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Euler strut: a mechanical analogy for dynamics in the vicinity of a critical point

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Abstract

An anchored elastic filament (Euler strut) under an external point load applied to its free end is a simple model for a second-order phase transition. In the static case, a load greater than the critical load causes a Euler buckling instability, leading to a change in the filament's shape. The analysis of filament dynamics with an external point load at its end shows that when approaching the critical end-load, the period of such an inverted pendulum diverges in a fashion analogous to a 'soft mode' critical slowing down in, for example, a ferroelectric phase transition of displacive type. We thus show that an advanced concept of solid state physics, i.e. 'soft mode' dynamics and critical slowing down, observable in a variety of second-order phase transitions, can be actualized in this simple mechanical system. The variable loads attached to a vertical spring allow for an experimental implementation and quantitative measurements as an illustration of this analogy.

1. Introduction

Students' conceptions of physical phenomena are usually best developed in mechanics. Everyday life and easily observable phenomena provide them with enough experiences that allow the construction of new knowledge. However, in the case of thermodynamic phenomena that can be observed only indirectly, the relevant theoretical concepts are seldom very close to everyday life, and many topics on the introductory level are presented for the first time without any preliminary knowledge. One such topic is *phase transitions*. Most students are familiar with only the three canonical thermodynamic states of water and transitions between them.

Usually they are not aware of any other states of matter characterized by different types of order and have not heard of the continuous phase transitions and phenomena associated with them.

Besides the phenomenological description of phase transitions and an introduction of associated latent heats, first-year physics students rarely hear more. They usually get better acquainted with thermodynamic phases and theoretical description of transitions between them in advanced courses on thermodynamics, solid state physics and soft matter physics. The simplest and most common technique of theoretical description of phase transition phenomena is the *Landau theory* [1], valid close to but not in the immediate vicinity of the phase transition, where it breaks down due to pronounced fluctuation phenomena[6]. The emerging order of matter is usually illustrated by sketches, but the experimental observations of material properties in the vicinity of the transition are difficult to grasp. The majority of suggested pedagogical models in this context consider mechanical analogues, which are conceptually simple but are usually prone to rather difficult physical realizations that do not allow straightforward observations of typical features associated with their behaviour [3–6]. This is even more true for complex dynamical behaviour of matter close to a phase transition. For instance, the *critical slowing down* is a fundamental property of dynamics close to a second-order transition [7] which would be very difficult to explain without any anchoring in simple mechanical phenomena.

In this paper we suggest a new, intuitive and simple mechanical model based on the Euler buckling instability of an elastic filament, which allows for an intuitive understanding of a mechanical phenomenon that is closely related to the second-order phase transition as well as the dynamical properties close to this transition. It can be realized as a hands-on type laboratory experiment and can be included as an example within the course on the theory of elasticity or classical mechanics. This mechanical model can serve as a theoretical and experimental example introducing various concepts related to phase transitions: a continuous transition, an order parameter as well as a typical 'soft mode' dynamics in the vicinity of the phase transition exhibiting critical slowing down. The setup of the corresponding experiments is straightforward; the experiments themselves are visually appealing and can easily be implemented by rather standard means. The slow dynamics in the vicinity of the buckling instability exhibited by this mechanical model can be demonstrated easily and beautifully, even in an introductory level physics course.

The paper is organized as follows. After the introduction, section 2 sets forth with the description and geometry of the Euler strut, as well as the theoretical and conceptual framework of the second-order phase transitions by introducing the Landau theory of phase transitions and the Landau–Khalatnikov theory of dynamics close to the second-order phase transition for the over and under-damped cases. Section 3 then deals with the static configuration of the Euler strut, i.e. the shape of the end-loaded strut as a result of its elastic free energy minimization. In section 4 the dynamical equations for the motion of the strut are derived and discussed. In section 5 it is shown that the elastic energy of the Euler strut in the vicinity of the buckling instability is completely analogous to the free energy of a thermodynamic system close to the second-order phase transition. Furthermore, we find that the analogue of the order parameter, which is a deviation of the strut from the straight shape, depends on the weight in exactly the same way as the order parameter close to the transition depends on the temperature. Therefore, the weight of the end load of a Euler strut close to a buckling transition is analogous to the temperature close to a second-order phase transition. In order to complete the analogy,

[6] The breakdown of the Landau theory is important, experimentally discernible and well established for the gas–liquid critical point and for many magnetic systems; see, e.g., [2]. The region of breakdown of the Landau theory is experimentally less accessible in the case of liquid crystals.

