbf{x} = \mathbf{x}(\xi, t)$$

when the functions $\mathbf{x}(\xi, t)$ are linear, the deformation is said to be *homogeneous*; in this case planes remain planes and lines remain lines.

2.0.1 Deformation of an element of arc



In general, near P, if $\boldsymbol{\xi}+\delta\boldsymbol{\xi} \to \mathbf{x}+\delta\mathbf{x}$ and $\mathbf{x}(\boldsymbol{\xi}, t)$ is differentiable

$$\mathrm{d}x_i = \left(\frac{\partial x_i}{\partial \xi_j}\right)\mathrm{d}\xi_j, \tag{1}$$

or, symbolically,

$$\mathrm{d}\mathbf{x} = F(\boldsymbol{\xi}, t)\,\mathrm{d}\boldsymbol{\xi}$$

The matrix $F \equiv (\partial x_i/\partial \xi_j)$, is called the local deformation gradient, and for a (1,1) mapping the Jacobian $J = \det F \neq 0$.

Note: under change of background coordinates F transforms like a second rank tensor.

The inverse transformation

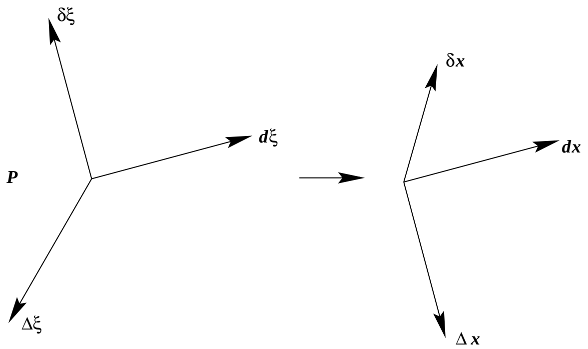
$$\mathrm{d}\boldsymbol{\xi} = F^{-1}(\boldsymbol{\xi}, t)\,\mathrm{d}\mathbf{x}, \quad \text{with} \quad F^{-1}_{kl} = \partial \xi_k/\partial x_l$$

2.0.2 Successive Deformations

If $\mathbf{x} = \mathbf{x}(\boldsymbol{y})$ with deformation gradient $F_1 = \partial\mathbf{x}/\partial\boldsymbol{y}$, and $\boldsymbol{y} = \boldsymbol{y}(\boldsymbol{\xi})$ with deformation gradient $F_2 = \partial\boldsymbol{y}/\partial\boldsymbol{\xi}$, then $\mathbf{x} = \mathbf{x}(\boldsymbol{y}(\boldsymbol{\xi})) = \mathbf{x}(\boldsymbol{\xi})$ with deformation gradient $F = \partial\mathbf{x}/\partial\boldsymbol{\xi}$, where

$$F = F_1\,F_2 \quad \text{(matrix multiplication)}$$

2.0.3 Deformation of an element of volume



The volume is given by the scalar triple product

$$\mathrm{d}V = [d\mathbf{x}, \delta\mathbf{x}, \Delta\mathbf{x}]$$

$$= [Fd\xi, F\delta\xi\ F\Delta\xi]$$

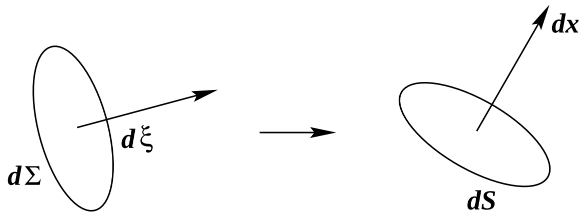
$$= \det F[d\xi, \delta\xi\ \Delta\xi] = J\mathrm{d}V_{o,} \qquad (2)$$

i.e. as would be expected the scaling is by the Jacobian of the transformation.

If we think in terms of a fixed volume (2) specifies the density after transformation, since $\mathrm{d}M = \rho\mathrm{d}V = \rho_o\mathrm{d}V_o$,

$$\rho(\mathbf{x}) = J^{-1}\ \rho_o(\xi) = (\det F)^{-1}\rho_o(\xi)$$

2.0.4 Deformation of an element of area



Consider the volume change of a skew cylinder with base $\mathrm{d}\mathbf{\Sigma}$, generators $\mathrm{d}\boldsymbol{\xi}$

$$\mathrm{d}V = \mathrm{d}\mathbf{x}.\,\mathrm{d}\mathbf{S} = \mathrm{d}\mathbf{x}^T\mathrm{d}\mathbf{S}$$

so

$$\mathrm{d}V = \mathrm{d}\mathbf{x}^T\mathrm{d}\mathbf{S} = J\mathrm{d}V_o = \det F\mathrm{d}\boldsymbol{\xi}^T\mathrm{d}\mathbf{\Sigma}$$

but $\mathrm{d}\boldsymbol{\xi} = F^{-1}\mathrm{d}\mathbf{x}$, and so

$$\mathrm{d}\mathbf{x}^T\mathrm{d}\mathbf{S} = \mathrm{d}\mathbf{x}^T\ \det F(F^{-1})^T\mathrm{d}\mathbf{\Sigma}$$

but this must be true for any d**x** and so for any area

$$\mathrm{d}\mathbf{S} = \det F\ F^{-T}\mathrm{d}\mathbf{\Sigma} \qquad (3)$$

Note: $F^{-T} \equiv (F^T)^{-1} \equiv (F^{-1})^T$.

2.0.5 Homogeneous deformation

Since only the ratios of differentials occur in (1), such relations at any point P are formally the same as those representing *homogeneous* deformation relative to P

$$\mathbf{x} = F\xi$$

with F independent of ξ.

This mapping may be regarded as between the positions of either **a particle** (material point) or **a fibre** (material line segment). The deformation gradient F is uniquely determined by the mappings of any 3 non-coplanar fibres and this provides a convenient way to find F experimentally.

2.1 Strain

The differential deformation is defined by $\mathrm{d}\mathbf{x} = F\mathrm{d}\boldsymbol{\xi}$

2.1.1 Stretch:

Consider a fibre in the direction $\boldsymbol{\nu}$

$$\mathrm{d}\boldsymbol{\xi} = |\mathrm{d}\boldsymbol{\xi}|\boldsymbol{\nu}$$

then define

$$\lambda_{(\boldsymbol{\nu})} = \frac{|\mathrm{d}\mathbf{x}|}{|\mathrm{d}\boldsymbol{\xi}|} = \text{stretch in direction } \boldsymbol{\nu} \; = \frac{\text{final length}}{\text{initial length}}$$

The length

$$|\mathrm{d}\mathbf{x}|^2 = \mathrm{d}\mathbf{x}^T\mathrm{d}\mathbf{x} = \mathrm{d}\boldsymbol{\xi}^T(F^TF)\mathrm{d}\boldsymbol{\xi} \tag{4}$$

2.1.1 Change in included angle

consider the transformation of two vectors

$$\mathrm{d}\mathbf{x} = F\mathrm{d}\boldsymbol{\xi}, \quad \mathrm{d}\mathbf{y} = F\mathrm{d}\boldsymbol{\eta}$$

then

$$\mathrm{d}\mathbf{x}.\,\mathrm{d}\mathbf{y} = \mathrm{d}\mathbf{x}^T\mathrm{d}\mathbf{y} = \mathrm{d}\boldsymbol{\xi}^T(F^TF)\mathrm{d}\boldsymbol{\eta}$$

The final angle is $\mathrm{d}\mathbf{x}.\,\mathrm{d}\mathbf{y}/|\mathrm{d}\mathbf{x}|\,|\mathrm{d}\mathbf{y}|$

Thus: F^TF is the metric for the deformed state relative to the reference state. Lengths and angles are unchanged if and only if F is proper orthogonal (i.e. a rotation).

The *stretch* is to be found from

$$(\lambda_{(\boldsymbol{\nu})})^2 = \frac{|\mathrm{d}\mathbf{x}|^2}{|\mathrm{d}\boldsymbol{\xi}|^2} = \frac{\mathrm{d}\boldsymbol{\xi}^T(F^TF)\mathrm{d}\boldsymbol{x}}{|\mathrm{d}\boldsymbol{\xi}|^2} = \boldsymbol{\nu}^T(F^TF)\boldsymbol{\nu} \tag{5}$$

A corresponding form can be found by working with the deformed configuration

$$|\mathrm{d}\boldsymbol{\xi}|^2 = \mathrm{d}\boldsymbol{\xi}^T\mathrm{d}\boldsymbol{\xi} = \mathrm{d}\mathbf{x}^T(F^{-1})^TF^{-1}\mathrm{d}\mathbf{x}$$

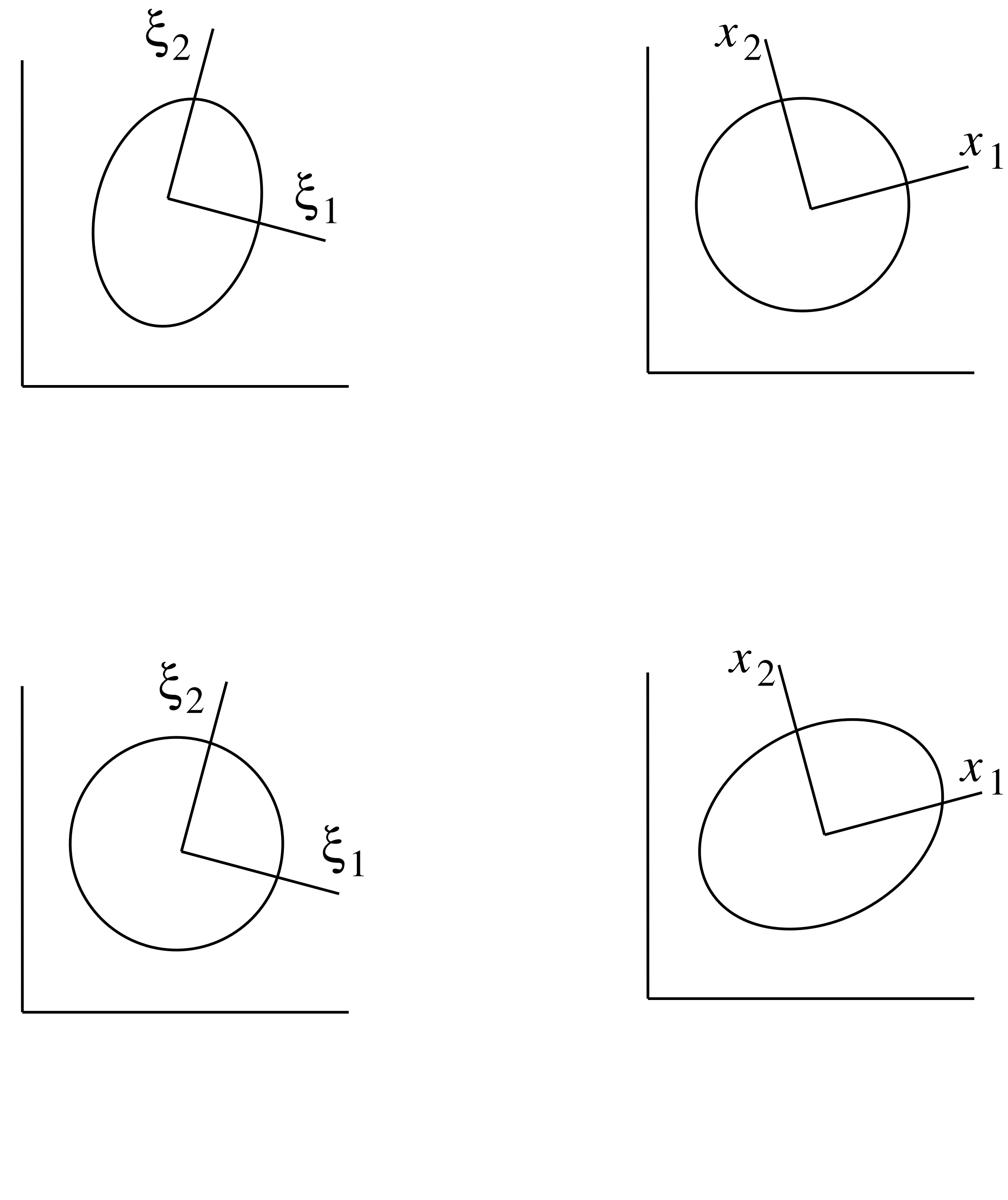
$$= \mathrm{d}\mathbf{x}^T(FF^T)^{-1}\mathrm{d}\mathbf{x}$$

If $\mathrm{d}\mathbf{x}$ lies in the direction $\mathbf{n}$,

$$(\lambda_{(\boldsymbol{\nu})})^2 = [\mathbf{n}^T(FF^T)^{-1}\mathbf{n}]^{-1}.$$

Relation of Lagrangian and Eulerian triad

Reference Deformed



2.1.2 Principal Fibres and Principal Stretches

If we consider the eigenvalues of the local metric tensor $F^T F$ we have

$$F^T F \mathrm{d}\boldsymbol{\xi}_r = \lambda_r^2 \mathrm{d}\boldsymbol{\xi}_r \tag{6}$$

and the eigenvalues λ_r^2 satisfy $\det|F^T F - \lambda_r^2 I| = 0$.

In the reference state $\mathrm{d}\boldsymbol{\xi}^T F^T F \mathrm{d}\boldsymbol{\xi} = const$ is an ellipsoid (the Lagrangian ellipsoid) with mutually orthogonal principal axes $\pm\mathrm{d}\boldsymbol{\xi}_r$.
Under deformation this ellipsoid transforms to a sphere - for a principal fibre from (4)

$$|\mathrm{d}\mathbf{x}_r|^2 = \mathrm{d}\boldsymbol{\xi}_r^T (F^T F) \mathrm{d}\boldsymbol{\xi}_r = \lambda_r^2 |\mathrm{d}\boldsymbol{\xi}_r|^2$$

and thus the λ_r are the principal stretches.
The triad of principal fibres $\mathrm{d}\boldsymbol{\xi}_r$ is termed the Lagrangian triad (or *material, referential* triad).

If we premultiply (6) by F

$$F F^T F \mathrm{d}\boldsymbol{\xi}_r = \lambda_r^2 F \mathrm{d}\boldsymbol{\xi}_r$$

and so

$$F F^T \mathrm{d}\mathbf{x}_r = \lambda_r^2 \mathrm{d}\mathbf{x}_r \tag{7}$$

so that the principal fibres are also finally orthogonal.
The $\mathrm{d}\mathbf{x}_r$ form a triad (the Eulerian or *spatial* triad) which are the principal axes of an ellipsoid $\mathrm{d}\mathbf{x}_r^T (F F^T)^{-1} \mathrm{d}\mathbf{x}_r = const$ which results from the deformation of a sphere in the reference state.

The Lagrangian triad (6) and the Eulerian triad (7) generally have different orientations.

2.1.3 Decomposition Theorem

We define the stretch tensor U to be coaxial with the Lagrangian triad of F and to have eigenvalues λ_r. The array of background components U can be constructed by the spectral formula

$$U = \sum_r \lambda_r \hat{\xi}_r \hat{\xi}_r^T, \qquad |\hat{\xi}_r| = 1$$

Considered as a mapping, U acts as a *pure strain*: it stretches the Lagrangian triad of F just like F itself but maintains the orientation of the triad. Hence F can differ from U only by a final rotation

$$F = RU \tag{8}$$

In the context of matrix algebra this is an example of a *polar decomposition*.

Algebraically: since $F^T F$ is coaxial with U and has eigenvalues λ_r^2, $F^T F = U^2$. When F is given, the equation $F^T F = U^2$ can be solved uniquely for a positive definite symmetric U. Then R defined by (8) is automatically proper orthogonal

$$F^T F = U^T R^T RU = UR^T RU = U^2$$

and so

$$R^T R = I$$

and $\det R > 0$ since $\det F > 0$ and $\det U > 0$.
There is also a similar decomposition with first a rotation of the principal fibres and then a stretching

$$F = RU = (RUR^T)R = VR$$

$V = RUR^T$ is the array of background components of the pure strain that is coaxial with the Eulerian triad of F and has principal values λ_r.

Note: The deformation gradient F can be specified independent of coordinate frame by using the triads and principal stretches

$$F = \sum_r \hat{\mathbf{x}}_r \hat{\xi}_r^T, \qquad |\hat{\xi}_r| = 1$$

which may be derived from the spectral form for U.

2.1.4 Pure Rotation

A rotation by an angle θ about an axis $\mathbf{n}$ transforms a vector $\boldsymbol{\xi}$ to

$$\mathbf{x} = \boldsymbol{\xi} + \sin\theta(\mathbf{n} \times \boldsymbol{\xi}) + (1 - \cos\theta)\mathbf{n} \times (\mathbf{n} \times \boldsymbol{\xi})$$

If we now introduce the skew matrix $N(\equiv \mathbf{n}\times)$ associated with $\mathbf{n}$, $N_{ik} = \varepsilon_{ijk} n_j$, we can write the rotation R in the form

$$R = I + N\sin\theta + N^2(1 - \cos\theta)$$

Now, $N^2 = \mathbf{nn}^T - 1$, $N\mathbf{n} = 0$ and so $N^3 = -N$, $N^4 = -N^2$.
This enables us to write R in the form

$$R = \exp(N\theta) = I + N\sum_r (-1)^r \frac{\theta^{2r+1}}{(2r+1)!} + N^2 \sum_r (-1)^r \frac{\theta^{2r}}{(2r)!}$$

$$= I + \sum_r N^r \frac{\theta^r}{r!}$$

Once R is given, the angle θ may be computed from

$$\operatorname{tr} R = 1 + 2\cos\theta$$

and then N and $\mathbf{n}$ can be found from

$$N \sin\theta = \tfrac{1}{2}\,(R - R^T)$$

2.1.5 Tensor Measures of Strain

We have seen that the change in shape of any cell can be calculated when we know which three fibres in the initial configuration are principal and by how much each is stretched. Moreover, this is the minimal information for this purpose.

In this sense a suitable strain measure can be found by looking at the change in the length of $\mathrm{d}\boldsymbol{\xi}$

$$|\mathrm{d}\mathbf{x}|^2 - |\mathrm{d}\boldsymbol{\xi}|^2 = 2\,\mathrm{d}\boldsymbol{\xi}^T\, E\,\mathrm{d}\boldsymbol{\xi} = 2\,\mathrm{d}\mathbf{x}^T\, e\,\mathrm{d}\mathbf{x}$$

where

$$E = \tfrac{1}{2}\,(F^T F - I) = \tfrac{1}{2}\,(U^2 - I)$$

and

$$e = \tfrac{1}{2}\,(I - (F\,F^T)^{-1})$$

associated with some specified background frame.

The components E of the *Green strain* generate a tensor whose axes define the reference directions of the Lagrangian principal fibres and whose principal values are

$$\tfrac{1}{2}\,(\lambda_r^2 - 1) \qquad r = 1, 2, 3$$

This strain measure is *objective* (i.e. unaffected by any final rotation) and it vanishes when, and only when, the shape is unchanged.

The normalising factor $\tfrac{1}{2}$ is included so that when the strain is infinitesimal, the principal values of the measure are just the fractional changes in length of the principal fibres; since, if $\lambda_r = 1 + \varepsilon$

$$\tfrac{1}{2}\,[(1+\varepsilon)^2 - 1] \approx \varepsilon$$

when ε is first order.

We have comparable properties for the *Cauchy strain* in terms of the Eulerian triad.

In terms of the displacement $\mathbf{u}$ between the deformed and reference configurations

$$\mathbf{u} = \mathbf{x} - \boldsymbol{\xi}$$

and its gradients, the components of the strain tensors are given by

i) Green strain tensor

$$E_{ij} = \tfrac{1}{2}\left[\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i} + \frac{\partial u_k}{\partial \xi_i}\frac{\partial u_k}{\partial \xi_j}\right]$$

and

ii) Cauchy strain tensor

$$e_{ij} = \tfrac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{\partial u_k}{\partial x_i}\frac{\partial u_k}{\partial x_j}\right]$$

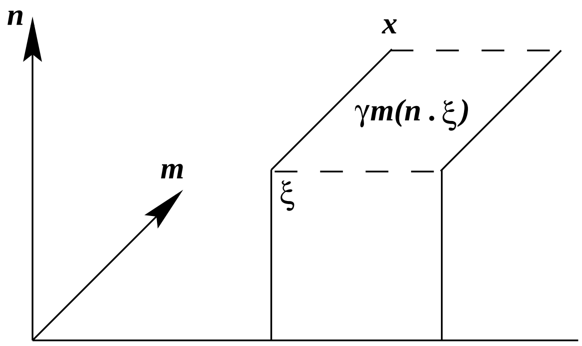
When the displacements and displacement gradients are sufficiently small, the distinction between the two small-strain definitions is usually ignored.

2.2 Plane deformation

Introduce a unit vector $\mathbf{n}$, and then in a deformation with displacement

$$\mathbf{x} - \xi = \gamma\ \mathbf{m}\,(\mathbf{n}.\,\xi) = \gamma\,\mathbf{m}(\mathbf{n}^T \xi) \tag{9}$$

particles are not displaced in the *basal plane* $\mathbf{n}^T \xi = 0$ irrespective of the direction of the unit vector $\mathbf{m}$.



