University of Ljubljana
Faculty of Mathematics and Physics

Seminar 4. letnik

Phase transition in an Ising economy

Author:
Miloš Borovšak

Mentor:
prof. dr. Rudolf Podgornik

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Abstract

In this seminar it is described Ising model of trading economy.

Contents

1 Introduction

Widely observed astonishing facts in economy is stability of the value of money. Economic models describe this because of a central agent or market maker who supervises the global dynamics and enforces market clearance. If we take a look on a real market, one can see that markets already function solely on the basis of the interaction between trading agents. Simple numerical models of exchange goods between agents as well as experimental studies support the scenario that the value of money appears as a dynamical variable that results from the dynamics of trading itself. We will take a look on behaviour of system (money and goods flows) of trading agents if we put them on a straight line. After that we will improve our model with a higher dimension and hierarchies between traders. We will take a look on a trading on a hierarchical network which allows us to include the interesting aspect of hierarchy in the monetary business. For explicit solution we will reformulate our model in terms of Ising type spin model.

2 Ising model in physics

Ising model was invented by physist William Lenz who gave it as a problem to his student Ernst Ising after whom it is named. The model consists of discrete variables called spins that can be in one of two states. The spins are arranged in a lattice or graph, and each spin interacts only with its nearest neighbors.
In feroelectric substances is so-called exchange interaction between neighbours atoms so strong that it comes to spontaneous arrangement of magnet moments. Lets take a look on a system of atoms with spin $s = \frac{1}{2}$. In addition to magnetic energy we get also the exchange interaction

$$H = -\sum_i (\gamma \hbar \vec{s}_i \cdot \vec{\mathcal{B}} + J \sum_{j=1}^{n.n.} \vec{s}_i \cdot \vec{s}_j) \tag{1}$$

where J is strength of exchange interaction and $n.n.$ is the number of the nearest neighbours.
For simplified description let use Ising model. We can calculate exact analitical

solution only for one- or two-dimensional case. While the one-dimensional Ising model is relatively easy to solve the two-dimensional Ising model is highly non-trivial. Lars Onsager found an analytical solution to the two-dimensional Ising model. To date, the three-dimensional Ising model remains unsolved.
Let us now take a look on a two dimensional case. Consider a two-dimension spin-lattice as shown on a Figure 1.

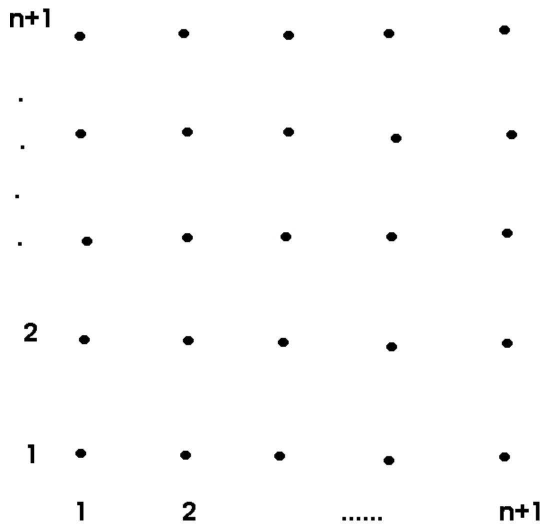

Figure 1: Two dimensional spin lattice

The Hamiltonian can be written as

$$H = -J \sum_{i,j} (\sigma_{i,j}\sigma_{i+1,j} + \sigma_{i,j+1}\sigma_{i,j}) - h \sum_{i,j} \sigma_{i,j} \tag{2}$$

where the spins are now indexed by two indices corresponding to a point on the 2-dimensional lattice. Introduce a shorthand notation for H

$$H = \sum_{j=1}^{n} [E(\mu_j, \mu_{j+1}) + E(\mu_j)] \tag{3}$$

where

$$E(\mu_j, \mu_k) = -\sum_{i=1}^{n} \sigma_{i,j}\sigma_{i,k} \tag{4}$$

$$E(\mu_j) = -J \sum_{i=1}^{n} \sigma_{i,j}\sigma_{i+1,j} - h \sum_{i,j} \sigma_{i,j} \tag{5}$$

and μ_j is defined to be a set of spins in a particular column

$$\mu_j = \sigma_{1,j}, \cdots, \sigma_{nj}. \tag{6}$$

Define now a transfer matrix P, with matrix elements

$$\langle \mu_j | P | \mu_k \rangle = e^{-\beta[E(\mu_j, \mu_k) + E(\mu_j)]} \tag{7}$$

which is a $2^n \times 2^n$ matrix. The partition function will be given by

$$\Delta = Tr(P^n) \tag{8}$$

and we are looking for the largest eigenvalue of P.
In the thermodynamic limit, the final result at zero field is

$$g(T) = -kTln[2\cosh 2\beta J] - \frac{kT}{2\pi}\int_0^\pi d\phi ln\frac{1}{2}(1+\sqrt{1-K^2\sin^2\phi}) \tag{9}$$

where

$$K = \frac{2}{\cosh 2\beta J \coth 2\beta J}. \tag{10}$$

The energy per spin is

$$\eta(T) = -2J\tanh 2\beta J + \frac{K}{2\pi}\frac{dK}{d\beta}\int_0^\pi d\phi \frac{\sin^2\phi}{\Delta(1+\Delta)} \tag{11}$$

where

$$\Delta = \sqrt{1-K^2\sin^2\phi}. \tag{12}$$

The magnetization, then, becomes

$$m = 1 - [\sinh 2\beta J]^{-4\frac{1}{8}} \tag{13}$$

for $T < T_c$ and 0 for $T > T_c$, indicating the presence of an order-disorder phase transition at zero field. The condition for determing the critical temperature at which the phase transition occurs turns out to be

$$2\tanh^2 2\beta J = 1 \tag{14}$$

and from here

$$kT_c \approx 2.269185J. \tag{15}$$

3 One-dimensional problem of money dynamics

In our model, we consider N agents, $n = 1, 2, \cdots, N$, placed on a one-dimensional lattice with periodic boundary conditions. This geometry is choosen in order to have a simple and specific way of defining who is interacting with whom. We assume that agents cannot consume their own output, so in order to consume they have to trade, and in order to trade they need to produce. Each agent produces a quantity q_n, of one good, which is sold at a unit price p_n, to his left neighbour $n-1$. He next buys and consumes one good from his neighbour to the right, who subsequently buys the good of his right neighbour, etc., until all agents have made two transactions. All agents are given utility function of the same form

$$u_n = -c_n(q_n) + d(q_{n+1}) + I_n(p_n q_n + p_{n+1}q_{n+1}). \tag{16}$$

The first term, $-c$, represents the agent's cost, or displeasure, associated with producing q_n units of the good he produces. The displeasure is an increasing function of q, and c is convex. The second term d, is his utility of the good he can obtain from his neighbour. Its marginal utility is decreasing with q, so d is concave.