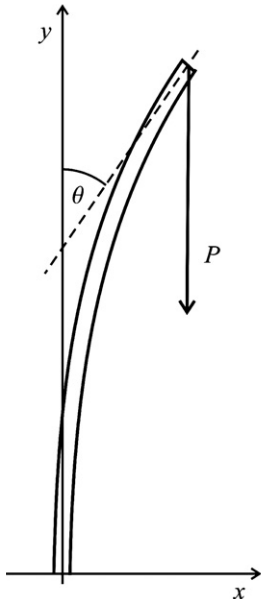

Figure 1. An elastic filament anchored to a solid horizontal support. P is the external vertical load at the free end and θ is the angle between the filament tangent and the vertical axis. The free energy can be expressed in terms of the displacement of the free end of the filament.

we also show that the dynamics of the strut, quantified by the functional dependence of the frequency of the strut on the end-loading weight, is exactly the same as that of the Landau system quantified by the dependence of the relaxation frequencies on the temperature. This completes the proof that both systems can be described by exactly the same equations and are therefore complete analogues. In section 6 we present a practical implementation of the mechanical model, which can be used as a demonstration or as a laboratory experiment. The very last section concludes the paper and discusses suggestions for using the material presented in an educational setting.

2. Theoretical background

The problem of elastic filament deformation has been analysed since the very early stages of the theory of elasticity because it has an important role in engineering and construction applications [8]. The elastic (free) energy (F_e) of a filament of length L is given by a simple expression involving the square of the local curvature [18]:

$$F_e = \frac{1}{2}\, EI \int_0^L \left(\frac{\mathrm{d}\theta(\ell)}{\mathrm{d}\ell}\right)^2 \mathrm{d}\ell. \tag{1}$$

Here ℓ is the arclength and $\theta(\ell)$ is the angle between the tangent to the filament and the vertical axis (see figure 1). E denotes Young's modulus and I is the moment of inertia of the profile of an isotropic rod. To obtain the deformed shape of the filament, Euler wrote the differential equations known now as Euler–Lagrange equations and qualitatively described all possible (planar) solutions [10]. Today these solutions are known as *elasticae Eulerii*.

In a related buckling problem of a Euler strut, see figure 1, an elastic filament is anchored vertically into a solid support at one end while the other is loaded by an external point force

P, in the vertical direction [8]. Euler calculated that there exists a critical force P_c at which the rod starts to bend, given by

$$P_c = \frac{\pi^2 EI}{4L^2}. \tag{2}$$

If the force is too small, $P < P_c$, the rod will remain straight and undeformed, but when the critical value of the vertical force is reached the rod will bend [10]. This type of behaviour is known as the *Euler buckling instability* which is historically the first example of a bifurcation phenomenon and a paradigm of a phase transition. Its detailed description can be found e.g. in books on elastic and structural stability [11].

If one now displaces the Euler strut from its equilibrium position, one obtains an *inverted Euler pendulum*. We expect that for small deviations from the undeformed configuration, with external point force below the critical value, it will exhibit harmonic oscillations around equilibrium. What we want to investigate in detail is how the period of these oscillations depends on the external end load generated by a point mass of variable weight.

A closer look shows that the dynamics of the Euler inverted pendulum with a point load attached to its free end is analogous to the dynamics of an order parameter in the vicinity of a second-order phase transition. The correspondence is as follows: the increasing point load plays the role of the decreasing temperature, and the deviation of the free end of the filament from the straight shape or equivalently of the angle formed by the end of the filament and the vertical direction corresponds to the order parameter, and the motion of the Euler strut around its straight shape when the end load is increased corresponds to the order-parameter dynamics in the parent phase on approach to the transition into the symmetry-broken phase. We now prove this correspondence exactly by first introducing the dynamics in the vicinity of the phase transition, then the buckling transition of the elastic filament and finally connecting the two.