The deformation gradient

$$F = I + \gamma\ \mathbf{m}\mathbf{n}^T$$

and the amount of deformation varies with the height of ξ above the basal plane $(\mathbf{n}.\,\xi)$

1.2.1 Shear

An equivoluminal plane deformation is called a shear, the principal stretches are then such that

$$\lambda_3 = 1 \quad \text{with} \quad \lambda_1\lambda_2 = 1$$

If $\mathbf{m}^T\mathbf{n} = 0$ the translation components in (9) are parallel to the basal plane and volume is conserved. This is simple shear

$$F = I + \gamma\ \mathbf{m}\mathbf{n}^T, \quad \mathbf{m}^T\mathbf{n} = 0$$

γ is called the amount of shear and a plane $\mathbf{n}^T \boldsymbol{\xi} = const$ a shearing plane.

2.2.2 Example of a shear process

Consider a type of shearing

$$x_1 = s\xi_1 + t\xi_2/s, \quad x_2 = \xi_2/s, \quad x_3 = \xi_3$$

with $s \geq 1$, $t \geq 0$ ($s \equiv 1$ is a path of simple shear). Show that any *resultant* simple shear can be obtained by a path of this type in (s, t) space such that the same fibres are principal at *every* stage.

SOLUTION

We have a homogeneous deformation gradient

$$F = \begin{pmatrix} s & t/s \\ 0 & 1/s \end{pmatrix}, \qquad \det F = 1$$

We require the principal axes of

$$F^T F = \begin{pmatrix} s^2 & t \\ t & (t^2+1)/s^2 \end{pmatrix}$$

to be fixed. For a matrix of the form

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix}$$

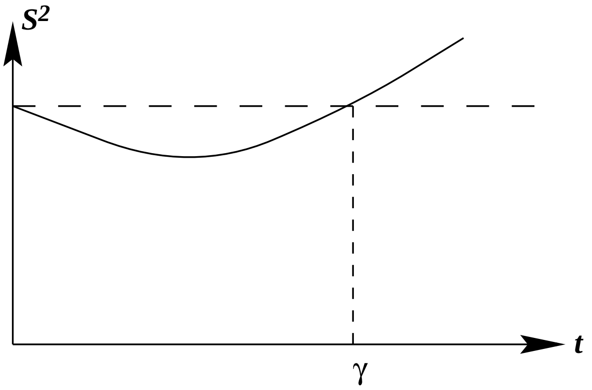
the inclinations of the principal axes depend only on $(a - b)/c$.

Because we require the final state to be a simple shear the trajectory in (s, t) space has to pass through $s = 1$, $t = \gamma$ and $s = 1$, $t = 0$ to include the undeformed state. To keep the inclination of the principal axes fixed we must have

$$\frac{1}{t}\left(\frac{t^2+1}{s^2} - s^2\right) = \frac{1}{\gamma}\left(\gamma^2 + 1 - 1\right) = \gamma$$

and so the path is given by

$$s^4 + \gamma s^2 t - t^2 = 1, \quad 0 \leq t \leq \gamma$$



2.3 Motion

Suppose we follow a given particle ξ (a Lagrangian viewpoint) then the time rate of change of interest is

$$\left.\frac{\partial}{\partial t}\right)_{\xi}$$

but, using the chain rule we find that the 'material time derivative'

$$\frac{D}{Dt}\psi \equiv \left.\frac{\partial}{\partial t}\right)_{\xi}\psi = \left.\frac{\partial}{\partial t}\right)_{\mathbf{x}}\psi + \left.\frac{\partial}{\partial t}\right)_{\xi} x_i \frac{\partial}{\partial x_i}\psi$$

in terms of an Eulerian decomposition.

We define the *velocity field* corresponding to a particle initially at ξ as

$$\mathbf{v} = \left.\frac{\partial}{\partial t}\right)_{\xi}\mathbf{x}(\xi, t) \equiv \frac{D}{Dt}\mathbf{x}$$

The *acceleration field* is then

$$\mathbf{f} = \left.\frac{\partial}{\partial t}\right)_{\xi}\mathbf{v}(\xi, t) \equiv \frac{D}{Dt}\mathbf{v}$$

and in the Eulerian viewpoint

$$\mathbf{f} = \left.\frac{\partial}{\partial t}\right)_{\mathbf{x}}\mathbf{v}(\mathbf{x}, t) + \left(\mathbf{v}.\frac{\partial}{\partial \mathbf{x}}\right)\mathbf{v}$$

where we have written $\partial/\partial\mathbf{x}$ for the gradient operator in the Eulerian configuration (for fluids this is the most common viewpoint to use)

In general, then the mobile derivative following the particles is

$$\frac{D}{Dt} \equiv \left.\frac{\partial}{\partial t}\right)_{\mathbf{x}} + \left(\mathbf{v}.\frac{\partial}{\partial \mathbf{x}}\right) \tag{10}$$

2.4 The Continuity Equation

Consider a volume V_o in the undeformed state and monitor its state during deformation, the conservation of mass requires that the integral

$$M_o = \int_{V_o} \rho_o(\xi)\, \mathrm{d}V_o$$

should remain invariant. In terms of the deformed state, the mass

$$M_o = \int_{V} \rho(\mathbf{x}, t)\, \mathrm{d}V = \int_{V_o} \rho(\mathbf{x}, t) J(t)\, \mathrm{d}V_o$$

where we have used (2). If we now equate the two expressions for M_o we have the expression for the conservation of mass in the Lagrangian representation

$$\int_{V_o} [\rho_o(\xi) - \rho(\mathbf{x}, t) J(t)]\, \mathrm{d}V_o = 0$$

This relation must be true for volume V_o and so the integrand must vanish, which gives an alternative derivation of the density relation deduced above in section 2.0.3 .

Now the density in the undeformed state is a constant whatever the deformation and so

$$\left.\frac{\partial}{\partial t}\right)_{\xi} \rho_o(\xi) = \left.\frac{\partial}{\partial t}\right)_{\xi} \rho(\mathbf{x}, t) J(t) = \frac{D}{Dt}\, \rho(\mathbf{x}, t) J(t) = 0$$

The material time derivative of density can therefore be found from

$$\frac{D}{Dt}\, \rho(\mathbf{x}, t) + \rho(\mathbf{x}, t) J^{-1}(t)\, \frac{D}{Dt}\, J(t) = 0$$

From the properties of the determinant of the deformation gradient

$$\frac{D}{Dt}\, J(t) = \left.\frac{\partial}{\partial t}\right)_{\xi} J(t) = J(t) \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t) \tag{11}$$

in terms of the velocity field $\mathbf{v}(\mathbf{x}, t)$. The material time derivative of the density is thus determined by

$$\frac{D}{Dt}\, \rho(\mathbf{x}, t) + \rho(\mathbf{x}, t)\, \frac{\partial}{\partial \mathbf{x}} \cdot \mathbf{v}(\mathbf{x}, t) = 0 \tag{12}$$

When we make use of the representation (10) for material time derivative we can write the Eulerian form of the equation of mass conservation in the form

$$\left.\frac{\partial}{\partial t}\right)_{\mathbf{x}} \rho(\mathbf{x}, t) + \frac{\partial}{\partial \mathbf{x}} \cdot [\rho(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t)] = 0 \tag{13}$$

which is known as the continuity equation. The relation (13) can also be deduced by considering a fixed volume in space and then allowing for mass flux across the boundary.

A consequence of (13) is that for any scalar, vector or tensor field ψ and a material volume V with surface S and outward normal $\hat{\mathbf{n}}$

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \rho\psi\,\mathrm{d}V = \int_V \rho\,\frac{D}{Dt}\,\psi\,\mathrm{d}V$$

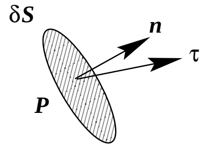
and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V \psi\,\mathrm{d}V = \int_V \frac{\partial}{\partial t}\,\psi\,\mathrm{d}V + \int_S \psi\mathbf{v}.\,\hat{\mathbf{n}}\,\mathrm{d}S$$

3 The Stress Field Concept

3.0.1 Traction

Consider any point P of a continuum and any infinitesimal plane element δS with P as centroid. Let $\mathbf{x}$ be the position vector of P, and $\mathbf{n}$ the unit normal to δS (in some consistent sense).



Then the mechanical influence exerted across δS **by** the material on the side to which $\mathbf{n}$ points is postulated to be represented by a force

$$\boldsymbol{\tau}(\mathbf{N}, \mathbf{x})\, \delta S$$

through P. The negative of this force is exerted across δS **on** this material.

$\boldsymbol{\tau}$ is called the *traction vector* and has the dimensions of force per unit area.

Note: a mutual couple could also be introduced across δS, but the need for it has never been indisputably established in any particular situation

The properties of traction can be determined from a hypothesis due substantially to Euler:

- the vector fields of force and mass-acceleration in a continuum have equal linear and rotational resultants over any part of the continuum.

Consider a volume V, bounded by a surface S with outward normal $\mathbf{n}$, and dm a generic mass element in V. Set

f - local acceleration
g - local body force per unit mass (e.g. gravitational or fictitious)
$\boldsymbol{\mu}$ - local body moment per unit mass (e.g. electrostatic or magnetic)

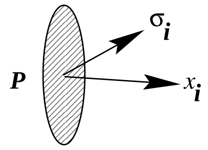
Then we have

$$\int_S \boldsymbol{\tau}\, \mathrm{d}S + \int_V \mathbf{g}\, \mathrm{d}m = \int_V \mathbf{f}\, \mathrm{d}m \tag{1}$$

$$\int_S \mathbf{x}\times\boldsymbol{\tau}\, \mathrm{d}S + \int_V \mathbf{x}\times\mathbf{g}\, \mathrm{d}m + \int_V \boldsymbol{\mu}\, \mathrm{d}m = \int_V \mathbf{x}\times\mathbf{f}\, \mathrm{d}m \tag{2}$$

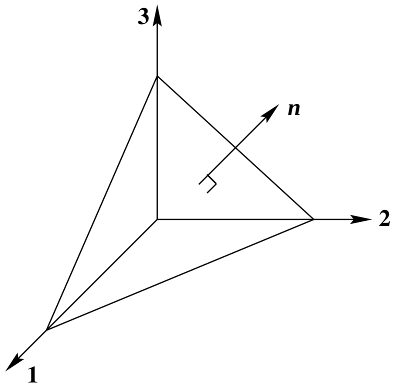
3.0.2 Tensor Character of Stress

At a point P consider a plane element perpendicular to the background axis x_i. Let $\boldsymbol{\sigma}_i$ denote the vector traction exerted over this by the material situated on its positive side.



The components σ_{ij} of $\boldsymbol{\sigma}_i$ constitute the *stress-matrix*. σ_{11}, σ_{22}, σ_{33} are called the *normal components* and the others *shear components*.

The stress matrix fully specifies the mechanical state at P, in the sense that it determines the traction $\boldsymbol{\tau}$ for any $\mathbf{n}$. Consider an infinitesimal tetrahedron at P with 3 faces parallel to the coordinate planes and the fourth with outward normal $\mathbf{n}$ and area ε^2.



We now apply the equation (1) for the linear force resultant to this volume. Suppose that $\mathbf{n}$ falls in the first octant; the facial areas are then

$$(n_1, n_2, n_3, 1)\,\varepsilon^2$$

the surface integral over the tetrahedron is

$$\int_S \boldsymbol{\tau}\,\mathrm{d}S = (\boldsymbol{\tau} - n_i\boldsymbol{\sigma}_i)\,\varepsilon^2 + \mathrm{O}(\varepsilon^3) = \int_V (\mathbf{f} - \mathbf{g})\mathrm{d}m$$

The same value is obtained when $\mathbf{n}$ falls in any other octant, or when $\mathbf{n}$ is perpendicular to one axis (in which case the tetrahedron is replaced by a triangular prism).

Now the mass integrals over the small volume are of order $\mathrm{O}(\varepsilon^3)$ because we can take $\mathbf{f}$, $\mathbf{g}$ constant within the volume, and so as $\varepsilon \to 0$,

$$\int_S \boldsymbol{\tau}\,\mathrm{d}S = 0$$

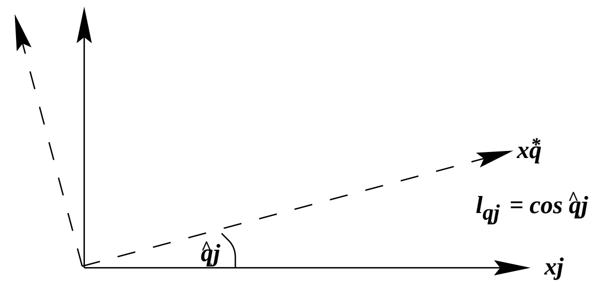
Thus

$$\boldsymbol{\tau}(\mathbf{n}) = n_i\,\boldsymbol{\sigma}_i, \quad \text{i.e.} \quad \tau_j(\mathbf{n}) = n_i\,\boldsymbol{\sigma}_{ij} \tag{3}$$

which is known as *Cauchy's relation*.

We note that (3) also implies that σ_{ij} transforms like a second rank tensor. Consider a new set of rectangular axes indicated by stars and set

$l_{pi} =$ direction cosine of pth starred axis and ith old axis



Then

$$\sigma^*_{pq} = \sigma_{ij} l_{pi}\, l_{qj} \tag{4}$$

since

$$\boldsymbol{\tau}^*_q = l_{qj}\boldsymbol{\tau}_j, \qquad \boldsymbol{\sigma}_i = l_{pi}\boldsymbol{\sigma}^*_p, \quad \text{as} \;\; l_{ip} l_{jp} = \delta_{ij}.$$

Alternatively, in terms of the rotation R which carries the unstarred axes into the starred ones

$$R_{ip} = l_{pi}$$

and

$$\boldsymbol{\sigma}^* = R^T\,\boldsymbol{\sigma}\,R \tag{4*}$$

3.1 Local equations of linear motion

We rewrite (1) in component form

$$\int_S \tau_j\,\mathrm{d}S = \int_S \sigma_{ij}\,n_i\,\mathrm{d}S = \int_V (f_j - g_j)\,\mathrm{d}m$$

and then use the tensor divergence theorem to obtain

$$\int_S \sigma_{ij}\,n_i\,\mathrm{d}S = \int_V \frac{\partial}{\partial x_i}\,\sigma_{ij}\,\mathrm{d}V = \int_V \frac{\partial}{\partial x_i}\,\sigma_{ij}\,\frac{\mathrm{d}m}{\rho}$$

where ρ is the local density and the stress field is assumed to be continuous and differentiable.

Thus we have

$$\int_V \frac{\partial}{\partial x_i} \sigma_{ij} \frac{\mathrm{d}m}{\rho} = \int_V (f_j - g_j)\,\mathrm{d}m$$

and since the region V is arbitrary

$$\frac{\partial}{\partial x_i} \sigma_{ij} + \rho g_j = \rho f_j$$

We see that it is the *stress gradient*, in conjunction with the body force that determines the local acceleration.

If $\partial\sigma_{ij}/\partial x_i = 0$, the stress field is said to be *self-equilibrated*

Note: the local equations (3) $[\tau_j = \boldsymbol{\sigma}_{ij} n_i]$ and (5) imply the global equation (1).

3.1.1 Symmetry of stress tensor

The component form of (2) is

$$\int_S \varepsilon_{ijk} x_j \tau_k \,\mathrm{d}S = \int_V \varepsilon_{ijk} \frac{\partial}{\partial x_l} (x_j \sigma_{lk}) \frac{\mathrm{d}m}{\rho}$$

$$= -\int_V \mu_i \,\mathrm{d}m + \int_V \varepsilon_{ijk} x_j (f_k - g_k)\,\mathrm{d}m$$

From (5)

$$\int_V \varepsilon_{ijk} \left[\frac{\partial}{\partial x_l} (x_j \sigma_{lk}) - x_j \frac{\partial}{\partial x_l} \sigma_{lk} \right] \mathrm{d} \frac{m}{\rho} = -\int_V \mu_i \,\mathrm{d}m$$

i.e.

$$\int_V \varepsilon_{ijk} \delta_{jl} \sigma_{lk} \,\mathrm{d}m = \int_V \varepsilon_{ijk} \sigma_{jk} \mathrm{d}m = -\rho \int_V \mu_i \,\mathrm{d}m$$

Since the region V is again arbitrary

$$\varepsilon_{ijk}\, \sigma_{jk} = -\mu_i \qquad (6)$$

and we may recover (2) from these local equations.

We will subsequently suppose that *body moments are absent* $(\mu_i \equiv 0)$ and then

$$\sigma_{jk} = \sigma_{kj}$$

and the stress matrix is *symmetric*.

3.1.2 Stress Jumps (Continuity Conditions)

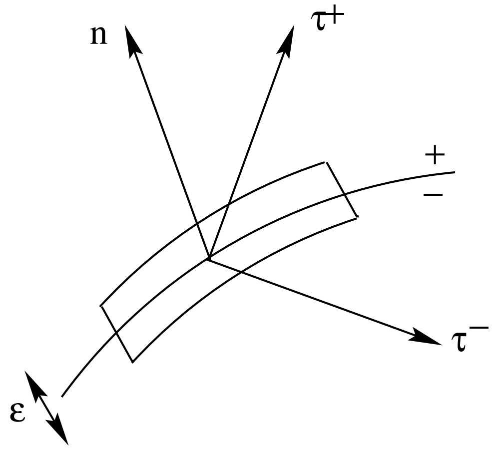
A discontinuity in material properties across an internal interface generally causes a discontinuity in the stress tensor.

Such interfaces are present in simple laminates like plywood and in composite solids of any kind. Well known composites are rocks, reinforced concrete, glass-fibre resins (used in the hull of boats), cermets - sintered mixtures of ceramic inclusions in a metallic matrix (used in cutting tools), ceramic fibre-reinforced materials (used in turbine blades).

Possible jumps in the stress components at an interface are, however, restricted by the inviolable continuity of the traction vector acting on the interface

$$[\tau_j]_-^+ = n_i[\sigma_{ij}]_-^+ = 0 \tag{8}$$

where $\mathbf{n}$ is the interface normal.



We may prove continuity of traction by applying (1) to a vanishingly thin disc enclosing an interfacial area A

$$\int_A \boldsymbol{\tau}^+ \,\mathrm{d}A^+ + \int_A \boldsymbol{\tau}^- \,\mathrm{d}A^- + \mathbf{O}(\varepsilon) = \mathbf{0}$$

where $\mathbf{O}(\varepsilon)$ arises from tractions on the edge, body forces and finite mass-accelerations (we exclude idealised 'shocks' with a jump in velocity).

In the limit as $\varepsilon \to 0$

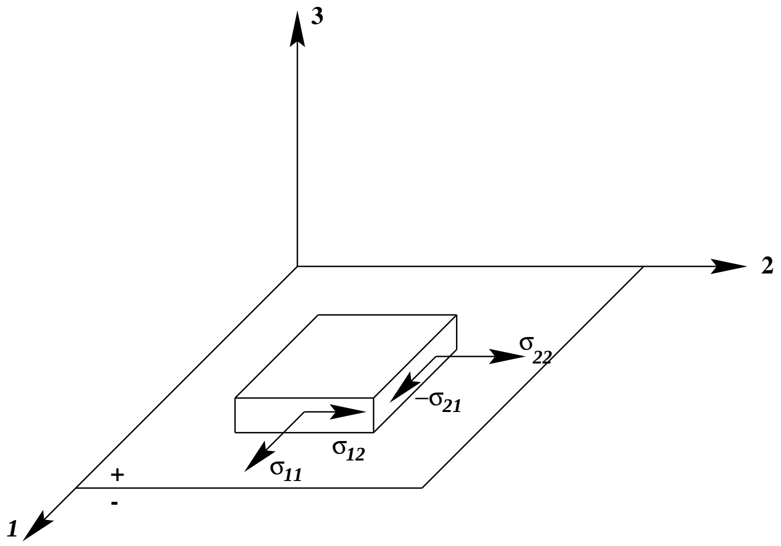
$$\boldsymbol{\tau}^+(\mathbf{n}) + \boldsymbol{\tau}^-(-\mathbf{n}) = \mathbf{0}, \quad \text{i. e.} \quad [\boldsymbol{\tau}(\mathbf{n})]_-^+ = \mathbf{0}$$

Since (8) imposes only 3 constraints on 6 independent components of σ_{ij}, the tensor is not necessarily continuous at an interface. Consider a local coordinate scheme with x_1, x_2 in the tangent plane and x_3 normal:

Then $\boldsymbol{\tau}$ in (8) becomes $\boldsymbol{\sigma}_3$ and so

$$[\sigma_{31}] = [\sigma_{32}] = [\sigma_{33}] = 0$$

consequently $[\sigma_{13}] = [\sigma_{23}] = 0$, but we cannot require $[\sigma_{11}]$, $[\sigma_{22}]$, $[\sigma_{12}]$ to be zero, simply from (8).



In a similar way, at a body surface, such 'interior' tractions are not constrained by the applied loads. In particular, even where the load is zero, the local stress tensor does not have to vanish completely - such a case is provided by a rod under uniform tension.

3.2 Principal Basis for Stress

The traction vector $\boldsymbol{\tau}(\mathbf{n})$ over a plane element is purely normal when $\boldsymbol{\tau} = \sigma \mathbf{n}$ for some σ. Thus since $\tau_j = \sigma_{ij} n_i$, we have

$$(\sigma_{ij} - \sigma \delta_{ij}) n_i = 0 \tag{9}$$

The eigenvectors $\pm\mathbf{n}_{\mathrm{r}}$ ($\mathrm{r} = 1, 2, 3$) are called the *principal axes of stress* and the eigenvalues σ_{r} given by the roots of $\det(\sigma_{ij} - \sigma\delta_{ij}) = 0$ are called the *principal stresses*.