An explicit example is choosen for illustration and analysis

$$c(q_n) = aq_n^\alpha \quad d(q_{n+1} = bq_{n+1}^\beta) \tag{17}$$

where values a, b, α and β are not important for general result, as long as c remains convex and d concave so we can choose $a = \frac{1}{2}$, $b = 2$, $\alpha = 2$ and $\beta = \frac{1}{2}$. Each agent has knowledge only about utility functions of his two neighbours, as they appeared on previous transaction. The agents are monopolic, i.e., agent n sets the price of his good, and agent $n-1$ then deciede how much q_n he will buy at that price. The goal of each agent is to maximize his utility by adjusting p_n and q_{n+1} while maintaining a constant amount of money. There is no point in keeping money, all they needed is what it takes to complete the transaction. The agents aim to achieve a situation where the expenditures are balanced by the income

$$p_n q_n - p_{n+1} q_{n+1} = 0 \tag{18}$$

When the value of money is fixed $I_n = I$, the agents optimize their utility by charging a price

$$p = 2^{\frac{1}{3}} I^{-1} \tag{19}$$

and selling an amount

$$q = 2^{-\frac{2}{3}} \tag{20}$$

at that price. This is the monopolistic equilibrium. The resulting quantities q, are independent of the value of money. Agent n tries to achieve his goal by estimating the amount og goods q_n, that his neighbour will order at a given price, and the price p_{n+1}, that his other neighbour will change at the subsequent transaction. Knowing that his neighbours are rational beeings like himself, he is able to deduce the functional relationship between the price p_n that he demanda and the amount of goods q_n that will be ordered in response to this. He is also able to estimate the size of p_{n+1} based on previous transaction with his right neighbour. This enalbes him to decide what the percived value of money should be, and hence how much he should buy and his price should be. This process is then continued indefinitely, at times $\tau = 1, 2, \cdots$. The strategy we investigate contains the assumption that agents do not change their valuation of money I between their two transactions, and they maximize their utility accordingly. The process is initiated by choosing some initial values for the I. That could be related to some former gold standard. In fixing his price at his first transaction at time τ agent n exploits the knowledge he has of his neighbours' utility function, i.e., he knows that the agent to the left will maximize his function with respect to $q_{n,\tau}$

$$\frac{\partial u_{n-1,\tau}}{\partial q_{n,t}} = 0 \tag{21}$$

hence the left neighbour will order the amount

$$q_{n,\tau} = (I_{n-1,\tau} p_{n,\tau})^{-2}. \tag{22}$$

This functional relationship between the amount of goods $q_{n,\tau}$, ordered by agent $(n-1)$ at time τ and the price $p_{n,\tau}$, set by agent n, allows agent n to gauge the effect of his price policy. Lacking knowledge about the value $I_{n-1,\tau}$, agent

n instead estimated it to equal the value it had in the previous transaction $I_{n-q,\tau-1}$, which he knows. Eliminating $q_{n,\tau}$ from equation (16) we obtain

$$u_{n,\tau} = -\frac{1}{2}(I_{n-1,\tau}^{-4} p_{n,\tau}^{-4} + 2\sqrt{q_{n+1,\tau}} + I_{n,\tau}(p_{n,\tau}^{-1} I_{n-1,\tau-1}^{-2} - p_{n+1,\tau} q_{n+1,\tau}). \quad (23)$$

Maximizaing this utility $u_{n,\tau}$ with respect to $p_{n,\tau}$ and $q_{n+1,\tau}$ yields

$$p_{n,\tau} = 2^{\frac{1}{3}} I_{n,\tau}^{-\frac{1}{3}} I_{n-1,\tau-1}^{-\frac{2}{3}} \quad (24)$$

and

$$q_{n+1,\tau} = (I_{n,\tau} p_{n+1,\tau})^{-2}. \quad (25)$$

By argument of symmetry

$$p_{n+1,\tau} = 2^{\frac{1}{3}} I_{n+1,\tau}^{-\frac{1}{3}} I_{n,\tau-1}^{-\frac{2}{3}} \quad (26)$$

is the price agent $(n + 1)$ will demand of agent n in the second transaction. Since agent n does no yet know the value of $I_{n+1,\tau}$ he instead uses known value of $I_{n+1,\tau-1}$ when estimating $p_{n+1,\tau}$. In the constraint (18) the following expressions are used

$$q_{n,\tau} = q_{n,\tau}^{guess} = (I_{n-1,\tau-1} p_{n,\tau})^{-2}. \quad (27)$$

$$p_{n+1,\tau} = p_{n+1,\tau}^{guess} = 2^{\frac{1}{3}} I_{n+1,\tau-1}^{-\frac{1}{3}} I_{n,\tau-1}^{-\frac{2}{3}} \quad (28)$$

$$q_{n+1,\tau} = q_{n+1,\tau}^{guess} = (I_{n,\tau} p_{n+1,\tau}^{guess})^{-2}. \quad (29)$$

and p_n is given by (24). Solving for $I_{n,\tau}$ and evaluating at time $\tau + 1$ we find (23)

$$I_{n,\tau+1} = (I_{n-1,\tau}^4 I_{n,\tau}^2 I_{n+1,\tau})^{\frac{1}{7}} \quad (30)$$