Theoretical analysis of the second-order phase transition and the dynamics close to the transition temperature usually starts with the Landau expansion of the free energy [12] close to a second-order phase transition:

$$\Omega(\eta, T) = \Omega(0, T) + \tfrac{1}{2}\, a\, (T - T_c)\, \eta^2 + \tfrac{1}{4}\, b\, \eta^4, \tag{3}$$

where η is the order parameter, a is a constant dependent on the system, T is the temperature and T_c is the phase transition temperature. The equilibrium value of the order parameter $(\overline{\eta})$ in the symmetry-broken phase $(\overline{\eta} \neq 0)$ is obtained from the condition of thermodynamic equilibrium

$$\frac{\partial \Omega(\overline{\eta}, T)}{\partial \overline{\eta}} = 0 \quad \text{wherefrom} \quad \overline{\eta} = \sqrt{-\frac{a(T - T_c)}{b}}, \tag{4}$$

valid for $T_c - T > 0$. For $T_c - T < 0$, the order parameter is 0. Dynamics of the order parameter close to equilibrium is given by the modified *Landau–Khalatnikov equation* [13]. To this effect one needs to add an 'inertial' term to the expansion equation (3) of the form

$$\Omega(\eta, \dot{\eta}, T) = \Omega(0, T) + \tfrac{1}{2}\, a\, (T - T_c)\, \eta^2 + \tfrac{1}{4}\, b\, \eta^4 + \tfrac{1}{2}\, A\, \dot{\eta}^2, \tag{5}$$

with the dot indicating derivative with respect to time t and where the constant A quantifies the 'inertial' effects. There exist many phase transitions that can be described by this type of theory, most notably the ferroelectric phase transitions of displacive type, ferromagnetic transitions with precession of the magnetic moment or many of the transitions found in liquid crystals [7, 13, 14, 16]. The dynamics close to equilibrium is then guided by the Landau–Khalatnikov equation of the system [13]:

$$\frac{\partial \eta}{\partial t} - \chi = \frac{\partial \Omega(\eta, \dot{\eta}, T)}{\partial \eta} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \Omega(\eta, \dot{\eta}, T)}{\partial \dot{\eta}}\right). \tag{6}$$

Written explicitly, considering the free energy close to the transition equation (3), the equation above becomes

$$A\ddot{\eta} + \chi\dot{\eta} + a\ (T - T_c)\eta + b\eta^3 = 0. \tag{7}$$

Here χ is a phenomenological constant characterizing the dynamic response of the system. To find the dynamical relaxation of the order parameter displaced from equilibrium, the instantaneous order parameter is written as $\eta(t) = \overline{\eta} + \delta\eta(t)$, leading to

$$A\delta\ddot{\eta}(t) + \chi\delta\dot{\eta}(t) + a(T - T_c)\delta\eta(t) + 3b\overline{\eta}^2\delta\eta(t) = 0 \tag{8}$$

which has the form of a damped oscillator equation of motion.

The dynamics close to the phase transition is usually investigated in two extreme cases. In soft matter physics, the systems are strongly overdamped and the inertial part of equation (7) characterized by A can be neglected. The deviation of the order parameter relaxes exponentially to equilibrium with the relaxation rate

$$\begin{aligned} \frac{1}{\tau} &= \frac{a}{\chi}\ (T - T_c) \quad \text{for} \quad T_c < T \\ \frac{1}{\tau} &= \frac{a}{\chi}\left((T - T_c) + 3\ \frac{b}{a}\overline{\eta}^2\right) = 2\frac{a}{\chi}(T_c - T) \quad \text{for} \quad T_c > T, \end{aligned} \tag{9}$$