Since σ_{ij} is symmetric, the principal stresses σ_{r} are real and the axes are orthogonal.

Figure Not Available.

Note that each σ_{r} acts in opposite directions on opposite faces of a 'principal' box. The vectors $\mathbf{n}_{\mathrm{r}}$ can also be thought of geometrically as the principal axes of the stress quadric surface

$$\sigma_{ij}\, x_i x_j = const$$

When the principal axes and stresses are known, stress components on background axes are given by the spectral formula

$$\boldsymbol{\sigma} = \sum_r \sigma_r\, n_r\, n_r^T \tag{10}$$

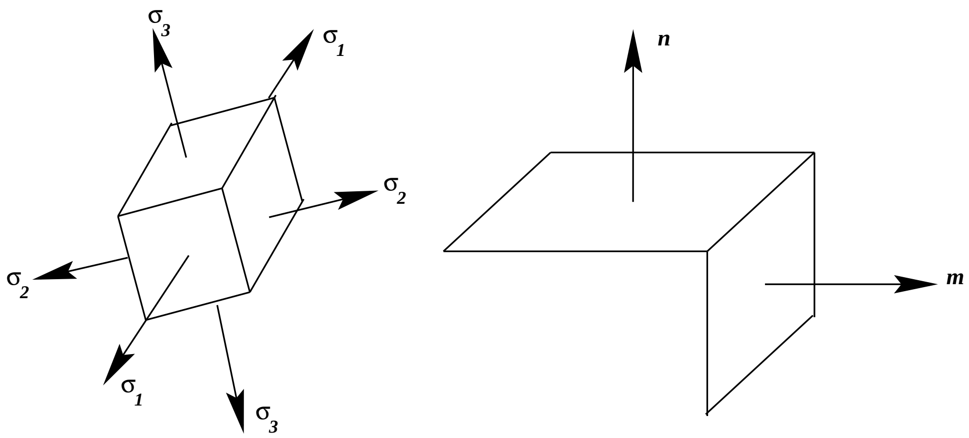
Note: any quantity which is solely a function of the σ_r is an invariant of σ_{ij}. The cubic equation for the principal stresses σ_r can be written as

$$\det(\sigma_{ij} - \sigma) = -\sigma^3 + I_\sigma \sigma^2 - II_\sigma \sigma + III_\sigma = 0$$

in terms of the invariants:

$$I_\sigma = \sigma_1 + \sigma_2 + \sigma_3 = \sigma_{kk}$$
$$II_\sigma = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1 = \tfrac{1}{2}\,[\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji}]$$
$$III_\sigma = \sigma_1\sigma_2\sigma_3 = \det \boldsymbol{\sigma}$$

On occasion one may need the stress tensor components with respect to a particular set of axes, and it is often easier to use the general tensor transformation rule (4) fron the principal basis, rather than use the spectral form (10).



If, in particular, we seek the normal component of traction on an element with normal **n**, and the shear component in a direction **M** with $\mathbf{m}.\,\mathbf{n} = 0, |\mathbf{m}| = 1$, then we should use

$$\boldsymbol{\tau}(\mathbf{n}) = (n_1\sigma_1, n_2\sigma_2, n_3\sigma_3)$$

using $\boldsymbol{\tau} = n_i \boldsymbol{\sigma}_i$, $\mathbf{n} = (n_1, n_2, n_3)$ in a principal basis. Then

$$\mathbf{n}.\,\boldsymbol{\tau}(\mathbf{n}) = n_1^2\sigma_1 + n_2^2\sigma_2 + n_3^2\sigma_3 \tag{11}$$

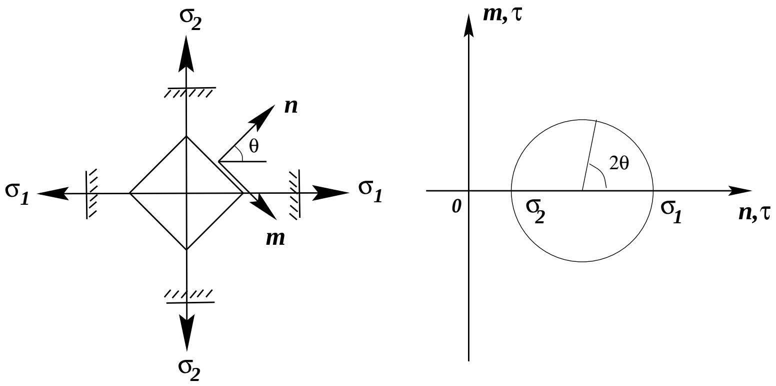
$$\mathbf{m}.\,\boldsymbol{\tau}(\mathbf{n}) = m_1 n_1 \sigma_1 + m_2 n_2 \sigma_2 + m_3 n_3 \sigma_3 \tag{12}$$

$$= \mathbf{n}.\,\tau(\mathbf{m})$$

From (12), any normal stress lies between the algebraically least and greatest principal stresses.

3.2.1 Stress Circle

Consider the tractions on a box with one pair of faces perpendicular to a principal axis. Then the traction on this pair is purely normal, σ_3, and the tractions on the other faces have no components in that direction.



Consider the normal in the $\widehat{12}$ plane an angle θ to the σ_1 axis, then

$$\mathbf{n} = (\cos\theta, \sin\theta, 0)$$
$$\mathbf{m} = (\sin\theta, -\cos\theta, 0)$$

so that the normal and shear stress components in the $\widehat{12}$ plane are

$$\mathbf{n}.\boldsymbol{\tau} = \cos^2\theta\,\sigma_1 + \sin^2\theta\,\sigma_2$$

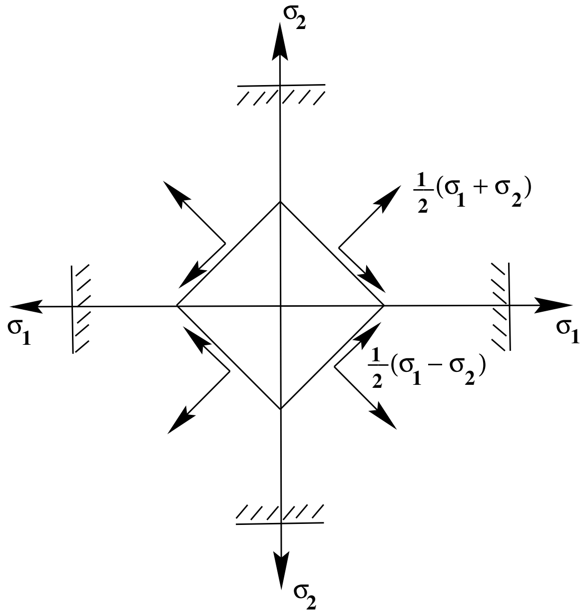
$$= \tfrac{1}{2}(\sigma_1+\sigma_2) + \tfrac{1}{2}(\sigma_1-\sigma_2)\cos 2\theta$$

$$\mathbf{m}.\boldsymbol{\tau} = \tfrac{1}{2}(\sigma_1-\sigma_2)\sin 2\theta$$

As θ is varied from 0 to π, the locus of $\mathbf{m}.\boldsymbol{\tau}$ versus $\mathbf{n}.\boldsymbol{\tau}$ is a circle: centre $\tfrac{1}{2}(\sigma_1+\sigma_2)$, radius $\tfrac{1}{2}(\sigma_1-\sigma_2)$, where we have assumed that σ_1 is greater than σ_2.
The shear stress is largest when $\theta = \pi/4$

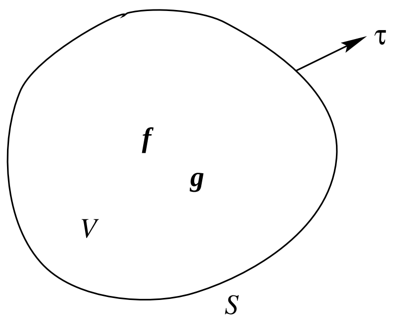
$$(\mathbf{m}.\boldsymbol{\tau})_{\max} = \tfrac{1}{2}(\sigma_1-\sigma_2)$$

and then the normal tractions are equal to $\tfrac{1}{2}(\sigma_1+\sigma_2)$ on all four faces.



3.3 Virtual Work Rate Principle

Consider a material in a bounded region with volume V and surface S, subject to surface tractions $\boldsymbol{\tau}$ and body forces $\mathbf{g}$.



The entire body must satisfy the equations of linear and angular motion

$$\int_S \boldsymbol{\tau}\,\mathrm{d}S + \int_V \mathbf{g}\,\mathrm{d}m = \int_V \mathbf{f}\,\mathrm{d}m \tag{1}$$

$$\int_S \mathbf{x}\times\boldsymbol{\tau}\,\mathrm{d}S + \int_V \mathbf{x}\times\mathbf{g}\,\mathrm{d}m = \int_V \mathbf{x}\times\mathbf{f}\,\mathrm{d}m \tag{2}$$

A stress field σ_{ij} is said to be compatible with these constraints if it possesses the properties we have just established

$$\begin{aligned} &\sigma_{ij} \text{ is symmetric} \\ &\frac{\partial}{\partial x_i}\sigma_{ij} = \rho(f_j - g_j) \quad \text{in } V \\ &n_i\sigma_{ij} = \tau_j \qquad \text{on } S \\ &n_i[\sigma_{ij}]_-^+ = 0 \quad \text{on any jump surface } J \end{aligned} \tag{13}$$

For a piecewise continuous and piecewise continuously differentiable stress field σ_{ij} and a field $\mathbf{v}^*$ which is continuous and piecewise continuously differentiable:

$$\int_S \boldsymbol{\tau}.\mathbf{v}^*\,\mathrm{d}S + \int_V \mathbf{g}.\mathbf{v}^*\,\mathrm{d}m = \int_V \sigma_{ij}\frac{\partial}{\partial x_i}v_j^*\,\mathrm{d}V + \int_V \mathbf{f}.\mathbf{v}^*\,\mathrm{d}m \tag{14}$$

Usually we would take $\mathbf{v}^*$ to be a velocity or an infinitesimal displacement. When $\mathbf{v}$ is velocity then as we shall see later (14) corresponds to the balance of externally applied and internal work rates.

PROOF

$$\int_S \boldsymbol{\tau}.\mathbf{v}^*\,\mathrm{d}S = \int_S n_i\sigma_{ij}v_j^*\,\mathrm{d}S$$

$$= \int_V \frac{\partial}{\partial x_i}(\sigma_{ij}v_j^*)\,\mathrm{d}V + \int_J n_i[\sigma_{ij}]_-^+ v_j^*\,\mathrm{d}J$$

$$= \int_V \left(v_j^*\frac{\partial}{\partial x_i}\sigma_{ij} + \sigma_{ij}\frac{\partial}{\partial x_i}v_j^*\right)\mathrm{d}V$$

$$= \int_V \sigma_{ij}\frac{\partial}{\partial x_i}v_j^*\,\mathrm{d}V + \int_V v_j^*\rho(f_j - g_j)\,\mathrm{d}V$$

using the conditions (13).•

3.3.1 Converse Principle

If σ_{ij} is a symmetric tensor field such that (14) holds for arbitrary $\mathbf{v}^*$, then σ_{ij} is compatible with the constraints and conforms to (1) and (2).

Now

$$\int_S n_i\sigma_{ij}v_j^*\,\mathrm{d}S = \int_V \frac{\partial}{\partial x_i}(\sigma_{ij}v_j^*)\,\mathrm{d}V + \int_J n_i[\sigma_{ij}]_-^+ v_j^*\,\mathrm{d}J$$

and subtracting this from (14) we finally obtain

$$\int_S (\tau_j - n_i\sigma_{ij})v_j^*\,\mathrm{d}S + \int_J n_i[\sigma_{ij}]_-^+ v_j^*\,\mathrm{d}J + \int_V [\frac{\partial}{\partial x_i}\sigma_{ij} - \rho(f_j - g_j)]v_j^*\,\mathrm{d}V$$

and since v_j^* is arbitrary, each integrand must vanish. We are therefore able to recover (13) and hence (1) and (2).

4 Constitutive Relations

The balance laws for linear and angular momentum (3.1), (3.2) do not make any reference to the properties of the body. The full specification of the mechanical and thermal properties of a material requires further information as to the relation between stress and strain. This is provided by a constitutive equation which give a mathematical representation of the functional dependence between stress and strain, which is designed to give agreement with the observed behaviour.

A constitutive relation will therefore be of the form

$$stress = functional\,(deformation)$$

and the possible behaviour is restricted by a set of postulates which are designed to reconcile the mathematical and physical behaviour.

a) Determinism

The stress at time t depends only on the history of the deformation up to t and not on future values.

b) Local Action

The stress at ξ at some time t depends only on the deformation in the neighbourhood of ξ

c) Material Objectivity

Constitutive equations are form invariant with respect to rigid motions of the spatial frame of reference.

d) Material Invariance (Symmetry)

Constitutive equations are form invariant with respect to a symmetry group. For *fluids* the symmetry group is the full unimodular group. For *solids* the symmetry group is restricted to a group of orthogonal transformations of the material coordinates.

It is only at this stage that we meet any distinction between fluids and solids.

The first principle a) allows for material with memory as in visoelastic behaviour e.g. plastics, the earth's upper mantle.

4.0.1 Simple Materials

We will here consider a special class of materials called *simple materials* for which

$$\sigma(\xi,t) = f(F^t,\xi,t) \tag{1}$$

where F^t denotes the history up to time t of the deformation gradient $F(\xi,t)$ and f is a functional in time alone.

In this case the principles a) and b) are satisfied automatically. In order to meet the requirement of material objectivity c) we consider the effect of a rigid rotation. We rotate the spatial frame by Q^T, then

$$\mathbf{x}^* = Q\mathbf{x}, \quad F^* = QF$$

and the stress transforms as a second rank tensor

$$\sigma^* = Q\,\sigma\,Q^T$$

After rotation we must have the same functional form as in (1) i.e.

$$\sigma^*(t) = f(F^{*t},\xi,t)$$

and thus

$$Q\,f(F^t)Q^T = f([QF^t]), \qquad \text{for all } Q \tag{2}$$

We can proceed further by making the polar decomposition $F = RU$. Since (2) is an identity in Q, choose $Q = R^T$. Then

$$R^T\,f(F^t,\xi,t) = f([R^T F]^t,\xi,t) = f(U^t,\xi,t)$$

and so

$$f(F^t,\xi,t) = R\,f(U^t,\xi,t)\,R^T$$

satisfies a), b) and c) with

$$\sigma(\xi,t) = R\,f(U^t,\xi,t)\,R^T \tag{3}$$

The stress therefore depends on the history of deformation only through the stretch U or equivalently on any strain measure such as

$$E = \tfrac{1}{2}(F^T F - I) = \tfrac{1}{2}(U^2 - I)$$

If we write

$$f(U^t) = U\,g(E^t)\,U^T$$

then the stress tensor

$$\sigma = RU\,g(E^t)\,U^T R^T = F\,g(E^t)F^T$$

We may therefore write the stress behaviour in the form

$$\sigma(\xi, t) = F\, g(E^t, \xi, t)\, F^T \tag{3a}$$

in terms of a functional of strain.

4.0.2 Material Symmetry

When we use the represenation $F = RU$ we are representing the form of the deformation in terms of stretching an element and then rotating it. If the material possesses some intrinsic symmetry, then when certain rotations are applied before the deformation RU, the resulting stress would be the same.

Suppose the material has symmetry with respect to the orthogonal transformation P, then states FP and F will be indistinguishable. We therefore require that

$$\sigma = f(F) = f(FP) \tag{4}$$

for all F and any P in the symmetry group of the material. If the body is *isotropic* then (4) holds for any P.

With the strain representation of the stress (3a), the symmetry property (4) gives

$$Fg(E)F^T = FPg(P^T EP)P^T F^T$$

since $E = \frac{1}{2}(F^T F - I)$. As a result we find that

$$P^T g(E)P = g(P^T EP) \tag{4a}$$

4.0.3 Functional dependence

We can describe different classes of material by the functional dependence appearing in the constitutive relation. For viscoelastic materials we need the full dependence on the history of deformation, but for many materials such effects are not important.

We can characterise a *fluid* by the assumption that the deviatoric part of the stress depends on the current strain rate $\dot{\varepsilon} = \partial \mathbf{v}/\partial \mathbf{x}$ i.e.

$$\sigma = -\,pI + f(\dot{\varepsilon}, \rho)$$

where I is the unit tensor. If the dependence on ε is a linear function we have a Newtonian viscous fluid.

For solid materials, the deformation itself is a more natural variable than strain rate. Thus elastic behaviour can be characterised by

$$\mathit{stress} = \mathit{function}(\mathit{deformation})$$

$$\text{i.e. } \sigma = f(F)$$

and so from (3) $\sigma = Rf(U)R^T$.

This approach (Cauchy elasticity) may be used with material symmetry arguments to find the form of the function f. However, a slightly more restrictive development based on the idea of a work function [due to Green] seems to be in close accord with experimental viewpoints and we will adopt this approach in our subsequent development.

4.1 Energy balance

We cannot separate the mechanical aspects of deformation from thermal and other changes and have therefore to introduce the concepts of continuum thermodynamics. In many applications, thermal effects are small and are often neglected but we should be aware how they may enter into the consideration of the deformation of a medium.

As in section 3.3 we consider a volume V with surface S (and outward normal $\hat{\mathbf{n}}$) subject to surface tractions $\boldsymbol{\tau}$ and body forces $\mathbf{g}$. The work rate of the body and surface forces for a velocity field $\mathbf{v}$ is then

$$\frac{\mathrm{d}}{\mathrm{d}t} Y = \int_S \boldsymbol{\tau}.\mathbf{v}\, \mathrm{d}S + \int_V \mathbf{g}.\mathbf{v}\, \mathrm{d}m \tag{5}$$

We can recast the right hand side of (5) in terms of the acceleration field $\mathbf{f}$ and the stress tensor σ_{ij} by using the virtual work-rate principle (2.14) with $\mathbf{v}^*$ set equal to the actual velocity $\mathbf{v}$.

$$\frac{\mathrm{d}}{\mathrm{d}t} Y = \int_V \sigma_{ij} \frac{\partial}{\partial x_i} v_j\, \mathrm{d}V + \int_V \mathbf{f}.\mathbf{v}\, \mathrm{d}m \tag{6}$$

We can recognise $\int_V \mathbf{f}.\mathbf{v}\, \mathrm{d}m$ as the rate of change of kinetic energy

$$\int_V \mathbf{f}.\mathbf{v}\, \mathrm{d}m = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \tfrac{1}{2} \rho \mathbf{v}^2\, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} \int_V \tfrac{1}{2} \mathbf{v}^2\, \mathrm{d}m \tag{7}$$

The rate of increase of thermal energy for a material volume is the resultant of the heat production by internal sources, h per unit volume, and the heat conducted across the surface S:

$$\frac{\mathrm{d}}{\mathrm{d}t} H = \int_V h\, \mathrm{d}V - \int_S \mathbf{q}.\hat{\mathbf{n}}\, \mathrm{d}S \tag{8}$$

$\mathbf{q}$ is the heat flux vector which can be related to the temperature gradient by Fourier's law of heat conduction

$$\mathbf{q} = -\kappa \frac{\partial}{\partial \mathbf{x}} T$$

where κ is the thermal conductivity of the material. We can recast the expression for the rate of change of the thermal energy in terms of a volume integral by using the divergence theorem

$$\frac{\mathrm{d}}{\mathrm{d}t} H = \int_V \left[h + \frac{\partial}{\partial \mathbf{x}} \cdot \left(\kappa \frac{\partial}{\partial \mathbf{x}} T \right) \right] \mathrm{d}V \tag{9}$$

Conservation of energy requires that the gain in the internal and kinetic energy balance the sum of the work rate by external forces and the rate of increase of thermal energy. In terms of the internal energy density U per unit mass, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \rho (U + \tfrac{1}{2} \mathbf{v}^2) \, \mathrm{d}V = \frac{\mathrm{d}}{\mathrm{d}t} (H + Y)$$

Combining the expressions (6) for the work rate and (9) for the rate of change of thermal energy we obtain

$$\int_V \rho \frac{D}{Dt} U \, \mathrm{d}V = \int_V \left[h + \frac{\partial}{\partial \mathbf{x}} \left(\kappa \frac{\partial}{\partial \mathbf{x}} T \right) + \sigma_{ij} \frac{\partial}{\partial x_i} v_j \right] \mathrm{d}V$$

Since the volume is arbitrary we may equate the integrands on the two sides of this equation and so the local form of the *principle of conservation of energy* is

$$\rho \frac{D}{Dt} U = h + \frac{\partial}{\partial \mathbf{x}} \cdot \left(\kappa \frac{\partial}{\partial \mathbf{x}} T \right) + \sigma_{ij} \frac{\partial}{\partial x_i} v_j \tag{10}$$

4.2.1 Neglect of Thermal Effects

The internal heat generation h from e.g. radioactive materials in the Earth occurs sufficiently slowly that it is not important except for very long-term deformation.

The heat transport contribution to the rate of change of internal energy

$$\frac{\partial}{\partial \mathbf{x}} \cdot \left(\kappa \frac{\partial}{\partial \mathbf{x}} T \right)$$

can be neglected in discussions of deformation in two simplifying circumstances:

i) Static Problems

if the time scale over which the deformation is occurring is sufficiently long, heat flow will occur to equalise temperature and conditions will be *isothermal*. Such is the case in static deformation in the laboratory or in the long term deformation of Earth materials.

ii) Dynamic Problems

if the time scale of deformation is short enough, the thermal state has no time to adjust to the disturbance and the effect is conditions comparable to thermal isolation. This is an *adiabatic* or *isentropic* state, appropriate to most elastic wave propagation, but not e.g. for rubber-like materials. For seismic waves in the Earth with periods of around a second and wavelengths of a few kilometres, the thermal time constant is many thousands of years and so the *adiabatic* assumption is assured. However for small specimens of the same material around 5 mm in size, the thermal time scale is much shorter (about 20 seconds) and the adiabatic assumption requires high-frequency waves (1 kHz or greater).