which sets agent n's value of money at $\tau + 1$ equal to a weighted geometic average of the value agent n and his two neighbours prescribed to their money the previous day. Using this value of I_n, agent n can fix his price p_n and decide which quantity q_{n+1} he should optimally buy. This simple equation completely specifies the dynamics of our model. The entire strategy can be reduced to an update scheme involving only the value of money - everything else follows from this. Thus, the value of money can be considered the basic strategic variable. Although (30), has been derived for a specific simple example, we submit that the structure is much more general. In order to optimize his utility function, the agent is forced to accept a value of money, and hence prices, which pertain to his economic neighbourhood. Referring again to a situation from physics, the position of an atom on a general lattice is restricted by the positions of its neighbours, despite the fact that the entire lattice can be shifted with no physical consequences. Even though there is no utility in the possession of money, as explicitly expressed by (18), the strategies and dynamics of the model nevertheless leads to a value being ascribed to the money. The dynamics in this model is driven by the need of the agents to make estimates about the coming transactions. In a sense, this models the real world where agents are forced to make plans about the future, based on knowledge about the past and, in practise, only a very limited part of the past. In short: the dynamics is generated by the bounded rationality of the agents. In the steady state, where the homogeneity of the utility functions give $I_n = I_{n+1}$, we retrieve the monopolistic equilibrium equations (19) and (20).

4 A network trading model

If we want to describe our economy model that will be similar to Ising model we need first intoduce the basic trading model on a network with dimension greater than one. Let us consider a model where an agent N sells goods whish are traded via $N-1$ intermediary agents to customers at level $n=0$. We call this selling mode. The goods are returned by a second chain where agent N buys goods, which are traded via $N-1$ different intermediaries from $n=0$ (buying mode). Combining both, selling and buying chain, we obtain the circular geometry from one-dimensional problem we were solving before. Let use now more general scenario. In selling or buying mode each agent can sell to or buy from $z-1$ agents. We replace linear chains ($z=2$) with so-called Cayley tree with z neighbors shown on Figure 5 for $z=3$.The agents are located at the sites or nodes, while goods and money flow along the links of the tree. The agents (n,i) are indexed by the distance n from the right-hand side of the tree. The index i distinguishes different agents at the same distance and will be used only if necessary.

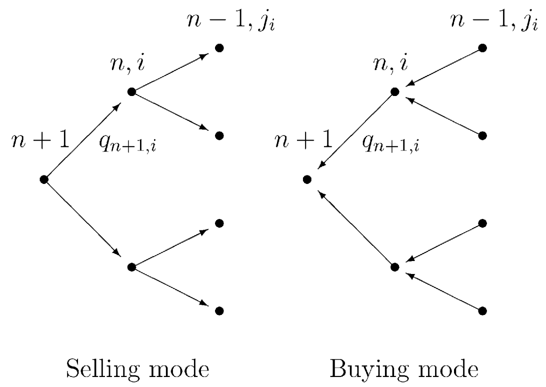

Figure 2: Selling and buying mode on a Cayley tree ($z=3$)

For the amount of goods $q_{n,i}$ owing between agents n and $(n-1,i)$ we use the normalized variable

$$\bar{q}_{n,i} = \frac{1}{(z-1)^{n-1}} q_{n,i}. \tag{31}$$

Define now economic term we were just talking about - utililty function u_n. Utility is a concept referring to the precise degree of personal satisfaction, pleasure, or sense of want-fulfillment an individual derives from consuming some quantity of a good or service at a particular point in time.

The amount of trading goods is described by two utility functions. If agent n sells q at price p he gains the utility

$$u_n^S = I_n \bar{q} p - \tilde{c}(\bar{q}) \tag{32}$$

where I_n denotes the value of money and $\tilde{c}(\bar{q})$ the decrease of utility by losing q.

Similary, if the agent buys q at price p the utility reads

$$u_n^B = d(\bar{q}) - I_n \bar{q} p. \tag{33}$$

It is important to use the normalized flow of goods (31) instead of q for the following reason. In the monopolistic equilibrium all the money values I_n are the same and all goods are conserved. Therefore, the goods q_n increase with $(z-1)^{n-1}$. Then, utilities (32) and (33) express the assumption that an agent level of n gets the same utility by trading $q_n = (z-1)q_{n-1}$ as the agents at level $n-1$ trading q_{n-1}.

Each agent n can choose its own money value I_n , the amount of bought goods q_{n+1} and the price $p_{n,i}$ for sold goods $q_{n,i}$. We must also use the assumption that the time scale on which the q and p change is much shorter than the scale of changing the value of money. Therefore we can optimize the coupled system (32) and (33) with fixed value of I_n. For the utility function we must define $\tilde{c}$ and d. It is sufficient for $\tilde{c}$ to increase faster than, and for d less than linearly for large $\bar{q}$. To avoid algebraic complications, we here use for d a power law

$$d(x) = \frac{1}{\beta} x^\beta \quad \beta < 1 \tag{34}$$

and for $\tilde{c}$ similar

$$\tilde{c}(x) = \frac{1}{\alpha} x^\alpha \quad \alpha > 1. \tag{35}$$

In general case $\tilde{c}$ must have a positive first and second derivate.

Having performed the optimization all quantities can be expressed by the Legendre transformation of $\tilde{c}(s^{1/\beta})$ denoted by $c(r)$. For the power law (35) we get

$$c(r) = \frac{\alpha - \beta}{\alpha\beta} (\beta r)^{\alpha/(\alpha-\beta)}. \tag{36}$$

The optimization is slightly different in the buying or selling mode. In the latter we have for each agent n

$$u_n^B = \frac{1}{\beta} \bar{q}_{n+1}^\beta - I_n \bar{q}_{n+1} p_{n+1} \quad n = 0, \cdots, N-1, \tag{37}$$

$$u_n^S = I_n \sum_i [\bar{q}_{ni} p_{ni} - \tilde{c}(\bar{q}_{ni})] \quad n = 0, \cdots, N-1. \tag{38}$$

The optimization of selling mode begins at $n = 0$, where only u_n^B is present. The maximum (37) leads to value $\bar{q}_1$. This value $\bar{q}_{1i}(p_{1i})$ is used to optimize p_{1i} and u_1^S given by (38). This procedure is repeated to the top agent N. The resulting values of traded goods and money flow $qg_{n,i}$ from $(n-1, i)$ to n

$$g_{n,i} = q_{n,i} p_{n,i} \tag{39}$$

are given by

$$q_{n,i} = (z-1)^{n-1} [c'(\frac{I_n}{I_{n-1,i}})]^{1/\beta}, \tag{40}$$

$$g_{n,i} = (z-1)^{n-1}\frac{1}{I_{n-1,i}}c'(\frac{I_n}{I_{n-1,i}}). \quad (41)$$