which approaches zero at the transition temperature. The system approaching the phase transition therefore relaxes with longer and longer times once it is perturbed. On the other hand, when the system is not significantly damped, the kinetic term overwhelms the damping, and the linear term in the derivative of the order parameter can be neglected. The dynamics of the order parameter is in this case oscillatory and the square of the relaxation frequency depends in the same way on the temperature as the relaxation rate:

$$\begin{aligned} \omega^2 &= \frac{a}{A}(T - T_c) \quad \text{for} \quad T_c < T \\ \omega^2 &= \frac{a}{A}\left((T - T_c) + 3\frac{b}{a}\overline{\eta}^2\right) = 2\frac{a}{A}(T_c - T) \quad \text{for} \quad T_c > T. \end{aligned} \tag{10}$$

Obviously the inverse relaxation time and the oscillation time would vanish at the critical temperature in both cases. This behaviour is known as the *critical slowing down* of the 'soft mode' and is ubiquitous for second-order phase transitions [15] in general and in liquid crystals [16] in particular.

3. The shape of an elastic filament

We now list some of the salient features of the equilibrium properties of an elastic filament under point end load. The equilibrium shape of an anchored elastic filament confined to a two-dimensional plane with an external point load P acting at its free end in the vertical direction corresponds to the solution of the *Euler equation* obtained via minimization of the elastic energy equation (1) in the form [18]

$$EI\ \frac{\mathrm{d}^2\theta(s)}{\mathrm{d}s^2} - P\sin\theta(s) = 0. \tag{11}$$

This equation can be rewritten as

$$\frac{\mathrm{d}^2\theta(s)}{\mathrm{d}s^2} + q^2\sin\theta(s) = 0, \qquad q^2 = \frac{PL^2}{EI}, \tag{12}$$

where $s = \ell/L$. The differential equation is second order, and its solution depends on two integration constants, A and κ, that are set by the boundary conditions. Explicitly, the solution is given by *Jacobi elliptic functions* [19] in the form

$$\theta(s) = 2\arcsin\left(\kappa \,\mathrm{sn}\left[\kappa, qs + A\right]\right). \tag{13}$$

Equation 12 is related to the sine-Gordon equation of field theory, as well as to the Poisson–Boltzmann equation of statistical mechanics [20].

The two boundary conditions are trivial: the filament is anchored at $s = 0$ and is free at $s = 1$, i.e. $\theta(0) = 0$ and $\frac{\mathrm{d}\theta(1)}{\mathrm{d}s} = 0$. The first condition states that the deformation at the anchored end is zero, and the second says that there is no moment of force at the free end of the filament. We now write the solution in terms of the coordinates of the free end of the filament, $x(s = 1)$ and $y(s = 1)$, given by

$$\begin{aligned} x(s=1) = x_L &= \int_0^L \sin\theta(\ell)\,\mathrm{d}\ell = L\frac{2\kappa}{q} \\ y(s=1) = y_L &= \int_0^L \cos\theta(\ell)\,\mathrm{d}\ell = L\left(-1 + \frac{2}{q}(E\left[\kappa\right] - E\left[\kappa, 0\right])\right). \end{aligned} \tag{14}$$

where the indicated integrals have been evaluated in terms of the associated elliptic functions. The two boundary conditions for the free and the anchored end can be obtained as

$$0 = \mathrm{sn}(\kappa, A) \qquad \text{and} \qquad 0 = \mathrm{cn}(\kappa, qL + A). \tag{15}$$

From here, by taking note of the connections between the different elliptic functions [19], we deduce that

$$q = K[\kappa], \tag{16}$$

with $0 \leqslant \kappa \leqslant 1$. In general $K[\kappa] \geqslant \frac{\pi}{2}$ for all κ, which implies that

$$P \geqslant \frac{\pi^2 EI}{4L^2} = P_c. \tag{17}$$

From this we conclude that a non-trivial solution to the Euler equation exists only if the end-load is greater than some critical value. This is, of course, the essence of the *Euler buckling instability*. From the above solution, we can also derive the corresponding elastic free energy of the filament as

$$\begin{aligned} F_e(\kappa, q) &= \frac{1}{2}EI\int_0^L \left(\frac{\mathrm{d}\theta(\ell)}{\mathrm{d}\ell}\right)^2 \mathrm{d}\ell \\ &= \frac{2EI}{L}q\left((\kappa^2 - 1)q + E[\kappa] - E\left[\kappa, 0\right]\right). \end{aligned} \tag{18}$$