The form of the internal energy function will differ in the two cases, and so the mechanical response to deformation will be different. For intermediate time-scales, the roles of the deformation and thermal behaviour are closely linked and need to be treated with continuum thermodynamics.

4.3 Elastic Materials

In those cases where thermal effects can be neglected we can equate the rate of change of the energy of deformation (the *stress power*) to the deformation term in (6)

$$\int_{V_o} \frac{D}{Dt} W \,\mathrm{d}V_o = \int_V \sigma_{ij} \frac{\partial}{\partial x_i} v_j \,\mathrm{d}V \tag{11}$$

where W is called the work density or strain energy.

Now, if we work in terms of the current volume V we can use (2.2) and find

$$\int_V (\det F)^{-1} \frac{D}{Dt} W \,\mathrm{d}V = \int_V \sigma_{ij} \frac{\partial}{\partial x_i} v_j \,\mathrm{d}V$$

But, once again the volume V is arbitrary and so

$$(\det F)^{-1} \frac{D}{Dt} W = \sigma_{ij} \frac{\partial}{\partial x_i} v_j \tag{12}$$

Now using the chain rule

$$\frac{D}{Dt} W = \frac{\partial W}{\partial F_{jk}} \frac{D}{Dt} F_{jk} = \frac{\partial W}{\partial F_{jk}} \frac{D}{Dt} \left(\frac{\partial x_j}{\partial \xi_k} \right)$$

$$= \frac{\partial W}{\partial F_{jk}} \frac{\partial v_j}{\partial \xi_k} = \frac{\partial W}{\partial F_{jk}} \frac{\partial x_i}{\partial \xi_k} \frac{\partial v_j}{\partial x_i}$$

and so

$$(\det F)^{-1} \frac{\partial W}{\partial F_{jk}} F_{ik} \frac{\partial v_j}{\partial x_i} = \sigma_{ij} \frac{\partial v_j}{\partial x_i}$$

Since this is true for an arbitrary velocity field

$$\det F\, \sigma_{ij} = F_{ik} \frac{\partial W}{\partial F_{jk}} \tag{13a}$$

We already know that the stress σ depends only on the stretch U (or strain E), we therefore recast (13a) in terms of the Green strain E

$$2E_{kl} = F_{jk} F_{jl} - \delta_{kl}$$

and

$$\mathrm{d}W = \frac{\partial W}{\partial E_{kl}} \tfrac{1}{2} (F_{jk} \mathrm{d}F_{jl} + F_{jl} \mathrm{d}F_{jk})$$

$$= \frac{\partial W}{\partial E_{kl}} F_{jl} \mathrm{d}F_{jk} = \frac{\partial W}{\partial F_{jk}} \mathrm{d}F_{jk}$$

Thus we may write

$$\det F\, \sigma_{ij} = F_{ik} F_{jl} \frac{\partial W}{\partial E_{kl}} \tag{13b}$$

or

$$\det F\, \sigma = F \frac{\partial W}{\partial E} F^T = R \left[U \frac{\partial W}{\partial E} U \right] R^T \tag{13c}$$

We can now identify (13c) as having precisely the form (3a) with the functional $g(E)$ identified with the tensor derivative $\dfrac{\partial W}{\partial E}$.

From the symmetry properties (4a) we can deduce

$$\frac{\partial W}{\partial E} (P^T E P) = P^T \frac{\partial W}{\partial E} P \tag{14}$$

for all orthogonal P in the symmetry group of the elastic medium.

4.3.1 Material Symmetry

The strain energy function W will be invariant under the symmetry group of the material $\{P\}$ i.e.

$$W(F) \equiv W(FP) \tag{15}$$

and since a rigid post-rotation makes no difference

$$W(P^T FP) = W(F) = W(U)$$

4.4 Isotropic Elastic Material

If the material properties are unaffected by any rotation the solid is said to be isotropic

4.4.1 For all rotations Q

$$W(QFQ^{T)} = W(F) = W(U) = W(Q^T UQ)$$

If we choose Q as the rotation from background axes to the Lagrangian triad, then

$$Q^T FQ = \text{diag}\,\{\lambda_1, \lambda_2, \lambda_3\}$$

in some ordering of the principal stretches.

Thus for any isotropic solid, $W(F)$ is a symmetric function of the principal stretches, since it is unchanged by a rotation of $\pi/2$ about the 3 principal axis which takes $1 \rightarrow 2$ etc.

4.4.2 From the symmetry property of stress (4a) and the stress formula (13c) we have the relation (14)

$$\frac{\partial W}{\partial E}\,(Q^T EQ) = Q^T\,\frac{\partial W}{\partial E}\,Q$$

which now must hold for all rotations Q.

A consequence of isotropy is that $\partial W/\partial E$ and E are coaxial. The tensor E is symmetric with principal axes along the Lagrangian triad and principal values $\frac{1}{2}\,(\lambda_r^2 - 1)$, $r = 1, 2, 3$. In this principal basis consider a rotation P of π about the 3 axis: which takes $1 \rightarrow$ -1, and $2 \rightarrow$ -2, then $P^T EP = E$, and so from (15)

$$\frac{\partial W}{\partial E}\,(P^T EP) = P^T\,\frac{\partial W}{\partial E}\,P$$

and setting $g(E) = \partial W/\partial E$, we have $Pg = gP$ with the result that

$$g_{13} = g_{23} = 0.$$

Similarly we may show that all off-diagonal terms vanish, i.e. with respect to the Lagrangian triad

$$g(E) = \frac{\partial W}{\partial E} = \text{diag}\,\{g_1, g_2, g_3\}$$

and so $\partial W/\partial E$ and E are coaxial. This is in fact a general theorem for an isotropic tensor function W

Now from the stress relation (13c) we have

$$\det F\, \sigma = R\left[U\, \frac{\partial W}{\partial E}\, U\right] R^T$$

and so

$$R^T \sigma R = U\, \frac{\partial W}{\partial E}\, U (\det F)^{-1}$$

With the Lagrangian triad as basis the right hand side of this equation is diagonal. The action of the rotation R is to take the Lagrangian into the Eulerian triad and so, *for an isotropic solid*, the principal axes of σ lie along the Eulerian triad. Thus the Cauchy stress tensor and the Eulerian triad are coaxial.

4.4.3 Principal stresses

For a pure strain deformation, $F_{ij} = U_{ij}$, and referred to the Eulerian triad $F_{ij} = \mathrm{diag}\,\{\lambda_1, \lambda_2, \lambda_3\}$. From (13a) the principal stress σ_{r} is given by

$$\det F\, \sigma_{\mathrm{r}} = \lambda_1 \lambda_2 \lambda_3\, \sigma_{\mathrm{r}} = \lambda_{\mathrm{r}}\, \frac{\partial W}{\partial \lambda_{\mathrm{r}}} \quad \text{(no sum)} \tag{16}$$

Suppose the work function is given as $W(\alpha, \beta, \gamma)$ with $\alpha = \log(\lambda_1 \lambda_2 \lambda_3)$, $\beta = \lambda_1 + \lambda_2 + \lambda_3$, $\gamma = \frac{1}{2}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$. Then the principal stress

$$\sigma_{\mathrm{r}} = \frac{1}{\lambda_1 \lambda_2 \lambda_3}\left[\frac{\partial W}{\partial \alpha} + \frac{\partial W}{\partial \beta}\, \lambda_{\mathrm{r}} + \frac{\partial W}{\partial \gamma}\, \lambda_{\mathrm{r}}^2\right]$$

and since σ is coaxial with the Eulerian triad, λ_{r} are the principal values of RUR^T, since $F = RU = (RUR^T)R$. Since $\det F = \lambda_1 \lambda_2 \lambda_3$ we can equate principal values

$$\sigma_{\mathrm{r}} = (\det F)^{-1} R\left[\frac{\partial W}{\partial \alpha}\, I + \frac{\partial W}{\partial \beta}\, U + \frac{\partial W}{\partial \gamma}\, U^2\right]_{\mathrm{r}} R^T$$

Transferring to general axes we have

$$\sigma = (\det F)^{-1} R\left[\frac{\partial W}{\partial \alpha}\, I + \frac{\partial W}{\partial \beta}\, U + \frac{\partial W}{\partial \gamma}\, U^2\right] R^T$$

$$= (\det F)^{-1}\left[\frac{\partial W}{\partial \alpha}\, I + \frac{\partial W}{\partial \beta}\, (FF^T)^{\frac{1}{2}} + \frac{\partial W}{\partial \gamma}\, (FF^T)\right]$$

which as we would expect depends on FF^T.

4.4.4 Some Isotropic work-functions

The simplest type is $W=\phi(\lambda_1\lambda_2\lambda_3)$ depending only on change of volume. Then from (16) each $\sigma_\mathrm{r}=\phi'(\lambda_1\lambda_2\lambda_3)$, and so the stress tensor is purely hydrostatic.

A second type is $W=\psi(\lambda_{1)}+\psi(\lambda_2)+\psi(\lambda_3)$ for which $\sigma_\mathrm{r}=\lambda_\mathrm{r}\psi'(\lambda_\mathrm{r})/\lambda_1\lambda_2\lambda_3$. However this type has the defect that it predicts no change in lateral dimensions in a tension or compression test

$$\sigma_1 \neq 0,\ \sigma_2 = \sigma_3 = 0, \quad ☞ \quad \lambda_1 = \lambda_2 = 1$$

To remedy this we combine the two types

$$W = \phi(\lambda_1\lambda_2\lambda_3)+\psi(\lambda_1)+\psi(\lambda_2)+\psi(\lambda_3)$$

for which

$$\sigma_\mathrm{r} = \phi'(\lambda_1\lambda_2\lambda_3)+\lambda_\mathrm{r}\psi'(\lambda_\mathrm{r})/\lambda_1\lambda_2\lambda_3 \tag{17}$$

Such a representation is suitable for small deformations in most materials. For large elastic deformation it provides an adequate description of the static deformation of vulcanised or synthetic rubbers (i.e. random assemblages or cross-linked, long shain molecules) for dimensional changes of a factor of 10 or so.

The reference state is *stress-free* if $\phi'(1)+\psi'(1)=0$ Near this reference state we can put $\lambda_\mathrm{r}=1+e_\mathrm{r}$, where e_r is first order and can denote any normalised measure of strain. Then

$$\sigma_\mathrm{r} = [\phi''(1)-\psi'(1)](e_1+e_2+e_3)+[\psi''(1)+\psi'(1)]e_\mathrm{r}+[\phi'(1)+\psi'(1)]+\mathrm{O}(e_\mathrm{r}^2)$$

and if the ground state is stress free

$$\sigma_\mathrm{r} = [\phi''(1)+\phi'(1)](e_1+e_2+e_3)+[\psi''(1)+\psi'(1)]e_\mathrm{r}+\mathrm{O}(e_\mathrm{r}^2)$$

We may identify the elastic moduli in the ground state by comparison with a general linearised development.

Note that we may also make an expansion $\lambda_\mathrm{r}=\lambda_{\mathrm{r}o}+e_\mathrm{r}$ about a distorted state but then the moduli for incremental deformation need not be isotropic.

4.5 Fluids

We have so far considered elastic solids for which the stress depends on strain. We can characterise a fluid by a constitutive equation in which stress depends on the rate of strain

$$\sigma_{ij} = s_{ij}(\partial v_p/\partial x_q, \rho, T) \tag{18}$$

in terms of the velocity gradient $\partial v_p/\partial x_q$, density ρ and temperature T.

We introduce the rate of deformation tensor

$$D_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) \tag{19}$$

which represents the symmetric part of the velocity gradient in the Eulerian configuration. The corresponding antisymmetric part

$$\Omega_{ij} = \tfrac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right) \tag{20}$$

is called the *spin or vorticity tensor*.

We will now impose the requirement on the fluid that the stress σ is independent of amy superimposed rigid body rotation. We rewrite (18) in the form

$$\sigma = s(D, \Omega, \rho, T) \tag{21}$$

to separate out the dependence of the symmetric and antisymmetric part of the velocity gradient.

Suppose we have a deformation process

$$\mathbf{x} = \mathbf{x}(\boldsymbol{\xi}, t), \qquad \mathbf{v} = \mathbf{v}(\mathbf{x}, t)$$

and superpose a time dependent rigid rotation so that in the new flow

$$\bar{\mathbf{x}} = Q(t)\,\mathbf{x}(\boldsymbol{\xi}, t) \tag{22}$$

where $Q(t)$ is an orthogonal rotation tensor. The associated velocity

$$\bar{\mathbf{v}} = \frac{D}{Dt}\,\bar{\mathbf{x}} = \dot{Q}\,\mathbf{x} + Q\,\dot{\mathbf{x}} = \dot{Q}\,\mathbf{x} + Q\,\mathbf{v}$$

and the velocity gradient

$$\frac{\partial \bar{\mathbf{v}}}{\partial \bar{\mathbf{x}}} = \frac{\partial \bar{\mathbf{v}}}{\partial \mathbf{x}}\,\frac{\partial \mathbf{x}}{\partial \bar{\mathbf{x}}} = \left(\dot{Q} + Q\,\frac{\partial \mathbf{v}}{\partial \mathbf{x}}\right)Q^T$$

The rate of strain tensor for the new flow

$$\bar{D} = \tfrac{1}{2}\left(\frac{\partial \bar{\mathbf{v}}}{\partial \bar{\mathbf{x}}} + \frac{\partial \bar{\mathbf{v}}^T}{\partial \bar{\mathbf{x}}}\right)$$

$$= \tfrac{1}{2}\,(\dot{Q}Q^T + Q\dot{Q}^T) + QDQ^T$$

but since $QQ^T = I$, the first term vanishes and so

$$\bar{D} = QDQ^T \tag{23a}$$

The spin tensor for the new flow

$$\bar{\Omega} = \tfrac{1}{2}\left(\frac{\partial \bar{\mathbf{v}}}{\partial \bar{\mathbf{x}}} - \frac{\partial \bar{\mathbf{v}}^T}{\partial \bar{\mathbf{x}}}\right)$$

$$= \tfrac{1}{2}(\dot{Q}Q^T - Q\dot{Q}^T) + Q\Omega Q^T$$

$$= Q(Q^T\dot{Q} + \Omega)Q^T \tag{23b}$$

If σ is the stress from the original deformation process, the stress associated with the new flow

$$\bar{\sigma} = Q\sigma Q^T$$

and in terms of $\bar{D}$, $\bar{\Omega}$ we can express $\bar{\sigma}$ as

$$\bar{\sigma} = s(\bar{D}, \bar{\Omega}, \rho, T) \tag{25}$$

Now, equating the representations for $\bar{\sigma}$, from (24) with (18) and (25) using (23a,b), we require

$$s(QDQ^T, Q(Q^T\dot{Q} + \Omega)Q^T, \rho, T) = Qs(D, \Omega, \rho, T)Q^T \tag{26}$$

for all rotations Q.

Suppose now that $Q = I$, $\dot{Q} \neq 0$ then

$$s(D, \dot{Q} + \Omega, \rho, T) = s(D, \Omega, \rho, T)$$

and so s must be independent of the spin tensor Ω. We can therefore reduce (26) to the form

$$s(QDQ^T, \rho, T) = Qs(D, \rho, T)Q^T \tag{27}$$

for all rotations Q.

Following the arguments of section 4.4.2, this isotropy property requires that $\sigma = s(D, \rho, T)$ and D are coaxial. The most general form for σ is therefore

$$\sigma = aI + bD + cD^2 \tag{28}$$

where a, b, c are functions of ρ and T and the invariants of D $\{\mathrm{tr}\, D,\ \tfrac{1}{2}[\mathrm{tr}\, D^2 - (\mathrm{tr}\, D)^2],\ \det D$ which are symmetric functions of the principal values d_1, d_2, $d_3\ \}$.

In the absence of motion $D \equiv 0$ and

$$\sigma = s(0, \rho, T) = -\,p(\rho, T)I$$

where $p(\rho, T)$ is the hydrostatic pressure.

The classical Newtonian viscous fluid is a special case of (28) in which the stress is linear in the velocity gradient

$$\sigma_{ij} = -\,p(\rho, T)\delta_{ij} + b_{ijkl}D_{kl}$$

The b_{ijkl} are the components of a fourth order isotropic tensor with $i - j$, $k - l$ symmetry so

$$b_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

so that we can write

$$\sigma_{ij} = [-p(\rho, T) + \lambda(\rho, T)]\delta_{ij} + 2\mu(\rho, T)D_{ij}$$

For a dilatational flow from the origin

$$\mathbf{v} = d\mathbf{r}, \quad d_{ij} = d\delta_{ij}$$
$$\sigma_{ij} = [-p(\rho, T) + (\lambda + {}^{2}/3\mu)]\delta_{ij}$$

and so $\lambda + {}^{2}/3\mu$ is known as the *coefficient of bulk viscosity* (and is often assumed to be zero).

For a shear flow

$$v_1 = dx_2, \quad v_2 = v_3 = 0$$
$$d_{12} = d_{21} = \tfrac{1}{2}\,d, \quad \text{all other} \;\; d_{ij} = 0$$

and

$$\sigma_{12} = \sigma_{21} = \mu d, \quad \sigma_{11} = \sigma_{22} = \sigma_{33} = -\,p, \quad \sigma_{23} = \sigma_{13} = 0$$

so that μ is known as the coefficient of *shear viscosity* (more commonly just *viscosity*).

The explicit form of the stress tensor when the bulk viscosity vanishes (*a Stokes fluid*) is

$$\sigma_{ij} = -\,p\delta_{ij} + \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right) - {}^{2}/3\,\mu\,\mathrm{div}\,\mathbf{v}\delta_{ij}$$

and this gives an excellent description of many fluids e.g. water, air.

The additional dependence on D^2 in (28) is insufficient to provide an adequate description of many materials e.g. blood, ploymeric liquids which deviate from linear dependence on rate of strain, these materials have a memory of deformation which can be described using viscoelastic constitutive laws.

5 Linearised Elasticity

The only known continua which are capable of 'finite' elastic strain and which have also comparable resistance to shear and compression are the new strong fibres or whiskers of ceramics based on carbon, boron etc.

In common metals the elastic range of strain is only *infinitesimal* and may be treated by a linearised theory. In the earth as well, small additional disturbances can be treated by linearising about an existing stress state.

5.0.1 Linearisation of deformation

The asumption we shall make in subsequent work is that the strain is small, i.e. the difference $U - I$ is first order.

However for bodies such a slender columns or thin panels, the rotation can be large even though the strain is small and in this case we have to retain rotation terms R. In general, we can take $R - I$ to also be small and work with a fully linearised theory.

In terms of particle displacement

$$\mathbf{u} = \mathbf{x} - \xi = D\xi, \quad D = F - I \tag{1}$$

where D is the displacement gradient. To first order in D, the Green's strain

$$E = \tfrac{1}{2}(F^T F - I) = \tfrac{1}{2}(D + D^T) + \cdots \tag{2}$$

and also

$$(U - I) = \tfrac{1}{2}(D + D^T) + \cdots$$

Now,

$$D = RU - I = (R - I) + (U - I) + \cdots$$

so that

$$(R - I) = \tfrac{1}{2}(D - D^T) + \cdots \tag{3}$$

but from our study of pure rotation

$$(R - I) = \theta N = \theta \mathbf{n} \times$$

for a first order rotation.

To this order

$$E_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial \xi_j} + \frac{\partial u_j}{\partial \xi_i}\right) + \cdots \tag{4}$$

and the distinction between E_{ij} and the Cauchy strain e_{ij} is negligible. The fractional dilatation

$$E_{kk} = \frac{\partial u_k}{\partial \xi_k} + \cdots = \operatorname{div}\mathbf{u} + \cdots$$

and the rotation gives

$$\theta\mathbf{n} = \tfrac{1}{2}\operatorname{curl}\mathbf{u} + \cdots$$

In the linearised theory we will use $\mathbf{x}$ rather than ξ to denote the reference position of a particle.

5.1 Constitutive Relation

We make an expansion of the strain energy function about the reference state

$$2W(E) = a + b_{ij} + c_{ijkl}E_{ij}E_{kl} + \mathrm{O}(E^3) \tag{5}$$

and since E_{ij} is symmetric we can arrange $i-j$ and $k-l$ symmetry in the coefficients and also $ij-kl$ symmetry.

$$b_{ij} = b_{ji}, \quad c_{ijkl} = c_{jikl} = c_{ijlk} = c_{klij}$$

If the reference state is stress free $b_{ij} = 0$. In the earth, for example, there is a large hydrostatic pressure field due to self-gravitation and then $\frac{1}{2}\,b_{ij} = -\,p\delta_{ij}$. (*Note:* the fact that earthquakes occur means that there must also be non-hydrostatic stresses representable by the second term.)

Now from (3.13c)

$$\det F\sigma = F\,\frac{\partial W}{\partial E}\,F^T$$

and so, in the absence of non-hydrostatic pre-stress, when E is first order but R is unrestricted

$$\sigma = R\,\frac{\partial W}{\partial E}\,R^T + \mathrm{O}(E^2) \tag{6}$$

as noted above this semi-linearised relation is needed for slender bodies.