We can see that the flow of goods and the valued money flow $I_{n,i}g_{n,i}$ only depend on the ratio I_n/I_{n-1}, but not on the absolute scale of I. The value of utilities in (37) and (38) at the maximum are given by

$$u_n^B = \frac{1-\beta}{\beta}c'(\frac{I_{n+1}}{I_n}) \quad (42)$$

and

$$u_n^S = \sum_{i=1}^{z-1} c(\frac{I_n}{I_{n-1,i}}). \quad (43)$$

The buying mode can be treated with the same method. It can be obtained from the selling mode by interchanging at each link $(n; n-1, i)$ the adjecent I_n and I_{n-1}, what leads to

$$q_{n,i} = (z-1)^{n-1}[c'(\frac{I_{n-1,i}}{I_n})]^{1/\beta}, \quad (44)$$

$$g_{n,i} = (z-1)^{n-1}\frac{1}{I_n}c'(\frac{I_{n-1,i}}{I_n}) \quad (45)$$

and the utility function

$$u_n^B = \frac{1-\beta}{\beta}\sum_{i=1}^{z-1} c'(\frac{I_{n-1,i}}{I_n}) \quad (46)$$

$$u_n^S = c(\frac{I_n}{I_{n+1}}). \quad (47)$$

In the case of the linear chain $z = 2$, the only difference between selling and buying is a reordering of I by placing the agents on a circle. We can consider the normalized money ratio at site n given by (in the selling mode)

$$\Delta g(n+1; n, i) = (z-1)\frac{g_{n-1,i}}{g_n} = \frac{I_n}{I_{n-1,i}}[c'(\frac{I_n}{I_{n-1,i}})/c'(\frac{I_{n+1}}{I_n})]. \quad (48)$$

Note that the function Δg is only a function of the ratios

$$r_{n,i} = \frac{I_{n+1}}{I_{n,i}}. \quad (49)$$

Further money conservation at agent n implies

$$\sum_{i=1}^{z-1} \Delta g(n+1; n, i) = z-1. \quad (50)$$

For $r > 1$ money is accumulated at agent n, while $r < 1$ means that money has to be borrowed. Therefore in the selling mode $r < 1$ implies an inflation, while values $r > 1$ imply deflation.

Let us explain at that place two economic terms: inflation and deflation. Inflation is a rise in the general level of prices of goods and services in an economy over a period of time. When the price level rises, each unit of currency buys

fewer goods and services; consequently, inflation is also an erosion in the purchasing power of money, a loss of real value in the internal medium of exchange and unit of account in the economy.
Deflation is a decrease in the general price level of goods and services. Deflation occurs when the annual inflation rate falls below zero percent (a negative inflation rate), resulting in an increase in the real value of money, allowing one to buy more goods with the same amount of money.
In the buying mode r is essentially replaced by $1/r$ such that Δg is given by

$$\Delta g(n+1;n,i) = \frac{I_{n+1}}{I_n}[c'(\frac{I_{n-1,i}}{I_n})/c'(\frac{I_n}{I_{n+1}})] \tag{51}$$

and reversed statments are true.
In the case of $z = 2$ the condition of money conservation has been applied. In each step of the update of I_n the condition

$$\Delta g = 1 \tag{52}$$

is imposed. The change of I_n results in new p, q values and the procedure is repeated until convergence to (52) is reached.

For $z > 2$ money conservation involves a sum of Δg over i. To fix the money flow $g_{n,i}$ to agent n, extra conditions are needed. Such a condition may result from the cooperation between agents $(n-1,i)$ connected to n. Suppose agent n sells the amount $q = \sum_i q_{n,i}$ which is bought by agents $(n-1,i)$. If they do not cooperate, one agent may choose its $I_{n-1,i}$ such that the sum is exhausted. Then the system will collapse into a linear chain. If they cooperate, they optimize their common utility

$$u^B = \sum_i \frac{1-\beta}{\beta}\bar{q}_{n,i}^{\beta} \tag{53}$$

as function of $I_{n-1,i}$ subject to the condition $\sum q_{n,i} = q$ since the $q_{n,i}$ are unique functions of the $I_{n-1,i}$. For $\beta < 1$ u^B has a maximum for equal $q_{n,i}$ which implies $I_{n-1,i}$ is independent of i. Therefore we have condition (52) also valid for $z > 2$. In terms of ratios r it reads

$$c'(r_n) = r_{n-1}c'(r_{n-1}). \tag{54}$$

This recursion formula for the ratio r_n exhibits the stable fixed point $r_n = 1$, since both $c'(1)$ and $c''(1)$ are positive. The value r_{N_1} is arbitrary. For power laws we can solve the recursion explicitly. r_n depends only on the ratio of the exponents

$$\gamma = \frac{\beta}{\alpha} \tag{55}$$

which can lead to relative elasticty of the utility function, and is given by

$$\log r_n = \gamma^{N-n-1}\log r_{N-1}. \tag{56}$$

The same method can be applied in the buying mode. Now r_0' can be choosen arbitrarily due to replacement $r_n \to 1/r_n$:

$$\log r_n' = \gamma^n \log r_0'. \tag{57}$$

Both (56) and (57) can be used to obtain I_n or I'_n for buying mode. In the selling mode, money is accumulated at agent N and the agents at $n = 0$ have to borrow money. In order to 'recycle' the money, one can connect $n = 0$ and N with a second tree in the buying mode where agents $n = 0$ sell other goods q over this second tree to agent N . From $I_0 = I_0$ and $I_N = I_N$ the constants r_{N-1} and r_0 can be eliminated with the result

$$I_n = I_0(\frac{I_N}{I_=})^{\gamma^{N-n}}, \tag{58}$$

$$I'_n = I_N(\frac{I_0}{I_N})^{\gamma^n}. \tag{59}$$

In both (56) and (57) terms $\gamma^N << 1$ have been neglected in the exponent. The money values I_N of agent N and I_0 of the agents at $n = 0$ are free constants. Their choice depends on the relative weight the agents place on the utilities in the buying or selling mode. A seller dominated market leads to $I_N > I_0$. In Figure 3, we show I_n and I'_n as function of n for $\gamma = 1/4$ and $N = 11$. I_n (I'_n) are constant over a wide range and change in the last (first) two steps to the values imposed by the boundary conditions.