Usually this would be problem solved. However, in our case we need to express the elastic energy in terms of the coordinates of the free end of the filament via equations (14). We can then finally write the elastic energy in the form $F_e(x_L, y_L)$. It is this expression that we will use in what follows.

4. Equations of motion of an elastic filament

A weight with a mass M is now used as a point end load on a Euler strut. We assume that it is much larger than the mass of the strut itself, which thus by assumption does not contribute any inertia to the combined motion of the mass and the filament. In this case, the kinetic and potential (gravitational) energies of the filament itself can be neglected. This problem has had many reincarnations in the literature, but it has not been used as a pedagogical model of phase transition dynamics[7].

We write the *Lagrange function* of the filament loaded at its free end as the difference between the kinetic and potential energies of the system

$$\begin{aligned}\mathcal{L} &= T(\dot{x_L}, \dot{y_L}) - V(x_L, x_L) \\ &= \tfrac{1}{2}M(\dot{x_L}^2 + \dot{y_L}^2) - Mgy_L - F_e(x_L, y_L),\end{aligned} \tag{19}$$

where $V(x_L, x_L)$ is the total mechanical energy of the system. For economy of writing, from now on we will write (x, y) instead of (x_L, y_L). The Lagrangian of this system is thus formulated in terms of the position of the free end of the filament and its velocity. The corresponding equations of motion follow directly from this Lagrangian and describe the motion of the free end.

Because in general $F_e(x, y)$ is a nonlinear function, the equations of motion for the above Lagrangian cannot be solved explicitly. To gain some intuition about the solutions, we will first describe the limit of small-amplitude motion around the straight configuration and then expand the solution by using appropriate numerical methods.

5. Euler strut dynamics and order-parameter dynamics

We now prove the equivalence of the mechanical model and general order-parameter dynamics close to a second-order phase transition. The free energy of the general thermodynamic system is given by equation (3) and the equilibrium order parameter is given in equation (4). If the system is displaced or thermodynamically excited away from equilibrium, the pseudo-forces appear which return the system towards the equilibrium. For strongly underdamped systems, the motion is harmonic and the characteristic frequency is temperature dependent; see equation (10).

Can one describe the behaviour of the filament in a similar way so that it could serve as a model for a second-order phase transition? If the filament model is adequate, the elastic energy as a function of the free-end deviation from the straight line has to have the form of equation (3) and the free-end deviation has to play the role of the order parameter with an equilibrium value of the form of equation (4). If the external load for the Euler strut would play the role of the temperature, the analogy would be complete.

First of all the elastic energy $V(x, y)$ in equation (19) can be expanded with respect to small deviations in x, y up to the fourth order as

$$V(x, y) = \frac{1}{2}\,\frac{\pi^2}{8L}\,(P_c - Mg)\;x^2 + \frac{1}{4}\,3\pi\,\left(\frac{\pi}{8L}\right)^3\,P_c\,x^4 + \cdots. \tag{20}$$

Obviously the form of the elastic energy is the same as the free energy in equation (3), if a and b have the form

$$a = \frac{\pi^2}{8L}\,(P_c - Mg) \qquad \text{and} \qquad b = 3\pi\,\left(\frac{\pi}{8L}\right)^3\,P_c.$$

One can therefore find the equilibrium solution $\overline{x}$ of the energy minimization very close to the critical end load as

$$\begin{aligned}&\overline{x} = 0 \qquad \text{for} \qquad P_c > Mg \\ &\overline{x} = \frac{8L}{\pi\sqrt{3}}\sqrt{\left(\frac{Mg}{P_c} - 1\right)} \qquad \text{for} \qquad P_c < Mg.\end{aligned} \tag{21}$$

[7] According to Virgin *op. cit.*, Sommerfeld was the first to investigate the dynamics of the Eulerian strut and noticed that the eigenfrequencies of the strut differ markedly if the strut is directed up or down. A formula for the lowest eigenfrequency of the Euler strut is given also in Pippard *op. cit.* [8].