For full linearity $D-I$ is first order, and

$$\sigma = \frac{\partial W}{\partial E} + \mathrm{O}(E^2)$$

and so

$$\sigma_{ij} = \tfrac{1}{2}\, b_{ij} + c_{ijkl} E_{kl} + \mathrm{O}(E^2) \tag{7}$$

For a stress-free ground state (or working in terms of incremental stresses) we have the *generalised Hooke's Law*

$$\begin{aligned} \sigma_{ij} &= c_{ijkl} e_{kl} + \mathrm{O}(e^2) \\ W' &= \tfrac{1}{2}\, \sigma_{ij} e_{ij} + \mathrm{O}(e^3) \end{aligned} \tag{8}$$

where we have used the traditional notation for the strain, in view of the equivalence of the strain definitions.

In this full anisotropic case the fourth order tensor of elastic moduli c_{ijkl} is restricted by the $i-j$, $k-l$ and $ij-kl$ symmetries to 21 independent components.

The presence of material symmetry further reduces the number of independent moduli, e.g. for hexagonal symmetry there are 5 moduli.

5.1.1 Isotropic response

If the material is isotropic then c_{ijkl} must be an isotropic tensor subject to the required symmetries in permutation of indices, and so

$$c_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

Thus the stress-strain relation is

$$\sigma_{ij} = \lambda \delta_{ij} e_{kk} + 2\mu e_{ij} \tag{9}$$

λ and μ are known as the Lamé moduli.

The inverse relation to (9) is most conveniently expressed in terms of another set of constants

$$E e_{ij} = (1+\nu)\sigma_{ij} - \nu \sigma_{kk} \delta_{ij} \tag{10}$$

where E is the Young's modulus and ν is Poisson's ratio.

5.1.2 The interpretation of these moduli can be obtained by looking at simple deformations.

a)

Consider a purely volumetric strain

$$\begin{aligned} &e_{ij} = 0, \quad i \neq j \\ &e_{11} = e_{22} = e_{22} = e_{33} = e \end{aligned}$$

then

$$\sigma_{ij} = 0, \quad i \neq j \qquad \text{and} \quad \sigma_{11} = 3\kappa e, \ \text{etc}$$

$\kappa = \lambda + {}^2/3\mu$ is called the bulk modulus and we have a hydrostatic stress field.

b)

If we consider a simple shear deformation

$$x_1 = \xi_1 + \gamma\, \xi_2, \quad x_2 = \xi_2, \quad x_3 = \xi_3$$

the only non-zero infinitesimal strain component is $e_{12} = \frac{1}{2}\gamma$ and the corresponding stress $\sigma_{12} = \mu\gamma$, with all other $\sigma_{ij} = 0$. μ is called the *shear or rigidity modulus*. In general, if the dilatation $\sum_k e_{kk}$ vanishes $\sigma_{ij} = 2\mu e_{ij}$.

c)

Consider uniaxial tension $\sigma_{11} = \sigma_1$, $\sigma_{22} = \sigma_{33} = 0$, $\sigma_{ij} = 0 \ \ i \neq j$, then

$$\begin{aligned} e_{11} &= \sigma_1/E, \\ e_{22} &= -\nu\sigma_1/E, \qquad e_{ij} = 0 \quad i \neq j \\ e_{33} &= -\nu\sigma_1/E, \end{aligned}$$

so ν is the quotient of transverse contraction and longitudinal extension.

In terms of κ and μ the work function can be written as

$$W = \tfrac{1}{2}\kappa e_{kk}^2 + \mu(e_{ij} - {}^1/3 e_{kk}\delta_{ij})(e_{ij} - {}^1/3 e_{qq}\delta_{ij})$$

and this will be positive definite if $\kappa > 0$, $\mu > 0$.

5.1.3 The interrelations between the moduli may be obtained by comparing the descriptions of different deformations

$$\sigma_{kk} = 3\kappa e_{kk}, \qquad e_{kk} = \frac{1-2\nu}{E}\sigma_{kk} \tag{11}$$

and

$$\begin{aligned} 2\mu e_{ij} &= \sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kk}\delta_{ij} \\ Ee_{ij} &= (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij} \end{aligned} \tag{12}$$

Combining the results of (11) and (12)

$$\begin{aligned} \frac{E}{1+\nu} &= 2\mu, \qquad \frac{E}{1-2\nu} = 3\kappa \\ \frac{3}{E} &= \frac{1}{\mu} + \frac{1}{3\kappa}, \qquad 2\nu = \frac{3\kappa - 2\mu}{3\kappa + \mu} \end{aligned} \tag{13}$$

For positive definite W we note that $\kappa > 0$ with $\mu > 0$ is equivalent to $E > 0$ with $-1 < \nu < \frac{1}{2}$. Normally ν is positive since there is transverse contraction under compression.

The limiting case of incompressibility (or second-order volume change) is obtained by letting $\kappa \to \infty$ with μ fixed so that $E \to 3\mu$, and $\nu \to \frac{1}{2}$.

5.1.4 Example of linearisation

Suppose we have a work function specified in terms of principal stretches as (4.17)

$$W = \phi(\lambda_1 \lambda_2 \lambda_3) + \sum_{\mathrm{r}} \psi(\lambda_{\mathrm{r}})$$

with

$$\phi = \frac{c}{mn}\,[(\lambda_1 \lambda_2 \lambda_3)^{-mn} - 1] \qquad \psi(\lambda_{\mathrm{r}}) = \frac{c}{m}\,[\lambda_{\mathrm{r}}^m - 1]$$

for which the principal stress is

$$\sigma_{\mathrm{r}} = c[\lambda_{\mathrm{r}}^m - v^{-mn}]/v \qquad \text{with} \quad v = \lambda_1 \lambda_2 \lambda_3$$

Near the reference state

$$\sigma_{\mathrm{r}} = mc[n(e_1 + e_2 + e_3) + e_{\mathrm{r}}] + \mathrm{O}(e^2)$$

More general W may be constructed by summing over various (c, m) pairs keeping n fixed

Comparing with the isotropic constitutive relation in the principal frame

$$\sigma_{\mathrm{r}} = 2\mu e_{\mathrm{r}} + \lambda(e_1 + e_2 + e_3)$$

we can identify the shear modulus $\mu = \frac{1}{2}\sum mc$, and the Poisson's ratio $\nu = n/(1 + 2n)$.

Under a uniaxial load σ_1 with arbitrary magnitude

$$\lambda_2 = (\lambda_1 \lambda_2^2)^{-n}, \quad \text{since} \quad \lambda_2 = \lambda_3$$
$$\log \lambda_2 = -\,\nu \log \lambda_1$$

5.1.5 Elastic constants

Typical values of material parameters at 300 K are listed in the following table. They relate to random aggregates of anisotropic single crystals (at least 10^{12} m^{-3}). Such aggregates are effectively isotropic and homogeneous at a macroscopic level (the volume of a test specimen being of order 10^{-5} m^3. The values are means of the best data available and have been 'normalised' so as to be theoretically consistent (within the accuracy of the decimal places given).

	λ	μ	κ	E	ν
Aluminium	0.60	0.265	0.775	0.715	0.35
Copper	1.09	0.45	1.39	1.22	0.355
Nickel	1.39	0.76	1.90	2.01	0.32
Lead	0.38	0.074	0.43	0.21	0.42
Gold	1.48	0.28	1.67	0.795	0.42
Silver	0.82	0.27	1.00	0.755	0.37
Iron	1.18	0.83	1.73	2.145	0.29
Molybdenum	1.66	1.60	2.73	4.02	0.255

Material properties in the Earth

	λ	μ	κ	E	ν
2500 km deep	4.195	2.723	6.011	7.096	0.303
1000 km deep	2.180	1.820	3.420	4.630	0.272
20 km deep	0.429	0.323	0.645	0.830	0.285

Units for λ, μ, κ, E are 10^{11} Nm^{-2}.

In metals, the elastic range of strain is terminated by plastic yielding, e.g. for Cu and Al the tensile yield stress $\approx 17.5 \times 10^6$ Nm^{-2} so that

$$e_{yield} \approx 1.4 \times 10^{-4} \quad \text{for Cu}$$
$$e_{yield} \approx 2.4 \times 10^{-4} \quad \text{for Al}$$

Even for stronger constructional materials such as mild steel and aluminium alloys, the yield strains are around 10^{-3}.

For Earth materials, the linear approximation works well for strains less than 10^{-5} which is rarely exceeded except in the immediate vicinity of earthquakes and underground nuclear explosions.

5.2 Uniqueness Theorem

Locally the stress-field in a body satisfies (3.5)

$$\frac{\partial}{\partial x_i}\sigma_{ij} + \rho g_j = \rho f_j$$

where $\mathbf{g}dm$ is the body-force and $\mathbf{f}dm$ is the acceleration.

In the linearised approximation, the acceleration $\mathbf{f} = \partial^2\mathbf{u}/\partial t^2$ in terms of the displacement $\mathbf{u}$. The convected terms $\mathbf{v}.\nabla$ can be neglected.

Consider a bounded elastic region V with given boundary conditions on S

i)

τ specified on part of the surface

ii)

u specified

iii)

normal component of **u** and tangential component of τ (or vice versa)

and a locally positive definite strain energy density $W = \frac{1}{2}\,\sigma_{ij}e_{ij}$.

Suppose we have Cauchy initial conditions

$$\mathbf{u}(\mathbf{x},0) = \mathbf{U}(\mathbf{x}), \qquad \frac{\partial u}{\partial t}\,(\mathbf{x},0) = \mathbf{U}'(\mathbf{x})$$

where $\mathbf{U}$, $\mathbf{U}'$ and the body force $\mathbf{g}$ are bounded in V. Let $\mathbf{u}$, $\mathbf{u}'$ be two distinct solutions under these conditions and denote their difference by $\Delta\mathbf{u}$.

Then $\Delta\mathbf{u}$ satisfies

$$\rho\,\frac{\partial^2}{\partial t^2}\,\Delta u_i = \frac{\partial}{\partial x_j}\,\Delta\sigma_{ij} \qquad \Delta\mathbf{u} = \frac{\partial}{\partial t}\,\Delta\mathbf{u} = 0 \quad \text{at} \quad t = 0$$

and $\Delta\mathbf{u}.\,\Delta\tau = 0$ on the surface S.

Consider

$$0 = \int_0^t \mathrm{d}t \int_V \mathrm{d}V\,\frac{\partial}{\partial t}\,\Delta u_i\left[\rho\,\frac{\partial^2}{\partial t^2}\,\Delta u_i - \frac{\partial}{\partial x_j}\,\Delta\sigma_{ij}\right]$$

$$= \tfrac{1}{2}\int_V \mathrm{d}m\left(\frac{\partial}{\partial t}\,\Delta u_i\right)^2 + \int_0^t \mathrm{d}t\int_V \mathrm{d}V\,\frac{\partial}{\partial t}\left(\frac{\partial}{\partial x_j}\,\Delta u_i\right)\Delta\sigma_{ij} - \int_0^t \mathrm{d}t\int_S \mathrm{d}S\,n_j\Delta\sigma_{ij}\,\frac{\partial}{\partial t}\,\Delta u_i$$

$$= \int_V \mathrm{d}V\,K(\Delta\mathbf{u}) + \int_0^t \mathrm{d}t\int_V \mathrm{d}V\,\frac{\partial}{\partial t}\,\Delta e_{ij}\Delta\sigma_{ij} - \int_0^t \mathrm{d}t\int_S \mathrm{d}S\,\Delta\tau_i\,\frac{\partial}{\partial t}\,\Delta u_i$$

The last term vanishes identically, the first term is the positive definite kinetic energy and the second is the total elastic energy associated with $\Delta\mathbf{u}$

$$W = \int_V \tfrac{1}{2}\,\Delta e_{ij}\Delta\sigma_{ij}\,\mathrm{d}V$$

which by hypothesis is positive definite.

Thus $\partial\Delta\mathbf{u}/\partial t$ and Δe_{ij} are zero everywhere in V and so $\Delta\mathbf{u}$ is a static displacement with $\Delta\mathbf{u} = 0$ at $t = 0$. This can only be achieved if

$$\Delta\mathbf{u} = 0 \;\text{ in } V \;\; \text{for all } t > 0$$

i.e. the solution is unique.

5.2.1 Kirchhoff's Theorem

Positive definite W at every point **x** suffices for uniqueness.

For isotropic media this condition reduces to

$\kappa > 0$ with $\mu > 0$,

or $E > 0$ with $-1 < \nu < \frac{1}{2}$

5.3 Integral Theorems

Consider the displacement field in an elastic body body under different loadings

$\mathbf{u}(\mathbf{x})$ with τ on S, $\mathbf{g}$ in V

$\mathbf{u}^*(\mathbf{x})$ with τ^* on S, $\mathbf{g}^*$ in V

Then

$$\rho\partial_{tt}u_i = \rho g_i + \partial_j\sigma_{ij} \tag{a}$$

$$\rho\partial_{tt}u_i^* = \rho g_i^* + \partial_j\sigma_{ij}^* \tag{b}$$

Take the scalar product of (a) with u_i^* and (b) with u_i and then subtract. Now integrate throughout V and over time to get

$$\int_{t_o}^{t} \mathrm{d}t \int_V \mathrm{d}m\,(\partial_{tt}u_i u_i^* - u_i\partial_{tt}u_i^*)$$

$$= \int_{t_o}^{t} \mathrm{d}t \int_V \mathrm{d}m\,(g_i u_i^* - u_i g_i^*) + \int_{t_o}^{t} \mathrm{d}t \int_V \mathrm{d}V\,(\partial_j\sigma_{ij}u_i^* - u_i\partial_j\sigma_{ij}^*) \tag{14}$$

The volume integral over the stress

$$\int_V \mathrm{d}V\,(\partial_j\sigma_{ij}u_i^* - u_i\partial_j\sigma_{ij}^*) = \int_V \mathrm{d}V\,\partial_j(\sigma_{ij}u_i^* - u_i\sigma_{ij}^*) - \int_V \mathrm{d}V\,(\sigma_{ij}e_{ij}^* - e_{ij}\sigma_{ij}^*)$$

and

$$\sigma_{ij}e_{ij}^* = c_{ijkl}e_{kl}e_{ij}^* = c_{klij}e_{ij}^*e_{kl} = \sigma_{kl}^*e_{kl}$$

Thus (14) can be written as

$$\int_V \mathrm{d}V\,\rho(\partial_t u_i u_i^* - u_i\partial_t u_i^*)\Big]_{t_o}^{t} = \int_{t_o}^{t} \mathrm{d}t \int_V \mathrm{d}m\,(g_i u_i^* - u_i g_i^*) + \int_{t_o}^{t} \mathrm{d}t \int_S \mathrm{d}S\,(\tau_i u_i^* - u_i\tau_i^*) \tag{15}$$

5.3.1 Reciprocal Theorem

Take the time integration up to infinity and assume that $\mathbf{u}$, $\mathbf{u}^*$ are such that the left hand side of (15) vanishes, then

$$\int_{-\infty}^{\infty} \mathrm{d}t \left[\int_V \mathrm{d}m\, g_i u_i^* + \int_S \mathrm{d}S\, \tau_i u_i^* \right] = \int_{-\infty}^{\infty} \mathrm{d}t \left[\int_V \mathrm{d}m\, g_i^* u_i + \int_S \mathrm{d}S\, \tau_i^* u_i^* \right] \tag{16}$$

and for a *static* field we can drop the time integral

$$\int_V \mathrm{d}m\, g_i u_i^* + \int_S \mathrm{d}S\, \tau_i u_i^* = \int_V \mathrm{d}m\, g_i^* u_i + \int_S \mathrm{d}S\, \tau_i^* u_i^* \tag{17}$$

this is Betti's reciprocal theorem.

APPLICATIONS

In applications of Betti's Reciprocal theroem (17) we take the required field to be $\mathbf{u}$ and choose an appropriate starred field for which the solution is known and which has as close a character as we can achieve to the desired field.

i)

Self gravity is switched on in a homogeneous sphere - find the decrease in radius:

At the surface $r = a$, $\tau = 0$. Let g be the acceleration due to self gravity, then the interior body force per unit mass is $\mathbf{g} = -\, g\mathbf{x}/a$.

Choose $\mathbf{u}^* = \varepsilon \mathbf{x}$ implying uniform $\sigma_{ij}^* = 3\kappa\, \varepsilon \delta_{ij}$, then $\mathbf{g}^* = 0$ because the field is self-equilibrated. From (17)

$$\int_V -\frac{g\mathbf{x}}{a} \,.\, \varepsilon \mathbf{x}\, \rho \mathrm{d}V = 3\kappa\varepsilon\, 4\pi a^2\, u(a)$$

$$\frac{-g\rho\varepsilon}{a} \frac{4\pi}{5} a^5 = 3\kappa\varepsilon\, 4\pi a^2\, u(a)$$

i.e. the decrease in radius $= -\, u(a) = \rho g a^2 / 15\kappa$

ii)

Loaded cylinder

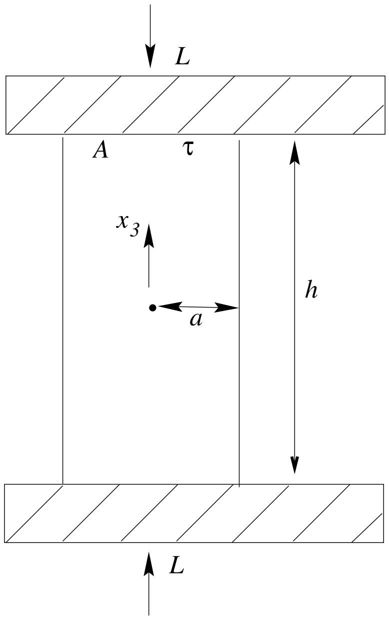
A solid circular cylinder height h and radius a is subjected to compressive loads L applied via rigid plates in contact with its ends. Radial displacements over the plates are opposed by a uniform frictional force τ

Find the reduction in height δh when L is large enough to cause sliding everywhere over the ends.

Let x_3 be measured axially from the centroid, and choose

$$\mathbf{u}^* = (\nu x_1, \nu x_2, -x_3)p^*/E$$

corresponding to a uniaxial pressure p^*, so $\mathbf{g}^* = 0$. Now also $\mathbf{g}=0$ in V, $\boldsymbol{\tau}^* = \boldsymbol{\tau} = 0$ over the curved surfaces. Thus denoting an end by A,



$$\int \boldsymbol{\tau}.\mathbf{u}^* \,\mathrm{d}A = \int \boldsymbol{\tau}^*.\mathbf{u}\,\mathrm{d}A$$

the load $L = -\int \tau_3 \,\mathrm{d}A$, and so

$$\frac{p^*}{E}\left(\tfrac{1}{2}HL - \nu\int_0^a r\tau 2\pi r\,\mathrm{d}r\right) = p^*\,\tfrac{1}{2}\,\pi a^2\,\delta h$$

i.e.

$$\delta h = \frac{h}{E}\left(\frac{L}{\pi a^2} - \frac{4\nu a\tau}{3h}\right)$$

5.3.2 Representation Theorem

Consider a delta function body force

$$g_i = \delta_{ik}\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\theta)$$

acting in a volume V. The resulting displacement at $\mathbf{x}, t$ is specified by the Green's tensor with components

$$G_i^k(\mathbf{x}, t; \boldsymbol{\xi}, \theta)$$

associated with the force in the k –direction at $\boldsymbol{\xi}, \theta$. Even when the properties of the region vary G_i^k will depend on the time difference $t-\theta$.

If we take g_i as above and set

$$g_i^* = \delta_{il}\delta(\mathbf{x}-\boldsymbol{\zeta})\delta(\sigma - t)$$

the displacement

$$u_i^* = G_i^l(\mathbf{x},-t;\boldsymbol{\zeta},-\sigma)$$

We assume that both G_i^k and G_u^l obey the *same* homogeneous boundary conditions on S, and then when we apply the reciprocal theorem (16) the surface integrals vanish

$$\int_{-\infty}^{\infty}\int_V \delta_{ik}\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\theta)G_i^l(\mathbf{x},-t;\boldsymbol{\zeta},-\sigma) = \int_{-\infty}^{\infty}\int_V \delta_{ik}\delta_{il}\delta(\mathbf{x}-\boldsymbol{\zeta})\delta(\sigma-t)G_i^k(\mathbf{x},t;\boldsymbol{\xi},\theta)$$

so that when $\boldsymbol{\xi}$, $\boldsymbol{\zeta}$ are in V

$$G_k^l(\boldsymbol{\xi},-\theta;\boldsymbol{\zeta},-\sigma) = G_l^k(\boldsymbol{\zeta},\sigma;\boldsymbol{\xi},\theta) \tag{18}$$

(18) expresses Green's tensor reciprocity.

Now set

$$g_i^* = \delta_{ik}\delta(\mathbf{x}-\boldsymbol{\xi})\delta(\theta - t)$$

in (15) and then

$$u_i^* = G_i^k(\mathbf{x},-t;\boldsymbol{\xi},-\theta)$$

and so

$$0 = \int_{t_o}\int_V \mathrm{d}V_x \{g_i(\mathbf{x})G_i^k(\mathbf{x},-t;\boldsymbol{\xi},-\theta) - u_i(\mathbf{x})\delta(\mathbf{x}-\boldsymbol{\xi})\delta(\theta-t)\}$$

$$+ \int_{t_o}\int_S \mathrm{d}S_x \{\tau_i(\mathbf{x})G_i^k(\mathbf{x},-t;\boldsymbol{\xi},-\theta) - u_i(\mathbf{x})H_i^k(\mathbf{x},-t;\boldsymbol{\xi},-\theta)\}$$

$$+ \int_V \mathrm{d}V_x [\partial_t u_i(\mathbf{x})G_i^k - u_i\partial_t G_i^k]_{t=t_o}$$

where H_i^k is the traction associated with G_i^k.