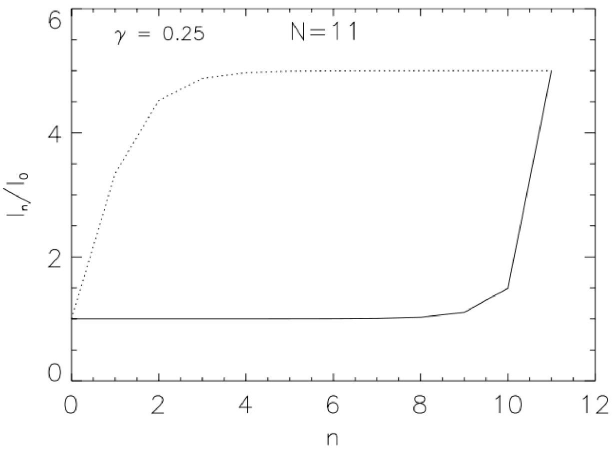

Figure 3: The money values I_n/I_0 for selling mode (solid line) and I_n/I_0 for buying mode (dotted line) as function of n. A seller dominated market with $I_N/I_0 = 5$ has been assumed.

Constant money values are achieved even when they are different in the selling and buying mode. This shows that the assumption of periodic boundary conditions is crucial for constant money values derived from money conservation $\Delta g = 1$ and not just minimizing the finite size effects as in physical problems.

Another consequence of the recursion is the 'peanuts effect'. Consider the normalized flows of goods $\bar{q}_n$ in a seller dominated market, using a power-law ansatz for utility functions. They are constant for $n << 1$ and increase near $n = N$. The ratio

$$\frac{\bar{q}_N}{\bar{q}_0} = (\frac{I_N}{I_0})^{\frac{1}{\alpha-\beta}} \tag{60}$$

may take large values, such that also $u_N^S = u_0^B$ becomes large. This remarkable feature seems to have induced an unfortunate german banker to publicly call the credits given to small customers at $n = 0$ as 'peanuts' (a statement that was not agreed upon by the broad public).
Up to now, the number $z-1$ of neighbors $n-1$ adjacent to agent n did not play any role given their money value $I_{n-1,i}$ has been chosen equally. In the next chapter we use a dynamics to reach the equilibrium condition (52) from an arbitrary initial state, including thermal noise. The utility function for updating the I_n may have other maxima besides the maximum described by (52).

5 Utility function for money

The dynamics of one-dimensional problem for the money value I_n is based on the conservation of money flux expressed by $\Delta g = 1$ in the case $z = 2$. This method has several disadvantages. It is completely deterministic and does not allow noise. More importantly, it does not involve the agents whose utility functions are minimal for the monopolistic equilibrium $r = 1$. Even a possible utility function for the dynamics would be rather complicated, since Δg on a Cayley tree connects agent $(n+1)$ with agents $(n-1, i)$ corresponding to a next to next neighbor interaction. In addition, we encounter for $z > 2$ the difficulty that money conservation in (50) does not determine the dependence of $g_{n,i}$ on i. To improve and to generalize the method in one dimension the dynamics of the money values will be based on an utility function H. Then the noise can be described by a Boltzmann distribution. H is the sum of two parts: first part H_M contains the effect of the money authorities, the second H_A is due to the agents. The latter should involve all agents equally. The simplest choice corresponds to a sum over all utilities $u^S + u^B$. The key observation is that the utilities depend on variables $q_{n,i}$ or $r_{n-1,i} = I_n/I_{n-1,i}$, which are defined on the links $x = (n; n-1, i)$ of the lattice. Moreover, the sum of utilities can be rearranged into a sum over links x

$$H_A = \sum_{agents} u^S + u^B = \sum_x u_A(r_x) \tag{61}$$

with u_A given in the selling mode by

$$u_A(z) = c(z) + \frac{1-\beta}{\beta} c'(z). \tag{62}$$

The money authority part must favor $g(n+1; n, i) = 1$. This establishes money conservation and a certain cooperation of the agents (n, i) to prefer equal money values $I_{n,i}$. Since due to (48) $\Delta g = 1$ only involves neighboring ratios, this suggests that one should consider the model on the dual lattice which is obtained by replacing the links $(n+1; n)$ of the Cayley tree by nodes x, and the nodes n by $z-1$ dimensional hypertetraeders. This dual lattice for $z = 3$ is called a cactus and is depicted in Figure 4. $\Delta g(x, y)$ are variables defined on the links x, y of the dual lattice. Nonvanishing values of Δg exist only on the links x, y depicted by the dotted lines in Figure 4 which are denoted by $x > y$.

We model H_M by a sum over all links $x > y$ of a utility function $u_M(\Delta g)$ having a maximum at $\Delta g = 1$. So we arrive at the following utility function for

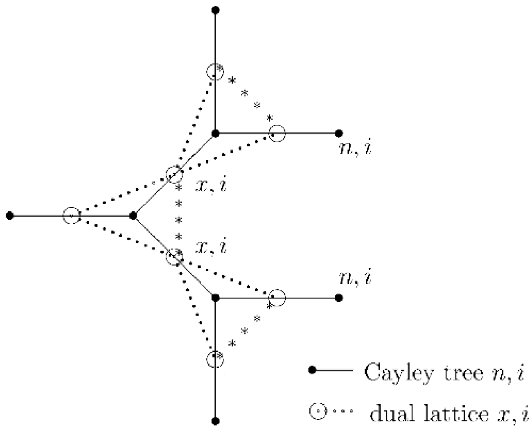

Figure 4: Cayley tree and its dual lattice for $z = 3$. On the links marked with $*$ the Δg variables are absent.

the dynamics of r:

$$H(r) = \sum_x u_A(x) + \sum_{x>y} u_M(\Delta g(x,y)) \tag{63}$$

Possible equilibrium states without noise correspond to the maximum of (63). Thermal noise is introduced by assuming that equilibrium distribution $w(r)$ for r is Boltzman distributed with the utility function (63)

$$w(r) \sim e^{\beta_T H(r)} \tag{64}$$

where β_T corresponds to the inverse temperature.
In the latter a randomly choosen agent n selects a new I'_n thereby changing its neighbouring ratios r_x and r'_x. The change I'_n is accepted with probability

$$p = e^{\beta_T min(0,\Delta H)} \tag{65}$$

where $\Delta H = H(r') - H(r)$ denotes the change in the utility function (63).