The analogy is thus complete. The horizontal end-deviation of the filament x is analogous to the order parameter and the external end load Mg to the temperature, with the critical buckling end load P_c standing for the critical temperature. Furthermore, it follows that the corresponding static critical exponent has the mean-field value of $1/2$ just as in the case of a second-order transition.

As the Landau free energy equation (3) has the same form as the mechanical energy equation (20) and since the mechanical system is strongly underdamped, the oscillation frequencies are obtained straightforwardly as

$$\begin{aligned} \omega^2 &= \frac{\pi^2}{8L}\frac{P_c}{}\left(1-\frac{Mg}{P_c}\right) \qquad \text{for} \qquad P_c > Mg \\ \omega^2 &= 2\,\frac{\pi^2 P_c}{8L}\left(\frac{Mg}{P_c}-1\right) \qquad \text{for} \qquad P_c < Mg. \end{aligned} \tag{22}$$

We can read off the above two equations that again the corresponding dynamical critical exponent for the Euler strut is equal to the mean-field prediction.

As an additional verification, we solved the exact equations of motion of the mechanical system numerically. From equation (19) we can derive an exact system of differential equations, which describes the movement of the weight on the free end of the filament:

$$\ddot{x} = -\chi\frac{\partial\tilde{F}}{\partial x} \qquad \ddot{y} = -\frac{g}{L} - \chi\frac{\partial\tilde{F}}{\partial y}, \tag{23}$$

where $\tilde{F} = F_e(x, y)L/EI$ is the elastic free energy and $\chi = \frac{4P_c g}{\pi^2 L M g}$. Though the equations of motion can be formulated straightforwardly, their numerical solution is non-trivial due to the complicated nature of the elastic energy function. Numerical algorithm has to be chosen appropriately, and we have implemented it on the level of the finite element method (*FEM*) using the software *COMSOL* [21].

In the numerical implementation, we chose the length of a square filament $L = 3.2$ m, width $d = 0.1$ m, Young's modulus $E = 210.0$ GPa, Poisson's ratio $\nu = 0$, and gravitational acceleration $g = 9.81$ m s^{-2}. For these values, we obtain the critical load $P_c = 4.217 \times 10^5$ N. Figure 2 shows how the frequency depends on the ratio Mg/P_c (numerical error is estimated to be lower than 1%, which we can safely neglect). For the calculated dependence of the frequency on this ratio, we found that the best fit is of the form $\omega = A\sqrt{\frac{g}{L}}\sqrt{\frac{P_c}{Mg} - 1}$ for $\frac{Mg}{P_c} < 1$, which was also the form obtained exactly in the limit of small oscillations. For the fitting parameter A, we obtain $A = 1.1049$, which is very close to the analytical solution $\pi/\sqrt{8} \approx 1.1107$ (compare with equation (22)). The relative error is only 0.5%.

6. Experimental demonstration

In the selection of materials to measure the filament's dynamics close to its buckling transition, one has to be aware that readily available filaments are made of various plastics such as optical fibres and various rubbers which are quite stiff, leading to rather high oscillation frequencies and therefore more sophisticated methods of measurement. On the other hand, the rubber filaments showed a strong damping that resulted in only a few oscillations before the system came to rest. It turned out that the best choice is a simple, rather long solenoid spring (between 10 and 20 cm long and around 0.5 cm in diameter), which is not prestrained and therefore allows for compression as well as extension. The spring is anchored at one end in the holder, with the deformable part short enough so that the stable unloaded filament shape is straight.