Now, using the reciprocity of the Green's tensor we can obtain a representation of the displacement as an integral over the boundaries, body forces and initial conditions

$$\Theta(\boldsymbol{\xi})\mathbf{u}(\boldsymbol{\xi},t) = \int_{t_o}\int_V \mathrm{d}V_x\, g_i(\mathbf{x})G_k^i(\boldsymbol{\xi},t;\mathbf{x},s)$$

$$+ \int_{t_o} \int_S \mathrm{d}S_x \{\tau_i(\mathbf{x}) G_k^i(\xi, t; \mathbf{x}, s) - u_i(\mathbf{x}) H_k^i(\xi, t; \mathbf{x}, s)\}$$

$$+ \int_V \mathrm{d}V_x [\partial_t u_i(\mathbf{x}) G_k^i - u_i \partial_t G_k^i]_{t=t_o} \tag{19}$$

where

$$\Theta(\xi) = 1, \quad \xi \text{ in } V$$

$$= 0, \quad \text{otherwise} .$$

5.4 Elastic Waves

The equation of motion and the stress-strain relation for small disturbances in an elastic medium, in the absence of body forces, can be combined to give

$$\frac{\partial}{\partial x_j}\left(c_{ijkl} \frac{\partial u_k}{\partial x_l} \right) = \rho \frac{\partial^2 u_j}{\partial t^2} \tag{20}$$

as the governing differential equation for displacement $\mathbf{u}$. Equation (20) controls the spatial and temporal development of the displacement field and admits solutions in the form of travelling waves. Consider then a plane wave travelling in the anisotropic medium

$$u_i = U_i \exp[\mathrm{i}\omega\, p\mathbf{n}.\mathbf{x} - \mathrm{i}\omega t]$$
$$= U_i \exp[\mathrm{i}\omega\, p n_k x_k - \mathrm{i}\omega t]$$

$\mathbf{n}$ represents the direction of travel of the phase fronts and p the apparent slowness (inverse of wave velocity) in that direction. On substituting this plane wave form into (20) we obtain

$$\omega^2 p^2 c_{ijkl} n_j n_l U_k = -\,\omega^2 \rho U_i$$

which constitutes an eigenvalue problem for the slowness p for with waves travelling in the direction $\mathbf{n}$

$$[p^2 c_{ijkl} n_j n_l - \rho \delta_{ik}] U_k = 0 \tag{21}$$

(21) has three roots for p^2 for each direction $\mathbf{n}$, associated with polarisations specified by the eigenvectors $\mathbf{U}^{(r)}(\mathbf{n})$.

In a general anisotropic medium, the slownesses vary with direction and can give quite complex slowness surfaces as e.g. those illustrated below:

Figure not available.

5.4.1 Isotropic media

For an isotropic medium, the eigenvalue equation for slowness takes the form

$$[p^2(\lambda+\mu)n_i n_k + p^2\mu\delta_{ik} - \rho\delta_{ik}]U_k = 0$$

where we have used the isotropic form for the elastic modulus tensor c_{ijkl}.

i) P waves

Consider a displacement field oriented along the propagation direction i.e.

$$\mathbf{U}_{\mathrm{P}} = c\mathbf{n}$$

then the slowness equation is

$$[(\lambda+2\mu)p^2 - \rho]n_i = 0$$

so that we have a wave disturbance with slowness

$$a^2 = \rho/(\lambda+2\mu)$$

and associated phase velocity

$$\alpha = [(\lambda+2\mu)/\rho]^{\frac{1}{2}}$$

This longitudinal wave solution is called the P wave

$$\mathbf{u}_{\mathrm{P}} = A_{\mathrm{P}}\mathbf{n}\exp[\mathrm{i}\omega(a\mathbf{n}.\mathbf{x} - t)]$$

ii) S waves

There is an alternative type of elastic wave motion in which the displacement is transverse to the direction of motion. Consider a displacement field

$$\mathbf{U}_{\mathrm{S}} = c\mathbf{s} \quad \text{with} \quad \mathbf{s}.\mathbf{n} = 0$$

for which the slowness equation reduces to

$$[\mu p^2 - \rho]s_i = 0$$

so that we have a wave disturbance with slowness

$$b^2 = \rho/\mu$$

and associated phase velocity

$$\beta = [\mu/\rho]^{\frac{1}{2}}$$

This form of solution holds for any direction orthogonal to the direction of motion i.e. we have a degenerate eigenvalue problem from which we can choose two orthogonal S wave vectors. It is conventional to choose one vector in the vertical plane (denoted SV) and the other purely horizontal (denoted SH). This choice simplifies the analysis of wave propagation in horizontally stratified media, and so we represent the S wave field as

$$\mathbf{u}_S = \{B_V \mathbf{s}_V + B_H \mathbf{s}_H\} \exp[i\omega(b\mathbf{n}.\mathbf{x} - t)]$$

$\mathbf{s}_V$ lies in the vertical plane through $\mathbf{n}$ and $\mathbf{s}_H$ in the horizontal plane $[\mathbf{s}_V.\mathbf{n} = \mathbf{s}_H.\mathbf{n} = 0]$

5.4.2 Green's tensor for isotropic media

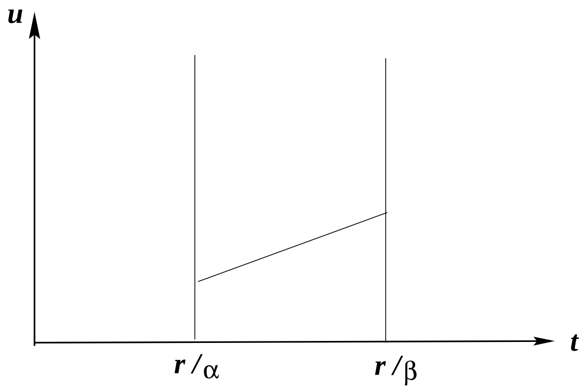
The wave disturbance at an observation point $\mathbf{r}$ produced by a delta-function point force in the direction $\mathbf{d}$ at the origin in an unbounded isotropic medium can be represented as

$$4\pi\mathbf{u}(\mathbf{r}) = (\mathbf{d}.\hat{\mathbf{r}})\hat{\mathbf{r}}\,\frac{\delta(t - r/\alpha)}{r\alpha^2} + \hat{\mathbf{r}} \times (\mathbf{d} \times \hat{\mathbf{r}})\,\frac{\delta(t - r/\beta)}{r\beta^2}$$

$$+ (3(\mathbf{d}.\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{d})\,\frac{t}{r^3}\,[H(t - r/\alpha) - H(t - r/\beta)]$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction $\mathbf{r}$ and $H(t)$ is the Heaviside step function. This result was first derived by Stokes in 1851 and can be deduced by many different techniques e.g. Fourier transform methods or the superposition of potential solutions.

The disturbance consists of two delta functions in time, one longitudinal travelling with the faster P-waves velocity α and the other with transverse displacement and the slower shear wave velocity β (this can be represented in terms of SV and SH components if required). Between these two wavefronts is a 'near-field' disturbance with intermediate character which decays more rapidly with distance r.



The specific form of the Green's tensor $G_i^k(\mathbf{x}, t; 0, 0)$ for an observation point with direction cosines γ_i is thus

$$G_i^k(\mathbf{x}, t) = (3\gamma_i\gamma_j - \delta_{ij})t\,r^{-3}[H(t - r/\alpha) - H(t - r/\beta)]$$

$$+ \gamma_i\gamma_j(r\alpha^2)^{-1}\delta(t - r/\alpha) + (\gamma_i\gamma_j - \delta_{ij})(r\beta^2)^{-1}\delta(t - r/\beta)$$

The radiation pattern for P waves depends on the cosine of the inclination to the force direction and the S wave pattern on the sine of the inclination.

More realistic sources can be obtained by using couples and dipoles whose associated displacements can be derived by differentiation of the Green's tensor.

Figure not available.

5.4.3 Interfaces

The solution presented in the previous section is for elastic waves in an unbounded medium, and more complicated behaviour results once material interfaces or a free surface is introduced.

At an interface between two different media the boundary conditions are that
i) the displacement $\mathbf{u}$ is continuous
ii) the traction $\boldsymbol{\tau}(\mathbf{n})$, associated with the normal $\mathbf{n}$ to the interface is continuous
These boundary conditions can only be satisfied by the coupling of P and S waves at the interface. For a plane wave incident on a horizontal interface the P and SV wave fields are coupled but SH is independent.

At a free surface, the requirement is that the traction should vanish. In this case there is a special form of wave which satisfies the vanishing traction condition called a Rayleigh wave. This consists of coupled exponentially decaying P and S waves travelling with a horizontal velocity about 0.9 of the shear wave velocity.

6 Some Fluid Flow Problems

We will consider a number of different types of fluid flow behaviour based on the assumption of a Stokes fluid, so that the stress σ is related to the fluid velocity $\mathbf{v}$ by

$$\sigma_{ij} = -\,p\delta_{ij} + \mu[\partial_i v_j + \partial_j v_i] - {}^2\!/3\,\mu\delta_{ij}\partial_k v_k \tag{1}$$

where p is the hydrostatic pressure and μ the viscosity, and we have written $\partial_i \equiv \partial/\partial x_i$. This assumes that there is no contribution from bulk viscosity.

6.1 The Navier-Stokes equation

The equation of motion for the fluid in the presence of body force $\mathbf{g}$ and mass-acceleration $\mathbf{f}$ is (3.5)

$$\partial_i\sigma_{ij} + \rho g_j = \rho f_j \tag{2}$$

and we can relate the acceleration $\mathbf{f}$ to the velocity field $\mathbf{v}$ using (2.10)

$$\mathbf{f} = \frac{D}{Dt}\mathbf{v} = \partial_t\mathbf{v} + (\mathbf{v}.\nabla)\mathbf{v} \tag{3}$$

and in component terms

$$f_j = \frac{D}{Dt}v_j = \partial_t v_j + (v_k\partial_k)v_j$$

Now, combining the constitutive equation (1) with the equation of motion (2) we require

$$\rho\frac{D}{Dt}v_j = \rho g_j - \partial_j p + \partial_i[\mu(\partial_i v_j + \partial_j v_i - {}^2\!/3\,\partial_k v_k\delta_{ij}] \tag{4}$$

which is usually called the Navier-Stokes equation for the fluid.

If the viscosity μ can be taken as constant over the fluid, then we can express the Navier-Stokes equation as

$$\partial_t v_j + (v_k\partial_k)v_j = \rho g_j - \partial_j p + \mu\partial_{ii}v_j + {}^1\!/3\,\mu\partial_{jk}v_k \tag{5a}$$

or in vector form as

$$\partial_t\mathbf{v} + (\mathbf{v}.\nabla)\mathbf{v} = \rho\mathbf{g} - \nabla p + \mu\nabla^2\mathbf{v} + {}^1\!/3\,\mu\nabla(\nabla.\mathbf{v}) \tag{5b}$$

which has explicit non-linearity through the convected derivative term.

If the viscosity varies with position (as may for example be associated with the effect of temperature) we have to include an additional term

$$+\partial_i \mu[\partial_i v_j + \partial_j v_i - {}^2/3\partial_k v_k \delta_{ij}]$$

on the right hand side of (5). We will also need to consider the equations for the distribution of temperature or any other factor which may influence the distribution of viscosity.

6.1.1 Heat flow

We can link the equations of heat flow and fluid flow by returning to the equation for the rate of change of the internal energy U (4.10)

$$\rho \frac{D}{Dt} U = h + \partial_k[\kappa \partial_k T] + \sigma_{ij}\partial_j v_i \tag{6}$$

where h is the rate of internal heat production, T is the temperature and κ is the thermal conductivity. From the first law of thermodynamics

$$\frac{D}{Dt} U = \frac{D}{Dt} Q + \frac{D}{Dt} W$$

where Q represents the thermal contribution to the internal energy and W the external work. We will isolate the contribution from $\sigma_{ij}\partial_j v_i$ in (6) due to the pressure term

$$p\partial_k v_k = -\frac{p}{\rho}\frac{D}{Dt}\rho$$

using the continuity equation (conservation of mass - 2.12). We will assume that the remaining contribution to the internal energy can be represented in terms of a thermal capacity C so that

$$\rho \frac{D}{Dt} U = \rho \frac{D}{Dt}(CT) - \frac{p}{\rho}\frac{D}{Dt}\rho \tag{7}$$

- the particular form of the heat capacity C to be employed will depend on the thermodynamic state of the material.

We recast the constitutive equation for the viscous fluid (1) as

$$\sigma_{ij} = -p\delta_{ij} + 2\mu \hat{D}_{ij} \tag{8}$$

in terms of the deviatoric strain rate

$$\hat{D}_{ij} = \tfrac{1}{2}[\partial_i v_j + \partial_j v_i] - {}^1/3\partial_k v_k \delta_{ij}$$

which is contstucted to have zero trace.

The stress-power contribution to the rate of change of internal energy is

$$\sigma_{ij}\partial_i v_j = -p\partial_k v_k + 2\mu \hat{D}_{ij}\partial_i v_j$$

$$= -p\partial_k v_k + 2\mu \hat{D}_{ij}\hat{D}_{ij} \tag{9}$$

(since the deviatoric term vanishes when the indices i and j coincide). We have already explicitly accounted for the pressure contribution in (7) so that we can recast (6) into an equation for the temperature field in the presence of fluid flow

$$\rho[\partial_t(CT) + v_k\partial_k(CT)] = h + \partial_k[\kappa\partial_k T] + 2\mu\hat{D}_{ij}\hat{D}_{ij} \tag{10}$$

which simplifies somewhat if C, κ are constant.

6.1.2 Prandtl number

A convenient measure of the flow properties of the fluid is provided by the *kinematic viscosity*

$$\nu = \mu/\rho \tag{11}$$

which can be regarded as a diffusion coefficient for linear momentum [with units m^2/s].

The equivalent measure for heat transport is provided by the *thermal diffusivity*

$$\kappa_H = \kappa/\rho C \tag{12}$$

[with units m^2/s]. The time required for thermal effects to propagate a distance l is of the order of l^2/κ_H.

The relative significance of momentum and heat transport can be measured by the ratio of ν to κ_H which is a dimensionless quantity P known as the Prandtl number

$$\mathrm{P} = \nu/\kappa_H \tag{13}$$

A fluid with a high Prandtl number has much weaker heat diffusion than momentum transport, and under this condition we will need take account of the influence of spatial variations in temperature on the viscosity of the fluid since the time scales of flow are very long as e.g. in convection in the mantle of the Earth.

For the earth's mantle

$$\mu \approx 10^{21}\,\mathrm{Pa\,s}, \qquad \rho \approx 4000\,\mathrm{kg\,m^{-3}}, \qquad \kappa_H \approx 10^{-6}\,\mathrm{m^2 s^{-1}}$$

and so the Prandtl number is approximately 10^{23}.

6.2 Rectilinear shear flow

For many fluids we can neglect the effects of compressibility (at least as a first approximation) and then the continuity equation reduces to

$$\nabla.\,\mathbf{v} = 0 \tag{14}$$

We consider a simple situation with flow in the z-direction induced by a pressure gradient

$$v_x = v_y = 0, \qquad v_z = v_z(x, y, t)$$

for which $(\mathbf{v}.\nabla)\mathbf{v} = 0$ and the Navier-Stokes equations reduce to

$$-\partial_x p = -\partial_y p = 0, \qquad -\partial_z p + \mu(\partial_{xx} + \partial_{zz})v_z = \rho\partial_{tt} v_z \tag{16}$$

so that there are no pressure variations with x or y and the pressure is a function of z alone $p(z)$. In the case of steady flow $\partial_{tt} v_z \equiv 0$ and so we have

$$\mu[\partial_{xx} v_z + \partial_{zz} v_z] = \mu[r^{-1}\partial_r(r\partial_r v_z) + r^{-2}\partial_{\theta\theta} v_z] = \partial_z p = const \tag{17}$$

where we have expressed the equation in both cartesian and cylindrical polar coordinates. The nature of the solution of (17) will depend on the flow field $v_z(x, y)$ in the $x - y$ plane.

We will adopt *no-slip* conditions and assume that a viscous fluid in contact with a solid boundary must have the same velocity as the boundary. For a free surface the traction must vanish and so the associated shear-stress components will be zero.

One of the simplest configurations we can envisage is a 2-D situation with solid boundaries at $x = \pm h$ and no dependence on y. In that case

$$\mu\partial_{xx} v_z = \partial_z p, \qquad v_z(h) = v_z(-h) = 0 \tag{18}$$

with solution

$$v_z = \frac{1}{2\mu}\frac{\partial p}{\partial z}(x^2 - h^2) \tag{19}$$

which is known as Couette flow, with a velocity profile in the form of a parabola symmetric about the centre line ($y = 0$).

This simple model gives a reasonable representation of flow through a volcanic conduit in a fissure eruption e.g. Hekla in Iceland. The flow rate per unit length of fissure

$$Q = \int_{-h}^{h} \mathrm{d}x\, v_z = -\frac{4}{3\mu}\frac{\partial p}{\partial z} h^3 \tag{20}$$

on the assumption that the pressure gradient is constant.

Where would the pressure gradient come from in such a case? The upward flow of the magma is driven by the natural buoyancy of the lighter magma relative to the denser surrounding rock. At a depth h the *lithostatic* pressure in the rock is $\rho_s gh$ for rock density ρ_s and the corresponding *hydrostatic* pressure for a magma column is $\rho_l gh$ for magma density ρ_l. If the walls of the conduit are free to deform, the pressure gradient available to drive the magma is

$$\frac{\partial p}{\partial z} = -(\rho_s - \rho_l)g$$

so that the flow rate would be ${}^{4}/3(\rho_s - \rho_l)gh^3/\mu$.

The solution here ignores the outflow condition at the top of the pipe and assumes that we do not have to worry about magma solidification in the dyke. A more thorough treatment requires a time-dependent solution with allowance for thermal effects and a better treatment of surface conditions.

6.3 Plane two-dimensional flow

In a situation with a very slow viscous flow the acceleration terms $\rho D\mathbf{v}/Dt$ can be ignored, if we also have incompressibility $\nabla.\,\mathbf{v} = 0$ as in (14) and the Navier-Stokes equations reduce to

$$-\nabla P + \mu\nabla^2\mathbf{v} = 0 \tag{21}$$

where P is the pressure produced by fluid flow

$$P = p - \rho g z$$

allowing for a gravitational body force. Since the divergence of the velocity vanishes, the pressure P must be a harmonic function i.e.

$$\nabla^2 P = 0$$

For a plane two-dimensional flow

$$v_x = v_x(x, z), \quad v_y = 0, \quad v_z = v_z(x, z) \tag{22}$$

we can satisfy the continuity equation by taking

$$v_x = -\,\partial_z\psi, \qquad v_z = \partial_x\psi$$

in terms of a stream function $\psi(x, z)$. The curves $\psi(x, z) = const$ represent streamlines since $\mathbf{v}.\,\nabla\psi = 0$ and $\nabla\psi$ will be normal to the surfaces $\psi = const$. The x and z components of (21) take the form

$$\partial_x P = -\,\mu\partial_z(\nabla^2\psi), \quad \partial_z P = \mu\partial_x(\nabla^2\psi)$$

and we can eliminate P to give

$$\nabla^2(\nabla^2\psi) = \nabla^4\psi = 0 \tag{24}$$

which is known as the biharmonic equation.

6.3.1 Glacial rebound

As a simple model of the recovery of the continents (e.g. Scandinavia) from the loading imposed by the ice sheets of the last Ice Age, we consider the response of a viscous fluid to a broadly distributed surface load.

We consider an initial vertical displacement

$$u_z = w_o \cos kz, \qquad k = 2\pi/\lambda \tag{25}$$

where the maximum displacement is small compared with the wavelength λ. Once we have determined the response to (25), more general deformation shapes can be generated by Fourier synthesis.

We note that the vertical velocity is to be found from $\partial_x \psi$ and so look for a stream function in the form

$$\psi = \sin kx\, Z(z)$$

and then

$$\frac{d^4}{dz^4} Z - 2k^2 \frac{d^2}{dz^2} Z + k^4 Z = 0$$

with solution

$$Z = B e^{-kz} + kDz e^{-kz}$$

where the growing exponentials have been excluded to give finite values for Z as $z \to \infty$. Thus

$$\psi = \sin kx e^{-kz}(B + kDz)$$

and the velocity components are

$$v_x = \sin kx e^{-kz}[kB + (kz - 1)kD]$$
$$v_z = k \cos kx e^{-kz}[B + kDz]$$

We will force the horizontal component of velocity to be zero at the surface, which since $u_z \ll \lambda$ can be taken as $z = 0$, and so

$$D = B$$

The final constant B can be found by equating the hydrostatic pressure associated with the topography to the normal stress at the upper boundary

$$\rho g u_z = -p + 2\mu \partial_z v_z \qquad \text{at} \quad z = 0$$

Now, at $z = 0$,

$$\partial_x p = \mu(\partial_{xx} v_x + \partial_{zz} v_x) = -2\mu B k^3 \sin kx$$

and so

$$p = 2\mu B k^2 \cos kx \qquad \text{on} \quad z = 0$$

and

$$\partial_z v_z = 0 \quad \text{on} \quad z = 0$$

The boundary condition at $z = 0$ is thus

$$\rho g u_z = -2\mu B k^2 \cos kx$$

The surface displacement is to be found from

$$\partial_t u_z = v_z(x, 0) = Bk \cos kx$$

and so the displacment at $z = 0$ is given by

$$\frac{\partial u_z}{\partial t} = -\frac{\rho g u_z}{2\mu k} = -\frac{g u_z}{2\nu k}$$

which can be integrated to give

$$u_z = w_o \cos kx \exp(-t/\tau_c)$$

where the decay time τ_c is given by

$$\tau_c = 2\nu k/g = 4\pi\nu/g\lambda$$

in terms of the kinematic viscosity ν and the wavelength of the initial disturbance λ. The decay time is therefore (for long wavelength disturbances) inversely proportional to the wavelength.