6 Phase transition in Ising model

In the deterministic limit $\beta_T \to \infty$ the utility function (63) should lead to the state $r_x = 1$, corresponding to the absolute maximum of H. However, there may be additional local maxima with $r \neq 1$ which are frozen if the thermal noise vanishes. To study this possibility we consider the following simplified version of (63). We allow only small deviations of r from 1 and parametrize r by a two valued function with one value 1 and the other r_0 close to 1

$$r_x = r_0^{(1+\sigma_x)/2} \tag{66}$$

with an Ising spin variable $\sigma_x = \pm 1$. In addition we assume for the utility function c a power law as in (35). The Boltzman weight (64) is a product of site factors

$$G^0(\sigma_x) = e^{\beta_T u_A(\sigma_x)} \tag{67}$$

and link factor

$$G^1(\sigma_x,\sigma_y) = e^{\beta_T u_M(\Delta g)} \tag{68}$$

with Δg derived from (51) for the buying mode and from (48) for selling mode. In the latter we obtain

$$\Delta g(\sigma_x,\sigma_y) = r_0^{(1-\gamma+\sigma_y-\gamma\sigma_x)/2(1-\gamma)}. \tag{69}$$

For r_0 close to 1, we can expand u_M around 1 and obtain

$$G^1(\sigma_x,\sigma_y) = (\,e\,)^{-K(1-\gamma)^2}\, e^{-K\gamma^2} e^{-K} 1_{\sigma_x,\sigma_y} \tag{70}$$

with the money conservation constant

$$K = -\frac{\beta_T u_M''(1)}{2}\left(\frac{\alpha \ln r_0}{\alpha-\beta}\right)^2. \tag{71}$$

In (70) the irrelevant factor $e^{\beta u_M(1)}$ has been omitted.
In the same way we obtain for G^0

$$G^0(\sigma_x) = e^{(z-1)L\delta_{\sigma_x,1}} \tag{72}$$

with the self-interest constant

$$L = \frac{\beta_T}{z-1}[u_A(r_0) - u_A(1)]. \tag{73}$$

Using (70) in (72) the Boltzman equilibrium distribution for the dynamical variables σ can be written as

$$w(\sigma) = \frac{1}{Z}\prod_x G^0(\sigma_x)\prod_{y<x} G^1(\sigma_x,\sigma_y). \tag{74}$$

The normalization factor Z follows from the condition $\sum_{\{\sigma\}} w(\sigma) = 1$. The distribution for a single spin $w_1(\sigma_x) = \sum_{\sigma\neq\sigma_x} w(\sigma)$ or for two spins $w_2(\sigma_x,\sigma_y) = \sum_{\sigma\neq\sigma_x,\sigma_y} q(\sigma)$ can be calculated recursively. For this purpose we introduce the tree distribution $T_n(\sigma_x)$ of length $|x| = n$ corresponding to the product of all factors G^0 and G^1 on a dual tree starting at x, which is summed over all spins σ_y with $|y| \geq |x|$

$$T_n(\sigma_x) = \frac{1}{Z_T}\sum_{\{\sigma_y, y>x\}}\prod_{|y|\geq|x|}(G^0(\sigma_y)\prod_{|y'|\geq|y|} G^1(\sigma_y,\sigma_{y'})). \tag{75}$$

Z_T is choosen such that $\sum_\sigma T(\sigma) = 1$. For agent N (75) yields the equilibrium distribution $w_1(\sigma_N)$, for agent $|x| < N$ the tree distribution $T_n(\sigma_x)$ is a conditional probability related to $w_1(\sigma_x)$. According to (75) a tree of length n can be expressed by trees of length $n-1$ in the following way

$$T_n(\sigma) = G^0(\sigma)\sum_{\sigma_1,\cdots,\sigma_{z-1}}\prod_{i=1}^{z-1} G^1(\sigma,\sigma_i)T_{n-1}(\sigma_i). \tag{76}$$

Any function $T(\sigma)$ depending on variable $\sigma = \pm 1$ can be parametrized as

$$T_n(\sigma) = a_n(w_n\delta_{\sigma,1} + \delta_{\sigma,-1}). \tag{77}$$

Carrying out the summation in (76) we find the recursion relation for a_{n+1} and w_{n+1} in terms of a_n and w_n. For the latter this reads

$$w_{n+1} = f(w_n) \tag{78}$$

$$f(w) = [e^{-K\gamma^2 + L\frac{1+e^{(2\gamma-1)K}w}{1+e^{-K}}w}]^{z-1} \tag{79}$$

which allows the recursive calculation of w_n if the values w_n at $n = 0$ are given. The mean value of r_x for the top agent $x = N$ is related to w_N by the inflation parameter

$$M = \langle\frac{\ln r_x}{\ln r_0}\rangle = \langle\delta_{\sigma_x,1}\rangle_T = \frac{w_N}{1 + w_N}. \tag{80}$$

Therefore $w_N \sim 0$ expresses the preference for $r_x = 1$, whereas $w_N >> 1$ leads to $r_x \sim r_0$. In physical problems $2M-1$ corresponds to the magnetization. Since our utility function is not symmetric under $\sigma \to -\sigma$ the disordered state $M = 1/2$ has no particular meaning. Here only the fully magnetized states are interesting. Inflation parameter $M = 0$ corresponds to the monopolistic state and $M = 1$ implies inflation ($r_0 < 1$) or deflation ($r_0 > 1$). Particularly interesting are stable fixed points w_n of recursion (79). These are solutions independent of n for $n >> 1$, especially independent of the boundary values w_0. They correspond to a homogeneous value of the inflation parameter on the lattice. If more than one fixed point exists, the system can exhibit different phases. It is a particular property of the Cayley tree that the values at the boundary decide which phase is adopted. On a normal finite dimensional lattice only one phase would be thermodynamically stable. The form of $f(w)$ shows that the fixed point equation $w = f(w)$ can have either one or two solutions satisfying the stability condition $|f'(w)| < 1$. Depending on the values of K, L and γ there can be a one-state phase (OSP) with a unique value of M or a two-state phase (TSP) with two possible values. In Figure 5, we show for the numerical solution of $w = f(w)$ with $z = 3$ neighboring agents the inflation parameter $M(w)$ as a function of K for several L values and $\gamma = 0.25$.