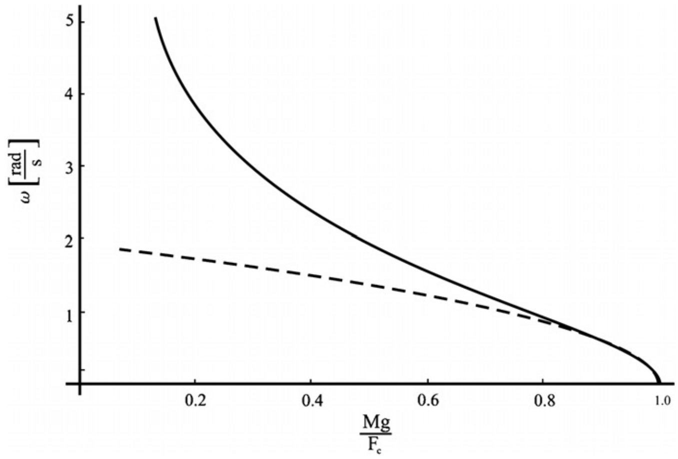

Figure 2. Frequency versus the ratio of the weight Mg of a point attached external load and the critical load P_c. The solid curve represents the numerical calculations, while the dashed curve is a best-fit function of the form $A\sqrt{\frac{g}{L}}\sqrt{\frac{P_c}{Mg}-1}$ with $A = 1.1049$. Close to the critical point, the theoretical prediction is very close to the numerical solution—the values differ by less than 0.5%.

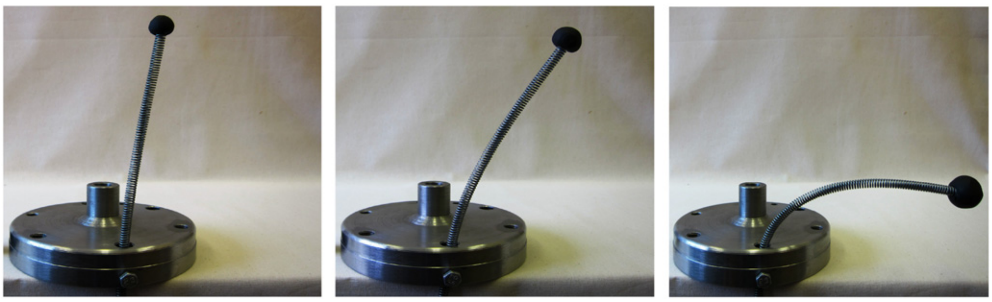

Figure 3. The solenoid spring end loaded with various plasticine loads. The shape of the deformed spring conforms nicely to the shapes of end-loaded filaments described by the solution of Euler's equation (13).

(This figure is in colour only in the electronic version)

The point load is added by pressing and fixing a plasticine ball to the free end. The variation of the load is obtained by adding or taking away some of the plasticine. Figure 3 shows the shapes of the spring when different masses were attached to its free end.

As an experiment for demonstration in the lecture room, we prepared a few equal springs with different loads fixed on the same supporting board. It is also important to prevent the rotation of the filament, which presents a degenerated shape of the filament when load exceeds the critical load. Two vertical sheets from thicker transparent foils sandwiching the spring allow for enough support to remove this unwanted degeneracy. The dynamics of the filament can than easily be followed by applying a sudden displacement. The periods of such Euler struts are on the order of a few per second, and the larger periods close to the critical points are evident even in the simplest observation.

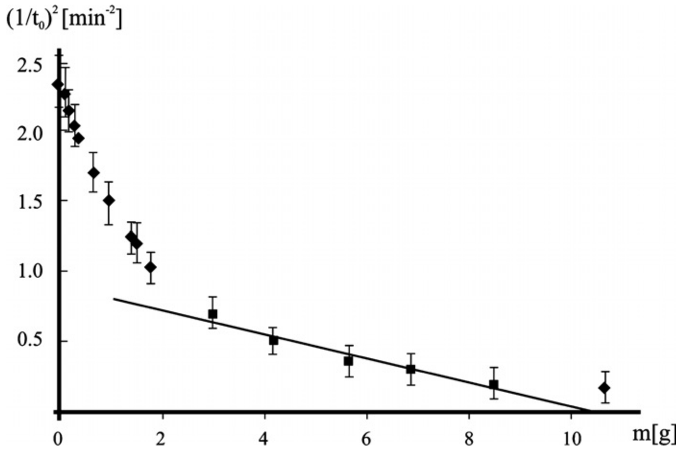

Figure 4. The measured inverse square of the frequency as a function of the spring end load. The linear fit close to the critical mass is added. The finite frequency at zero load appears to be due to the finite mass of the spring itself.