6.4 Thermal Convection

When a fluid is heated there is normally a decrease of density due to thermal expansion. A fluid with heating from below or internally with a cool upper surface will tend to have cool dense fluid near the upper boundary and lighter hot fluid at depth. Such a configuration is gravitationally unstable and so the hot fluid tends to rise and the cooler fluid tends to sink to give a thermal convection regime.

6.4.1 Boussinesq approximation

Because the principal driving force for convection comes from thermally induced density variations we must include these effects in the gravitational force terms, but the variations will be small enough that they can be neglected elsewhere. Set

$$\rho = \rho_o + \rho', \qquad \rho' \ll \rho_o$$

where ρ_o is a reference density and ρ' is the thermally induced density change

$$\rho' = -\rho_o \alpha_V (T - T_o)$$

where α_V is the volume coefficient of thermal expansion, and T_o the temperature corresponding to the reference density ρ_o. We can then write the Navier-Stokes equation in the form

$$-\nabla P + \rho' g \hat{\mathbf{z}} + \mu \nabla^2 \mathbf{v} = 0 \tag{26}$$

where P accounts for the hydrostatic pressure in the reference state

$$P = p - \rho_o g z$$

The appropriate thermal equation for weak flows can be derived from (10) by neglecting the work done by the flow and treating κ, ρ, and C as constants to yield

$$\partial_t T + (\mathbf{v}.\nabla)T = \kappa_H \nabla^2 T + h \tag{27}$$

in terms of the thermal diffusivity κ_H, h represents the rate of internal heating.

6.4.2 Onset of convection

We will consider a layer of fluid of thickness L without internal heat sources (i.e. $h=0$) with upper and lower boundaries maintained at fixed temperature T_u, T_l respectively.

In the absence of convection when the temperature gradient is small, the temperature equation reduces to

$$\partial_{zz} T = 0 \quad \text{with} \quad \begin{cases} T = T_u, & z = 0 \\ T = T_l, & z = L \end{cases}$$

The temperature profile is then

$$T_c(z) = T_u + (T_l - T_u)z/L = T_u + \theta z \tag{28}$$

When the lower temperature is raised sufficiently, the system will begin to convect and we can look at the onset of convection by a perturbation analysis about the non-convecting state in which $T = T_c(z)$ and $\mathbf{v} = 0$.

Thus, we set

$$T = T_c(z) + T'(x, z)$$
$$\mathbf{v} = \mathbf{v}'(x, z)$$

where T', $\mathbf{v}'$ are first order perturbations. The temperature equation then takes the form

$$\partial_t T' + v_z{}' = \kappa_H (\partial_{xx} T' + \partial_{zz} T') \tag{29}$$

since the nonlinear terms $v_x{}'\partial_x T'$, $v_z{}'\partial_z T'$ are second order quantities and can be neglected. The two-dimensional fluid flow equations under the assumption of incompressibility are

$$\begin{aligned}
&\partial_x v_x{}' + \partial_z v_z{}' = 0 \\
&-\partial_x P' + \mu(\partial_{xx} v_x{}' + \partial_{zz} v_x{}') = 0 \\
&-\partial_z P' - \rho_o g \alpha_V T' + \mu(\partial_{xx} v_z{}' + \partial_{zz} v_z{}') = 0
\end{aligned} \tag{30}$$

The boundary conditions are that the surfaces $z=0$ and $z=L$ are isothermal and that there is no vertical flow

$$T' = v_z{}' = 0 \quad \text{at} \quad z=0, L \tag{31}$$

We can simplify the subsequent analysis by taking free surface conditions at $z=0, L$ so that the shear stress vanishes and

$$\partial_z v_x{}' = 0 \quad \text{at} \quad z=0, L \tag{32}$$

There is no contribution from $\partial_x v_z{}'$ since $v_z{}'=0$ for all x on the boundaries.

As in previous 2-D flow problems, we introduce a stream function ψ such that

$$v_x{}' = -\,\partial_z \psi, \qquad v_z{}' = \partial_x \psi$$

and then we write (29), (30) in the form of two coupled partial differential equations

$$\begin{aligned}
&\partial_t T' + \theta \partial_x \psi = \kappa_H (\partial_{xx} T' + \partial_{zz} T') \\
&\nabla^4 \psi - \rho_o g \alpha_V \partial_x T' = 0
\end{aligned} \tag{33}$$

The solution of these two linear equations can be found by separation of variables and to satisfy the boundary conditions (31), (32) we can take

$$\begin{aligned}
&\psi = \psi_a \sin(\pi z/L) \sin kx \mathrm{e}^{\beta t} \\
&T' = T_a \sin(\pi z/L) \sin kx \mathrm{e}^{\beta t}
\end{aligned} \tag{34}$$

where the horizontal wavenumber k, the growth factor β and the constants ψ_a, T_a are related by the coupled equations

$$\begin{aligned}
&(\beta^2 + \kappa_H k^2 + \kappa_H^2 \pi^2/L^2)\, T_a = -\, k\theta\, \psi_a \\
&\mu(k^2 + \pi^2/L^2)\, \psi_a = -\, k\, \rho_o g \alpha_V\, T_a
\end{aligned} \tag{35}$$

For β positive any perturbation away from the reference state will grow in time and the heated layer will be convectively unstable. For β negative any perturbation will decay and so the layer will be stable against convection.

Eliminating ψ_a and T_a between the two equations (35) we can express the growth rate β in the form

$$\beta = \frac{\kappa_H}{L} \left(Ra\, \frac{k^2 L^2}{(k^2 L^2 + \pi^2)^2} - (k^2 L^2 + \pi^2) \right) \tag{36}$$

where the dimensionless Rayleigh number

$$Ra = \frac{\rho_o g \alpha_V \theta L^4}{\mu \kappa_H} = \frac{\rho_o g \alpha_V (T - T_o) L^3}{\mu \kappa_H} \tag{37}$$

The growth rate β will be positive, leading to convective instability, if

$$Ra > Ra_{cr} = \frac{(k^2L^2 + \pi^2)^3}{k^2L^2} \tag{38}$$

If $Ra < Ra_{cr}$, $\beta < 0$ and the layer will be stable against convection. The behaviour is determined by the dimensionless wavenumber kL relating the vertical and horizontal scales of the perturbation.

The critical Rayleigh number has an absolute minimum value when $kL = \pi/\sqrt{2}$ and

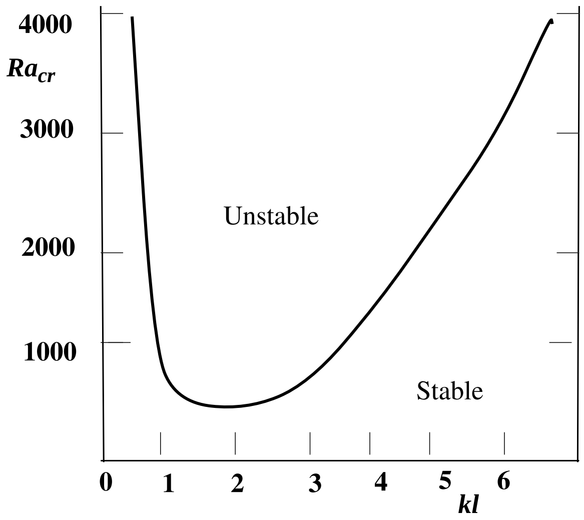
$$\min(Ra_{cr}) = 27\pi^4/4 \approx 657.51$$

If the Rayleigh number is below this value, convection cannot occur.

From the definition of the Rayleigh number (37) we can see that we can interpret the condition $Ra > Ra_{cr}$ in a variety of ways. In order to allow convection to begin we may require a sufficient temperature gradient or a temperature dependent viscosity to drop below a certain value.

A similar calculation may be carried out for no-slip boundry conditions or for a fluid heated from within, but in these cases the critical Rayleigh number has to be found numerically.

Most studies of time-dependent convection processes have been addressed by numerical solution of the governing equations, but that does not detract from utility of the linearised stability analysis in assessing the behaviour of different parts of the horizontal wavenumber spectrum of the fluid flow.



7 Linear Viscoelasticity

We can characterise a solid by a stress-state which depends on strain and a fluid by a stress-state which depends on the rate of strain. The intermediate situation in which stress depends on the strain and strain-rate via the history of deformation covers a wide variety of behaviour described as viscoelastic.

We will confine our attention to situations in which the strains are small so that the relation betwen stress and strain is linear. Even this development of *linear viscoelasticity* will allow the description of quite complex behaviour.

7.1 Constitutive relations for linear viscoelasticity

We assume that the stress tensor σ depends on the Cauchy strain tensor e by a linear transformation

$$\sigma(\mathbf{x}, t) = C\{e^t\} \tag{1}$$

where, as in chapter 4, e^t denotes the history of strain up to time t. This linear relation is assumed to be single-valued and to possess a unique inverse relation

$$e(\mathbf{x}, t) = G\{\sigma^t\} \tag{2}$$

Because the relation is linear C, G represent tensor functions e.g.

$$\sigma_{ij}(\mathbf{x}, t) = C_{ijpq}\{e^t_{pq}\} \tag{3}$$

with ij, pq symmetry associated with the symmetries of the stress and strain tensor

7.1.1 Creep and Relaxation

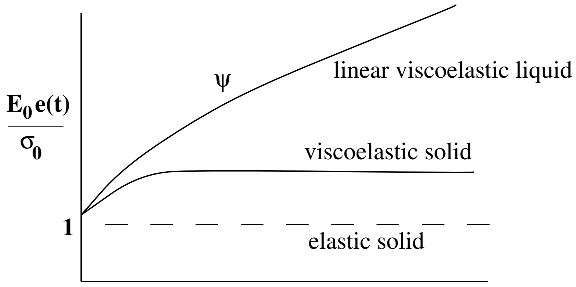
Consider imposing a step function stress cycle on a wire (i.e. a one-dimensional problem)

$$\sigma(t) = \sigma_o H(t)$$

then the resulting strain for $t > 0$ can be written as

$$e(t) = \frac{\sigma_o}{E_D}\,[1 + \psi(t)], \quad \psi(0) = 0$$

where E_D is the *dynamic modulus* and σ_o/E_D is an instantaneous elastic response. The monotonically increasing function $\psi(t)$ is known as the *creep function*. If $\psi(t)$ approaches a constant as $t \to \infty$, then we have long-term solid behaviour, whereas for materials like pitch whose ultimate behaviour is that of a (viscous) fluid $\psi(t)$ increases without limit.



We can extend this concept to a more general stress history, by considering the superposition of many small steps and then taking the limit as the step interval tends to zero, to obtain

$$e(t) = E_D^{-1}[\sigma(t) + \int_{-\infty}^{t} \mathrm{d}s\, \dot{\sigma}(s)\psi(t-s)] \tag{4a}$$

a convolution integral over stress rate. If the creep function is differentiable for $t \geq 0$, we may use the properties of the convolution to rewrite the strain as

$$e(t) = E_D^{-1}[\sigma(t) + \int_{-\infty}^{t} \mathrm{d}s\, \sigma(s)\dot{\psi}(t-s)] \tag{4b}$$

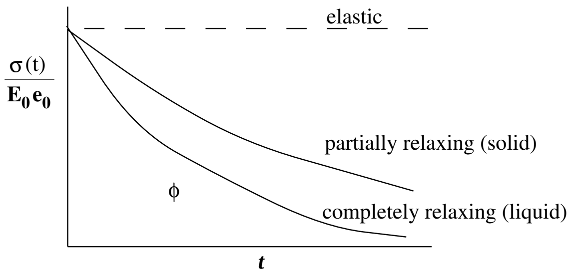
An alternative viewpoint is to consider a strain cycle for which

$$e(t) = e_o H(t)$$

with associated stress for $t > 0$

$$\sigma(t) = E_D e_o[1 - \phi(t)], \quad \phi(0) = 0$$

$\phi(t)$ is termed the relaxation function and is a monotonically increasing function with a monotonically decreasing slope. If the long-term behaviour $\phi(\infty) = 0$ we have a liquid.



For an arbitrary differentiable strain history we can represent the stress behaviour as

$$\sigma(t) = E_D[e(t) - \int_{-\infty}^{t} \mathrm{d}s\, \dot{e}(s)\phi(t-s)] \tag{5a}$$

Once again if the relaxation function is differentiable for $t \geq 0$, we can reorganise the convolution to give

$$\sigma(t) = E_D[e(t) - \int_{-\infty}^{t} \mathrm{d}s\, e(s)\dot{\phi}(t-s)] \tag{5b}$$

Since (4a), (5a) have to represent the same mechanical behaviour we can establish the relation between the creep and relaxation functions. Consider the application of a constant unit stress then

$$e(t) = E_D^{-1}[1 + \psi(t)]$$

and so from (5a) for $t > 0$

$$1 = 1 + \psi(t) - \int_{0}^{t} \mathrm{d}s\, \dot{\psi}(s)\phi(t-s)$$

i.e.

$$\psi(t) = \int_{0}^{t} \mathrm{d}s\, \dot{\psi}(s)\phi(t-s), \quad t > 0 \tag{6a}$$

and similarly we find

$$\phi(t) = \int_{0}^{t} \mathrm{d}s\, \dot{\phi}(s)\psi(t-s), \quad t > 0 \tag{6b}$$

These two coupled integral equations have their simplest solution in the Laplace transform domain. We define e.g.

$$\bar{\psi}(p) = \int_{0}^{\infty} \mathrm{d}t\, \psi(t)\mathrm{e}^{-pt}$$

and then (4) and (5) can be written as

$$\bar{e}(p) = E_D^{-1}[1 + p\bar{\psi}(p)]\bar{\sigma}(p) \tag{4c}$$

$$\bar{\sigma}(p) = E_D[1 - p\bar{\phi}(p)]\bar{e}(p) \tag{5c}$$

Since these relations are equivalent we require

$$[1 + p\bar{\psi}(p)][1 - p\bar{\phi}(p)] = 1$$

which connects the Laplace transforms of $\psi(t)$, $\phi(t)$. We can now define a transform modulus

$$E(p) = E_D[1 - p\bar{\phi}(p)] = E_D[1 + p\bar{\psi}(p)]^{-1} \tag{7}$$

in terms of which the one-dimensional constitutive relation for linear viscoelasticity can be written as

$$\bar{\sigma}(p) = E(p)\bar{e}(p) \tag{8}$$

The formal analogy between the Laplace transform domain result (8) and the equivalent elastic relation provides the formal basis for solving many viscoelastic problems via the *correspondence principle*.

7.1.2 Models of viscoelastic behaviour

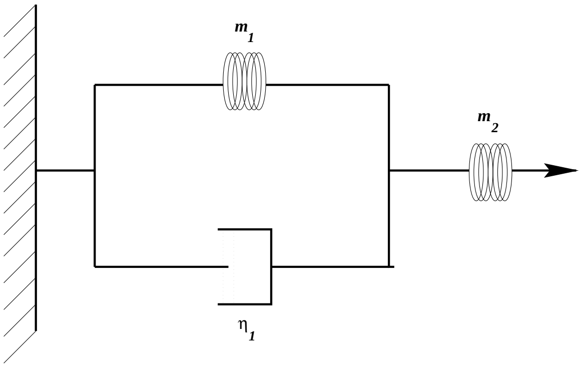
Elementary models of viscoelastic behaviour can be built up from two basic building blocks: the elastic spring for which

$$\sigma = me$$

amd the viscous dashpot for which

$$\sigma = \eta\dot{e}$$

The most useful model of this type is the *standard linear solid* consisting of a spring in series with a spring and dashpot combination.



If the strain in the parallel spring and dashpot is e_1 then the stress is

$$\sigma = m_1 e_1 + \eta_1 \dot{e}_1 = m_2 e_2$$

since this must also balance the strain e_2 in the second spring. The total strain is

$$e = e_1 + e_2$$

and thus

$$m_1 m_2 e + \eta_1 m_2 \dot{e} = (m_1 + m_2)\sigma + \eta_1 \dot{\sigma}$$

We may rewrite this relation as

$$\dot{\sigma} + \frac{\sigma}{\tau_R} = m_2\left(\dot{e} + \frac{e}{\tau_C}\right) \tag{9}$$

and the characteristic time scales for relaxation (τ_R) and creep (τ_C) are given by

$$\tau_R = \eta_1/(m_1 + m_2), \qquad \tau_C = \eta_1/m_1 \tag{10}$$

This model gives a three parameter description (m_2, τ_R, τ_C) of a viscoelastic material and covers all the mechanical features of such solids. For very rapid deformations the terms in $\dot{\sigma}$ and $\dot{e}$ are important and the material behaves like an elastic solid with dynamic modulus m_2; for very slow deformation the behaviour is again elastic with a static modulus $\tau_R m_2/\tau_C$.

The relaxation function for this standard linear solid model is

$$\phi(t) = [1 - (\tau_R/\tau_C)][1 - \mathrm{e}^{-t/\tau_R}] \tag{11}$$

and the corresponding creep function is

$$\psi(t) = [(\tau_C/\tau_R) - 1][1 - \mathrm{e}^{-t/\tau_C}] \tag{12}$$

Although the standard linear solid model goes a long way towards providing a description of a viscoelastic material it does not provide an adequate description of any real material. More satisfactory representations are provided by joining standard linear solid elements in series or parallel.

In the series case, the stress in each element is the same whilst the strain is the sum of the strains in each element. The creep function is then a sum of terms of the form (12)

$$\psi(t) = \sum_n f(\tau_n)[1 - \mathrm{e}^{-t/\tau_n}] \tag{13}$$

where $f(\tau_n)$ may be regarded as the contribution of a creep process with a characteristic time τ_n.

For a set of parallel elements, the stress is cumulative and the relaxation function can be built from a set of terms of the form (11)

$$\phi(t) = \sum_n g(\tau_n{}')[1 - \mathrm{e}^{-t/\tau_n{}'}] \tag{14}$$

In the most general case we can consider a spectrum of relaxation and creep processes.

7.1.3 Isotropic linear viscoelasticity

For an isotropic medium the tensor function C_{ijpq} in (3) can be represented as an isotropic tensor of the form

$$C_{ijpq} = L\delta_{ij}\delta_{pq} + M(\delta_{ip}\delta_{jq} + \delta_{iq}\delta_{jp}) \tag{15}$$

where L and M are scalar functions, and so

$$\sigma_{ij}(\mathbf{x}, t) = \delta_{ij} L\{e^t_{kk}\} + 2M\{e^t_{ij}\} \tag{16}$$

and we can separate the dependence of the dilatation component from the deviatoric stress and strain.

By analogy with the equations of isotropic elasticity we can write the stress strain relation as

$$\sigma_{ij}(\mathbf{x},t) = \lambda_0 \delta_{ij} e_{kk}(\mathbf{x},t) + 2\mu_0 e_{ij}(\mathbf{x},t)$$

$$+ \int_0^t \mathrm{d}s\, \delta_{ij} e_{kk}(\mathbf{x},s) \dot{R}_\lambda(t-s) + \int_0^t \mathrm{d}s\, 2e_{ij}(\mathbf{x},s) \dot{R}_\mu(t-s) \tag{17}$$

in terms of relaxation functions R_λ, R_μ and a representation analogous to (5b) for the one-dimensional case.

An equivalent representation to (17) expressing the strain as a function of the stress history can be made in terms of creep functions C_λ, C_μ. These are related to the relaxation functions via coupled integral equations similar to (6). The correspondence principle in the Laplace transform domain can be extended to these three-dimensional problems and provides a means of solving many viscoelastic problems.

7.2 Damping of harmonic oscillations

We consider a harmonic strain cycle of the form

$$e_{ij} = E_{ij} \mathrm{e}^{-\mathrm{i}\omega t}$$

and then the corresponding stress is

$$\sigma_{ij} \mathrm{e}^{\mathrm{i}\omega t} = \lambda_0 \delta_{ij} E_{kk} + 2\mu_0 E_{ij}$$

$$+ \int_{-T}^t \mathrm{d}s\, [\delta_{ij} E_{kk} \dot{R}_\lambda(t-s) + 2E_{ij} \dot{R}_\mu(t-s)] \mathrm{e}^{\mathrm{i}\omega(t-s)}$$

where $t = -T$ is the time at which the disturbance starts. Now let $T \to \infty$ and introduce the Fourier transforms of the time derivatives of the relaxation functions

$$\lambda_1(\omega) = \int_0^\infty \mathrm{d}t\, \dot{R}_\lambda(t) \mathrm{e}^{\mathrm{i}\omega t}, \qquad \mu_1(\omega) = \int_0^\infty \mathrm{d}t\, \dot{R}_\mu(t) \mathrm{e}^{\mathrm{i}\omega t}$$

We can express the stress as

$$\sigma_{ij} = \{[\lambda_0 + \lambda_1(\omega)] \delta_{ij} E_{kk} + 2[\mu_0 + \mu_1(\omega)] E_{ij}\} \mathrm{e}^{-\mathrm{i}\omega t} \tag{19}$$

with a stress-strain relation in the same form as for an elastic medium but with frequency dependent complex moduli

$$\bar{\lambda} = \lambda_0 + \lambda_1(\omega), \qquad \bar{\mu} = \mu_0 + \mu_1(\omega)$$

We may therefore employ the descriptions of P and S waves developed in section 5.4, but have to allow for the frequency dependent velocities and consequent energy loss.