At low L there will be a unique solution OSP in which M tends to zero for large values K of the money conservation term in H. M increases with the self-interest L of the agents. For suficiently large L a switch into the TSP with two possible values of M occurs. Still the monopolistic equilibrium can be achieved for large K. The fixed point equation can be only solved numerically, the calculation of the phase boundaries requires solution of quadratic equations. One finds that TSP only occurs if the following two conditions are satisfied. K has to be larger than a critical value given by

$$K_c = \frac{1}{\gamma}\ln\frac{z}{z-2} \tag{81}$$

and L has to be bounded by

$$L_-(K) < L < L_+(K). \tag{82}$$

For the following we need only the asymptotic form of $L_\pm(K)$ for $K >> 1$

$$L_\pm = K(\gamma^2 + \frac{1}{z-1}) - 2\gamma K \begin{array}{c}\frac{1}{z-1}\\ 1\end{array} . \tag{83}$$

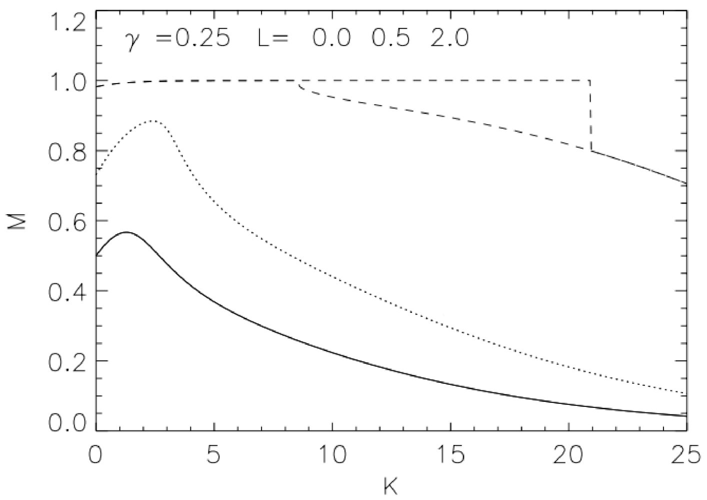

Figure 5: The inflation parameter M as a function of K for $\gamma = 0.25$ and $z = 3$ for three different values of L: 0 (solid line), 0.5 (dotted line), and 2.0 (dashed line).

For the linear chain ($z = 2$) discussed before K_c becomes infinte and a phase transition to TSP cannot occur. The second condition (82) explains why a window for TSP phase is observed in Figure 5. Another feature of the model is the dependece on the elasticy γ. For values $\gamma < \gamma_c$ with

$$\gamma_c = 1 - \sqrt{\frac{z-2}{z-1}}, \tag{84}$$

the lower bound L_- is always positive. For fixed L and $K \to \infty$ one always ends up in the OSP in agreement with what we have seen in Figure 5. Choosing a value $\gamma = 0.6 > \gamma_c$ we show in Figure 6, the inflation parameter M with $z = 3$ as function of K for various L values.

Above K_0 with $L = L_-(K_0) > 0$ the system is always in the TSP with M values near 0 or 1 corresponding to ratios of money values $I_n/I_{n-1} = 1$ or r_0. Even in the deterministic limit $K \to \infty$ the inflationary solution cannot be avoided.
The boundaries of the TSP phase in the $(K, L/K)$ plane are shown in Figure 7 ($\gamma = 0.25 < \gamma_c$) and Figure 8 ($\gamma = 0.6 > \gamma_c$) for $z = 3$.

In Figure 7, the regions where M is smaller than 0.5 (0.1) are indicated by the dotted (dashed) line, which occur outside of the TSP region. Therefore small values of M are guaranteed in the limit of large K. In contrast for $\gamma > \gamma_c$ the region of small M lies entirely in the TSP region, as seen from Figure 8. The OSP can be obtained only for negative L which implies $r_0 < 1$ which is against the agents interest in the selling mode. On the other side for $\gamma > \gamma_c$ small values of M can be obtained already at moderate K. The phase transition crossing the bounds $L_\pm$ from OSP to TSP will be in general a first-order transition, since M can change discontinuously by M. If one approaches the end points of TSP near K_c given by (81), the discontinuity vanishes with a power law according

$$\Delta M \sim (K - K_c)^{1/2} \tag{85}$$

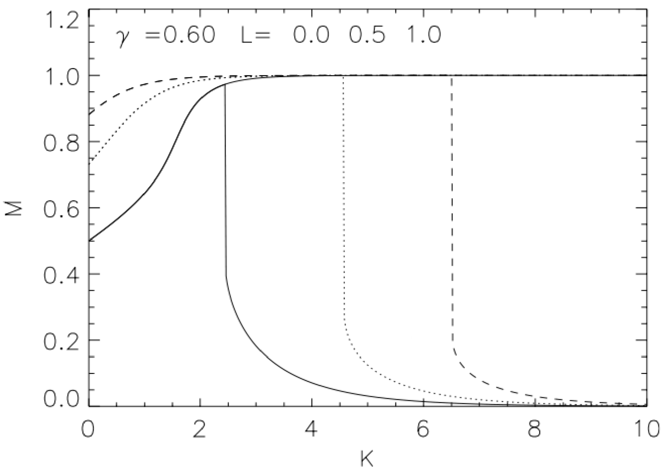

Figure 6: The inflation parameter M as a function of K for $\gamma = 0.6$ and $z = 3$ for three different values of L. There is a critical $K(L)$ where the system changes from the OSP into TSP with one value $M_1 \sim 1$ and $M_0 \sim 0$.

indicating a second-order phase transition of the mean field class. With increasing number z of neighboring agents the boundaries of the TSP degenerate into straight lines $L_+ = K\gamma^2$ and $L_- = K\gamma^{-2}$ implying presence of only the TSP for $L/K = \gamma^2$.
The same method can be applied to the buying mode, where agent N buys goods via the tree from agents at $n = 0$. One obtains a similar recursion formula as (79) with value L', K' and γ' obtained by the replacement

$$L' = -L, \quad K' = \gamma^2 K \quad \text{and} \gamma' = \frac{1}{\gamma}. \tag{86}$$

This leads to qualitatively similar phase transitions.