As laboratory or project work, students can measure the onset of Euler buckling instability as well as the period of oscillation for different plasticine loads more precisely. An example of frequency measurements as a function of the end loads expressed in grams for the spring is given in figure 4. The dependence of the frequency on the end load is clearly seen, and the square of the inverse oscillation time is proportional to the difference $(m - m_c)$, where m_c is the mass of the critical load that can be measured in a separate experiment, figure 4. The discrepancy with respect to the lowest order theoretical expression equation (22) is also evident, see figure 2. There is no divergence in the measurement of the frequency when the load goes to zero. In theoretical expressions the filament was considered massless; however, this is not true for the experiment leading to the finite frequency at the zero load corresponding to the inertia of the whole filament.

7. Conclusions

We presented a mechanical model which allows the observation of the second-order phase transition as well as critical dynamics of the order parameter close to this transition. The Euler strut mechanical model itself is not new and has been used in different engineering contexts [8]. It is easy to implement it experimentally and allows for simple measurements which clearly show the analogy with the critical behaviour of the typical second-order transition. As students usually have more experience in mechanics, such a model intuitively supports the abstract concepts such as critical slowing down related to phase transitions and dynamics close to phase transitions in very diverse contexts [7].

We studied the statical properties and dynamics of a Euler strut and investigated the bifurcation in the nature of its motion that appears when the load on the free end of the pendulum is varied. In the vicinity of the critical value of the external end load that corresponds to the Euler buckling transition, we calculated the corresponding critical exponent of the order parameter being identical to its mean-field value as predicted by the Landau theory. In addition, the dynamics of the Euler strut shows a behaviour analogous to the 'soft mode' dynamics in the vicinity of a second-order phase transition. When the load at the free end approaches the critical value, the oscillation period of the pendulum diverges. The critical exponent of this divergence was calculated analytically in the limit of small deviations as well as numerically. The results in the last case completely corroborate the approximate analytical theory.

We showed that the mechanical energy of the Euler strut close to the buckling instability can be expressed in a power series of the end deviation and is analogous to the Landau free energy of the thermodynamic system which exhibits a second-order phase transition. The parallel between the order parameter of the phase transition and the end deviation of the Euler strut is straightforward. The role of decreasing temperature in the thermodynamic systems is played by the mass of the end load. The behaviour of the oscillation frequency around the equilibrium position as a function of the end load close to its critical value, which follows from the small deviation approximation of the elastic energy, is completely analogous to the dependence observed in the critical slowing down close to the phase transition temperature.

The Euler strut and its dynamics represent a well-defined mechanical system that allows for simple practical demonstrations and can thus be used in order to explain in a straightforward and intuitive way a complicated and advanced concept such as critical slowing down that appears in a plethora of contexts [7]. One only needs a sufficiently stiff, not prestrained spring and a series of end loads. Fixing the spring into a solid support, adding the load and setting it in motion gives a clear demonstration of the dynamical properties of the system in the vicinity of a second-order phase transition. Apart from being revealing from a mechanical standpoint, this experiment has also an aesthetically pleasing quality as witnessed by the various forms of kinetic art where it plays a prominent role [17].

In this paper we assumed that the mass of the filament itself is much smaller than the mass of the attached weight. Very interesting phenomena can be observed also in the opposite limit of a clamped filament without any end load and have been elucidated only recently [22]. An interesting extension could be the introduction of torsion and a full 3D analysis of motion which based on our experiments we expect to be in general quite chaotic.

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