The rate of energy dissipation per unit volume due to deformation of the medium is from (4.10)

$$\rho \frac{D}{Dt} U = \sigma_{ij} \frac{\partial v_j}{\partial x_i} \approx \sigma_{ij} \dot{e}_{ij}$$

in the linear approximation. The average work dissipated per unit time and volume is thus

$$\bar{W} = \tfrac{1}{2} \,\mathrm{Re}\, [\mathrm{i}\omega \sigma_{ij} E^*_{ij} \mathrm{e}^{\mathrm{i}\omega t}]$$

$$= -\tfrac{1}{2}\, \omega \,\mathrm{Im}\, [\bar{\kappa} \Delta \Delta^* + 2\bar{\mu} \hat{E}_{ij} \hat{E}^*_{ij}]$$

where the dilatation

$$\Delta = E_{kk}$$

and the deviatoric strain

$$\hat{E}_{ij} = E_{ij} - {}^{1}\!/3 \Delta \delta_{ij}$$

The complex bulk modulus

$$\bar{\kappa} = \bar{\lambda} + {}^{2}\!/3 \bar{\mu} = \kappa_0 + \kappa_1(\omega)$$

A convenient measure of the rate of energy dissipation is the loss factor $Q^{-1}(\omega)$ which may be defined as

$$Q^{-1} = \frac{\Delta \bar{W}(\omega)}{2\pi W_{\mathrm{o}}(\omega)} \tag{20}$$

where $\Delta \bar{W}(\omega)$ is the energy loss in a cycle at frequency ω and $W_{\mathrm{o}}(\omega)$ is the 'elastic' energy stored in the oscillation i.e. the sum of the strain and kinetic energy associated with just the instantaneous elastic moduli. For purely deviatoric disturbances

$$Q^{-1}_{\mu}(\omega) = \frac{-\tfrac{1}{2}\,\mathrm{Im}\, \bar{\mu}(\omega) \hat{E}_{ij} \hat{E}^*_{ij}}{\tfrac{1}{2}\, \hat{E}_{ij} \hat{E}^*_{ij}} = \frac{-\mathrm{Im}\, \mu_1(\omega)}{\mu_0} \tag{21}$$

Similarly for purely deviatoric disturbances

$$Q^{-1}_{\kappa}(\omega) = -\mathrm{Im}\, \kappa_1(\omega)/\kappa_0 \tag{22}$$

Normally loss in pure dilatation is much less significant than loss in shear and $Q^{-1}_{\kappa} \ll Q^{-1}_{\mu}$

In general the loss factor for dilatation or shear can be represented as

$$Q_m^{-1}(\omega) = -\,\mathrm{Im}\, m_1(\omega)/m_0 \tag{23}$$

where m is the appropriate complex modulus. For the standard linear solid

$$\begin{aligned} m(\omega) &= m_2(\tau_C^{-1} - \mathrm{i}\omega)/(\tau_R^{-1} - \mathrm{i}\omega) \\ &= m_2(\omega^2 + \tau_C^{-1}\tau_R^{-1} - \mathrm{i}\omega(\tau_R^{-1} - \tau_C^{-1}))/(\omega^2 + \tau_R^{-2}) \end{aligned} \tag{24}$$

and the instantaneous modulus is m_2, and thus the loss factor

$$Q_m^{-1}(\omega) = \omega(\tau_R^{-1} - \tau_C^{-1})/(\omega^2 + \tau_R^{-2}) \tag{25}$$

By superposing a spectrum of relaxation times, a wide range of frequency behaviour can be simulated.

Note that the dissipation via $Q^{-1}(\omega)$ is accompanied by a frequency dependent correction to the real part of the complex modulus and hence to the apparent wavespeed

$$\mathrm{Re}\, m(\omega) = m_2(\omega^2 + \tau_C^{-1}\tau_R^{-1})/(\omega^2 + \tau_R^{-2}) \tag{26}$$

This property is a consequence of a causal dissipation mechanism. Since the relaxation contributions depend only on the past history of the strain $\dot{R}_m(t)$ vanishes for $t < 0$, so that the transform $m_1(\omega)$ must be analytic in the upper half plane. In consequence the real and imaginary parts of $m_1(\omega)$ are Hilbert transforms of each other

$$\mathrm{Re}\, m_1(\omega) = \frac{1}{\pi}\,\mathrm{P}\int_{-\infty}^{\infty} \mathrm{d}\omega'\, \frac{\mathrm{Im}\, m_1(\omega')}{\omega' - \omega} = -\frac{2m_0}{\pi}\,\mathrm{P}\int_{0}^{\infty} \mathrm{d}\omega'\, \frac{\omega' Q^{-1}(\omega')}{\omega'^2 - \omega^2} \tag{27}$$

where P denotes the Cauchy principal value of the integral.

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Problems I: Deformation

1. For a vector triad α_i, introduce the 'reciprocal triad' β_j such that $\alpha_i^T\beta_j = \alpha_i.\beta_j = \delta_{ij}$. Show that $\alpha_p\beta_p^T = I$ where I is the unit matrix.

   Under homogeneous deformation the triad α_i is transformed into the triad $\mathbf{a}_i$, show that the deformation gradient tensor F can be represented as $F = \mathbf{a}_p\beta_p^T$

2. Consider a crystal lattice whose elementary cell is a unit cube. A *lattice-plane* is defined to have intercepts ν_i^{-1} $(i=1,2,3)$ on the cube edges from node 0, where ν_i are positive or negative integers. Suppose the crystal is deformed so that each cell becomes a parallelepiped with edges $\mathbf{a}_i$, and let a *reciprocal lattice* be defined with cell edges $\mathbf{b}_j$ such that $\mathbf{a}_i.\mathbf{b}_j = \delta_{ij}$. Show that, under arbitrary deformation, the plane remains perpendicular to an embedded directions in the reciprocal lattice, while the distance of the plane from 0 varies as $|\nu_j\mathbf{b}_j|^{-1}$.

3. Given

$$F = \frac{1}{9}\begin{pmatrix} 7 & -5 & 2 \\ 4 & 4 & -1 \\ -5 & 7 & 2 \end{pmatrix}$$

   Show that its Lagrangian triad is coaxial with $(0,0,1)$, $(1,1,0)$, $(1,-1,0$ and find the principal stretches. Hence construct U and U^{-1} and by constructing $R = FU^{-1}$, verify that it is a rotation of $\pi/3$ about $(1,1,1)$. Finally show that the Eulerian triad is coaxial with $(2,-1,2)$, $(1,4,1)$, $(1,0,-1)$.

4. A unit cube is deformed homogeneously by the mapping $\xi \to \mathbf{x} = F\xi$ with respect to coordinates along the edge of the cube. Obtain formulae for the lengths of the corresponding edges of the resulting parallelepiped, their included angles, and the distances between pairs of opposite faces.

   Hence find the specific values of these quantities for a rhombohedron obtained by doubling the length of the long diagonal of the cube from the origin to $(1,1,1)$, without changing the directions in transverse planes.

   Verify that a congruent rhombohedron could be generated by a homogeneous deformation that maps the cube edges $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ onto $(0,1,1)$, $(1,0,1)$, $(1,1,0)$ respectively and find the required matrix F.

5. A plane deformation is specified by

$$x_1 = \lambda_1\xi_1 + \gamma\lambda_2\xi_2, \quad x_2 = \lambda_2\xi_2$$

which is a pure strain with principal stretches λ_1, λ_2 followed by a simple shear of amount γ. For $\lambda_1 \neq \lambda_2$ and small γ, verify that the overall principal stretches are $\lambda_1(1+\eta+\ldots)$ and $\lambda_2(1-\eta+\ldots)$ where

$$(\lambda_1^2/\lambda_2^2 - 1)\eta = \tfrac{1}{2}\gamma^2$$

Show that after the shear γ the Lagrangian diad has an inclination to the reference axes of $\lambda_1\lambda_2\gamma\,/\,(\lambda_1^2 - \lambda_2^2)$ to first order. Find the corresponding inclination of the Eulerian diad.

6. A simple shear $I + \gamma \mathbf{mn}^T$ is followed by a deformation F. Show that the final configuration could alternatively be produced by F followed by a simple shear. Determine its amount, direction and basal plane.

7. A simple model of a fibre reinforced composite is an incompressible matrix containing densely packed inextensible fibres, all initially inclined at an angle ϕ to the ξ_1 axis $(0 \le \phi \le \pi/2)$, and lying in the (ξ_1, ξ_2) plane.

   Consider a shear deformation

$$x_1 = s\xi_1 + r\xi_2, \quad x_2 = s^{-1}\xi_2, \quad x_3 = \xi_3$$

   and show that for the composite medium the path in (r, s) space is given by

$$s_2 \cos^2\phi + (s^{-2} + r^2)\sin^2\phi + 2rs\sin\phi\cos\phi = 1$$

   and hence that $s > |\sin\phi|$. Show that the fibres remain parallel but now have an inclination θ to the x_1 axis where $\tan\theta = \sin\phi\,/\,(s^2 - \sin^2\phi)^{\frac{1}{2}}$.

8. A body undergoes the deformation specified by

$$x_1 = \xi_1(1 + a^2t^2), \quad x_2 = \xi_2, \quad x_3 = \xi_3$$

   Find the displacement and velocity in both the material and spatial descriptions

9* Let F_1 and F_2 be the gradients of deformation when reckoned relative to different reference configurations 1 and 2 (differing by a rotation). Let E_1 and E_2 be the corresponding Green's measures of the relative strains, and let $\dot{E}_1$ and $\dot{E}_2$ be their rates of change as the deformation varies. Show that

$$B_1\dot{E}_1B_1^T = B_2\dot{E}_2B_2^T$$

   where B_1 and B_2 are the transposed inverses of F_1 and F_2.

10*.

Show that for any scalar,vector or tensor field ψ and any material volume V bounded by a surface S with outward normal $\mathbf{n}$

$$\frac{d}{dt}\int_V \rho\psi \,\mathrm{d}V = \int_V \rho\,\frac{D\psi}{Dt}\,\mathrm{d}V$$

$$\frac{d}{dt}\int_V \psi \,\mathrm{d}V = \int_V \frac{\partial\psi}{\partial t}\,\mathrm{d}V + \int_S \psi\,\mathbf{v}.\,\mathbf{n}\,\mathrm{d}S$$

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Problems II: Stress Analysis

1. Show that the stress field $\sigma_{ij} = p\delta_{ij} + \rho v_i v_j$ is self-equilibrated, where p, ρ and $\mathbf{v}$ are the pressure, density and velocity in a steady flow of some inviscid compressible fluid.

   Show that the stress field $\sigma_{ij} = v_i v_j - \frac{1}{2}\mathbf{v}^2\delta_{ij}$ exerts tractions on any closed surface which are statically equivalent overall to body forces $\mathbf{v}\nabla.\mathbf{v}$ per unit volume, when $\mathbf{v}$ is any irrotational field.

2. Verify that any stress of the type $\sigma_{ij} = r^{n-2}[(n+2)r^2\delta_{ij} - x_i x_j]$, where $r = |\mathbf{x}| \neq 0$ is self-equilibrated. Find the traction vector at any point on the surface of a hemispherical shell with internal and external radii a and b centred on the origin. Check your tractions are in overall balance by resolving in the direction of the axis of symmetry.

3. A cylinder whose axis is parallel to the x_3-axis and whose normal cross-section is the square $-a \le x_1 \le a$, $-a \le x_2 \le a$ is subjected to torsion by couples acting over its ends $x_3 = 0$ and $x_3 = L$. The stress components are given by $\sigma_{13} = \partial\psi/\partial x_2$, $\sigma_{23} = -\partial\psi/\partial x_1$, $\sigma_{11} = \sigma_{12} = \sigma_{22} = \sigma_{33} = 0$, where $\psi = \psi(x_1, x_2)$.

(a) Show that this stress tensor is self-equilibrated;

(b)

show that the difference between the maximum and minimum stress components is $2[(\partial\psi/\partial x_1)^2 + (\partial\psi/\partial x_2)^2]^{\frac{1}{2}}$, and find the principal axis which corresponds to the zero principal component;

(c) for the special case $\psi = (x_1^2 - a^2)(x_2^2 - a^2)$ show that the lateral surfaces are free from traction and that the couple acting on each end face is $32a^6/9$.

4*.

Show that every statically-admissable jump in the stress tensor is of the type

$$(\delta_{\mathrm{i}} - n_i n_r)(\delta_{js} - n_j n_s)p_{rs}$$

for some symmetric p_{rs}, where $\mathbf{n}$ is the unit normal to the jump surface.

5. Describe the properties of the strain energy for a body which has an axis of symmetry with respect to deformation from the reference state so that the strain energy of deformation is unchanged by rotation in the reference state about a direction fixed in the body. If the 1-axis of the Lagrangian triad of the strain coincides with the symmetry axis, show that the 23 component of the stress referred to the Eulerian triad is zero.

6. The work density of an isotropic material may be written as a function of the principal stretches in terms of

$$\alpha = \tfrac{1}{2}\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2), \quad \beta = \tfrac{1}{2}\,(\lambda_1^{-2} + \lambda_2^{-2} + \lambda_3^{-2}), \quad \gamma = \lambda_1\lambda_2\lambda_3$$

Show that the Cauchy stress is given by

$$\frac{1}{\gamma}\left(\frac{\partial W}{\partial \alpha}\, AA^T - \frac{\partial W}{\partial \beta}\, BB^T + \gamma\,\frac{\partial W}{\partial \gamma}\, I\right)$$

where A is the deformation gradient and B its transposed inverse.

7. An isotropic solid has the work function $W(\lambda_1, \lambda_2)$ in plane deformation. A simple shear $x_1 = \xi_1 + r\xi_2$, $x_2 = \xi_2$, $x_3 = \xi_3$, is applied to the material. Show that the stress components can be expressed compactly by

$$(\lambda_1 + \lambda_2)(\sigma_{11}, \sigma_{22}, \sigma_{12}) = (\lambda_1\sigma_1 + \lambda_2\sigma_2,\ \lambda_2\sigma_1 + \lambda_1\sigma_2,\ \sigma_1 - \sigma_2)$$

where

$$\sigma_1 = \lambda_1\,\frac{\partial W}{\partial \lambda_1}, \quad \sigma_2 = \lambda_2\,\frac{\partial W}{\partial \lambda_2}, \quad \lambda_1 - \lambda_2 = r, \quad \lambda_1\lambda_2 = 1.$$

8* A compressible material is characterised by the strain energy function

$$W = \tfrac{1}{2}\,\mu(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 - 3v^{2/3}) + \frac{k}{m}\left(v - \frac{m}{m-1} + \frac{v^{(1-m)}}{m-1}\right)$$

where $v = \lambda_1\lambda_2\lambda_3$, and $\mu\,(> 0)$, $k\,(> 0)$ and m are constants. Derive the equations for the stress components in the material.

A spherical ball made of this material has radius A before deformation, and is subjected to a uniform compression which reduced the radius to a. With respect to background axes through the centre of the ball, the deformation is represented as $x_r = \lambda\xi_r$. If the ball is held in equilibrium, in the absence of body forces, obtain a relation between a and the pressure p acting on the surface of the ball.i

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Problems III:

1. Let F_o be an arbitrary constant tensor for which $\det F_o > 0$. Show that in an incompressible elastic body there exists a motion for which the deformation gradient tensor $F = F_o$ throughout the body at time $t = t_o$, and for which the velocity gradient $(\partial \mathbf{v}/\partial \mathbf{x})$ takes an arbitrary constant value. Find the body-force field required to produce this motion.

2. The work density of an elastic material is given in terms of the principal stretches λ_1, λ_2, λ_3 by

$$W = a(\lambda_1 \lambda_2 \lambda_3)^{-p} + \tfrac{1}{2}\, b\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$$

Obtain the relation between a and b which leads to a stress-free ground state and deduce the Lame moduli for infinitesimal strain. Deduce also Young's modulus.

A right circular cylinder of height h and density ρ is placed upright on a smooth horizontal surface under uniform gravity g. Using the reciprocal theorem show that the mean reduction in height due to the force of gravity is

$$\left(\frac{p+1}{3p+2}\right)\frac{\rho g h^2}{2pa}$$

assuming infinitesimal strain.

3. Define the mean stress in a volume V by

$$V\bar{\sigma}_{ij} = \int_V \sigma_{ij}\, \mathrm{d}V$$

Show that for a volume with surface tractions τ, body force $\mathbf{g}$ and mass-acceleration $\mathbf{f}$ that

$$V\bar{\sigma}_{ij} = \int_S x_i \tau_j\, \mathrm{d}S + \int_V x_i (g_j - f_j)\, \mathrm{d}m$$

An isolated spherical planet, with radius a and uniform density ρ, has a constant spin ω against the stellar background. Show that the mean stress in the planet is a hydrostatic pressure $(1/5)\rho g a$ combined with a biaxial tension $(1/5)\rho a^2 \omega^2$ perpendicular to the polar axis, where g is the acceleration at the surface of the planet due to its own gravitation.

4. Show that the displacement field for elastic waves in an isotropic medium at angular frequency ω can be written in the form

$$\mathbf{u} = \{ A^P [p, 0, \pm q_\alpha] e^{\pm i\omega q_\alpha x_3}$$

$$+ B_V [q_\beta, 0, \mp p] e^{\pm i\omega q_\beta x_3}$$

$$+ B_H [0, 1, 0] e^{\pm i\omega q_\beta x_3} \} e^{i\omega(px - t)}$$

and find expressions for q_α, q_β. Demonstrate that the $\pm$ signs correspond to phase fronts progressing in the $\pm x_3$ direction repectively. What happens if $p > 1/\alpha$?

If the plane $x_3 = 0$ is free from traction, find the reflected waves due to an incident wavefield from $x_3 > 0$. Show that the SH components are decoupled from the P and SV waves.

Discuss how the behaviour would be modified for a linear visco-elastic medium.

5. The behaviour of certain viscous fluids can be modelled by the constitutive equation

$$\sigma_{ij} = - p\delta_{ij} + 2\mu(K_2)D_{ij}$$

where $K_2 = 2D_{ij}D_{ij}$, for rate-of strain tensor D_{ij}, and $\mu(K_2) = kK_2^{(n-1)/2}$ where k and n are positive constants ($n = 1$ corresponds to a Newtonian fluid).

Such a power-law fluid undergoes a simple shearing flow between two large parallel plates a distance h, such that one plate is held fixed and the other moves with a constant speed U in its plane. Show that the shearing force per unit are on the plates is $k(U/h)^n$ and that the apparent viscosity is $k(U/h)^{n-1}$ as a function of the shear rate U/h.

6. Formulate the linear stability problem for the onset of convection in a layer of fluid heated from within. Assume that the boundaries are stress-free. Take the upper boundary to be isothermal and the lower boundary to be insulating. Carry the formulation to the point where the solution of the problem depends only on the integration of an ordinary differential equation for the stream function, subject to appropriate boundary conditions.

7* Consider a flow in a region V bounded by S with $\mathbf{v}$ specified on S_1 and traction τ ($\boldsymbol{\sigma}.\mathbf{n}$) specified on S_2 (where $S_2 = S - S_1$). Introduce $\gamma = (2D_{ij}D_{ij})^{\frac{1}{2}}$ in terms of the rate-of-strain tensor D_{ij}, and define

$$\psi(\gamma) = \int_0^{\gamma} \mu(\gamma')\gamma' \,\mathrm{d}\gamma'.$$

Show that if

$$J = \int_V [\psi(\gamma) - pD_{ii}]\,\mathrm{d}V - \int_{S_2} \mathbf{V}.\tau\,\mathrm{d}S$$

then J is minimised over all choices of $\mathbf{v}$ satisfying the boundary conditions on S_1 by the velocity of a generalised Newtonian fluid with

$$\sigma = pI + \mu(\gamma)D$$

[Hint: perturb $\mathbf{v}$ by an amount $\delta\mathbf{v}$ and set $\delta J = 0$]

8* Consider a body of volume V, subjected to body forces $\mathbf{g}$ and with the displacement $\mathbf{u}$ specified on the portion of the surface S_u and the traction τ specified on S_τ. The strain energy

$$W = \tfrac{1}{2}\int_V e_{ij}\sigma_{ij}\mathrm{d}V.$$

Show that any displacement solution of the elastostatic equations makes

$$W(e_{ij}) - \int_S \tau.\mathbf{u}\,\mathrm{d}S_\tau - \int_V \mathbf{g}.\mathbf{u}\mathrm{d}m$$

stationary in the class of continuous and piecewise continuously differentiable fields taking the given values on S_u.

A homogeneous medium is subject to given traction τ all over S with no body forces acting. Consider a small change

$$\delta u_j = \varepsilon x_k \partial_k u_j$$

and equate the perturbation δW obtained i) by direct evaluation and ii) from the variational principle above, to find a representation of W in terms of a surface integral.