The money value regulating authorities can achieve a stable economy with an inflation parameter $M = 0$ for given agent parameters L and γ by the choice of large K. The success depends on the value of the elasticity ratio $\gamma = \beta/\alpha$. Very different utilities $\bar{c}(q)$ and $d(q)$ lead to $\gamma << 1$. In this case the system remains in the OSP and the desired result is obtained for large K. For similar utilities $\bar{c}(q)$ and $d(q)$ we expect $\gamma \sim 1$ and TSP occurs with $M_0 \sim 0$ and $M_1 \sim 1$. Which solution is obtained depends on the boundary values of the agents at $n = 0$. Since their utility function (42) increases with r_0 they prefer a value $r_0 > 1$ leading to increasing money values from $n = 0$ to N (deflation). Additional measures as indirect taxes are required to persuade the $n = 0$ agents to choose the solution M_0. Alternatively one can close the selling tree by a second tree in the buying mode, where the agents at $n = 0$ sell their goods (e.g. labor) through a tree to the top agents. In general, such a mechanism should exist in order to recycle the money flow from $n = 0$ to N in the selling mode. In this case an inflationary value $r_0 < 1$ is preferred. Combining both trees indeed allows the intermediate state $r_0 = 1$ to be reached, as desired in a stable economy.

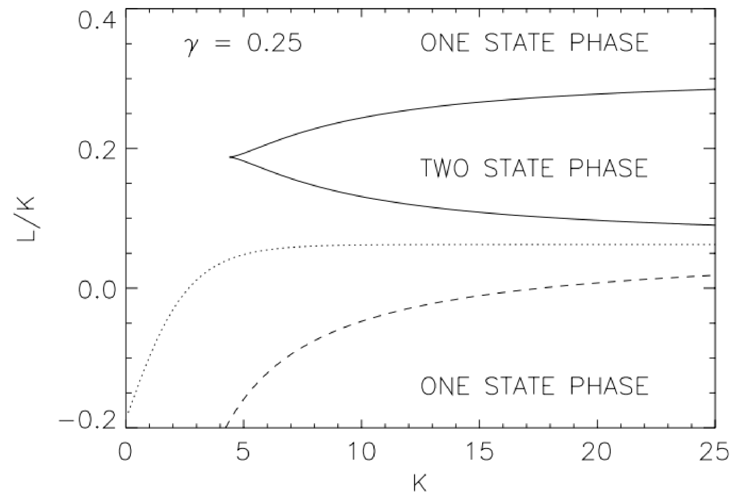

Figure 7: Phase diagram for $\gamma = 0.25$ and $z = 3$ in the $(K, L/K)$ plane.

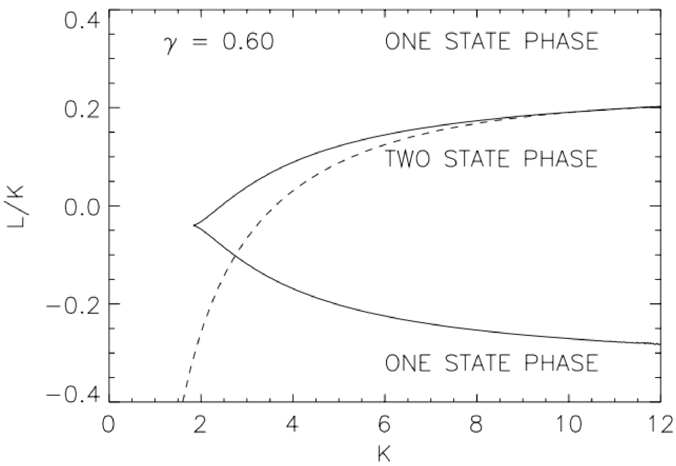

Figure 8: Phase diagram for $\gamma = 0.6$ and $z = 3$ in the $(K, L/K)$ plane.

7 Conclusions

In this article we presented the Ising model in physics and how we can use similar model in economy. We considered a trading model of agent for two different types: on a linear chain with periodical boundry conditions and the hiearchical network of a Cayley tree, treating money as a dynamical variables. In the case of a linear chain, imposing money conservation at each agent we find constant I, however, different in the selling and buying mode leading to the 'peanuts effect'. When agents are allowed to choose between neighbors, as for $z > 2$, additional dynamical phenomena may occur, dependent on whether agents cooperate or not. We include this as an optimization problem between nearest neighbors and next to nearest neighbors which, moving the model to the dual lattice (the cactus in $z = 3$) still can be described in terms of nearest neighbor interactions (now between links). An elegant simplification of this model in terms of an Ising model allows to include noise and to explicitly solve the model. The phases of this Ising version of the model correspond to different dynamical regimes of the economy. The main result is the existence of a TSP above a critical money conservation parameter $K_c = (1/\gamma)ln\frac{z}{(z\hat{a}'2)}$ with critical curves separating the OSP from the TSP. In the TSP one observes a first-order phase transition between an inflationary phase and a phase with stable money value. Whether such a phase transition can occur, depends on the exponents (or elasticities) of the utility functions for buying or selling only. For very different functions the elasticity parameter $\gamma = \beta/\alpha$ will be small and the system can remain in a OSP with stable money value at $K \to \infty$. If one increases beyond a critical value $\gamma_c = 1 - \sqrt{(z-2)/(z-1)}$ the TSP is inevitable. If the utility functions are similar ($\gamma \sim 1$) the inflationary phase can occur also in the limit $K \to \infty$. In contrast to the linear model ($z = 2$), the equilibrium properties depend on both, the utility functions (γ) and the geometry (z).

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