FILAMENT SHAPES GIVEN BY EULER-KIRCHHOFF EQUATIONS

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Abstract

The shapes of elastic rods with circular cross-section are described by solutions of static Kirchhoff equations. They are called the Euler-Kirchhoff filaments and can have visually very interesting forms. The Kirchhoff equations are formally equivalent to the Euler equations describing the motion of spinning tops. Principal object of this seminar is to investigate different possible configurations of rods and some interesting particular cases. At the same time, these configurations are compared with corresponding spinning top motions.

1 Introduction

The properties of elastic deformations in rods have been studied for a long time, since Lagrange and Euler. The topic has many applications in practice [1], for example twisting of electric cables, study of filamentary structures like DNA and bacterial fibres in biology and sun spot formation or vortex tube motion in hydrodynamics.

The Kirchhoff theory of deformation in rods [2] is a principal tool for studying this phenomenon and is one of the basic parts of theory of elasticity. It couples two possible deformations, i.e. torsion and bending. It assumes thin elastic filaments, that means unidimensional pieces of elastic material subject to internal stresses and boundary constraints. First we will introduce this model.

2 The Kirchhoff model

The shape of the filament is described as a curve in space by a function $\boldsymbol{R}(s,t)$, where the independent variables are arc length s and time t. We will treat only static case so we omit the time dependence. At every point of the filament, this means for every s we define a local basis $(\boldsymbol{n},\boldsymbol{b},\boldsymbol{t})$. Vector $\boldsymbol{t}$ is the unit tangent vector given by $\boldsymbol{t}=\frac{\partial \boldsymbol{R}}{\partial s}$. Vector $\boldsymbol{n}$ is called normal vector, defined as

$$\frac{\partial \boldsymbol{t}}{\partial s}=\kappa\,\boldsymbol{n} \tag{1}$$

by introducing the *curvature* $\kappa=\left|\frac{\partial \boldsymbol{t}}{\partial s}\right|$. The remaining vector is called binormal vector and is equal to $\boldsymbol{b}=\boldsymbol{t}\times\boldsymbol{n}$. These three vectors form an orthonormal basis on the curve $\boldsymbol{R}$.

If we look at dot products of $\frac{\partial \boldsymbol{b}}{\partial s}$ with $\boldsymbol{b}$ and $\boldsymbol{t}$ respectively :

$$\frac{\partial \boldsymbol{b}}{\partial s}\cdot\boldsymbol{b} = \frac{1}{2}\frac{\partial(\boldsymbol{b}\cdot\boldsymbol{b})}{\partial s}=0, \tag{2}$$

$$\frac{\partial \boldsymbol{b}}{\partial s}\cdot\boldsymbol{t} = \frac{\partial(\boldsymbol{t}\times\boldsymbol{n})}{\partial s}\cdot\boldsymbol{t}=(\kappa\,\boldsymbol{n}\times\boldsymbol{n})\cdot\boldsymbol{t}+(\boldsymbol{t}\times\frac{\partial \boldsymbol{n}}{\partial s})\cdot\boldsymbol{t}=0, \tag{3}$$

we see that $\frac{\partial \boldsymbol{b}}{\partial s}$ has only a component in the direction of $\boldsymbol{n}$, therefore we can write

$$\frac{\partial \boldsymbol{b}}{\partial s} = -\tau \, \boldsymbol{n}, \tag{4}$$

which defines the *torsion* τ. The torsion measures the rotation of the $(\boldsymbol{n}, \boldsymbol{b}, \boldsymbol{t})$ basis around the tangent vector $\boldsymbol{t}$ as the arc length increases. It can easily be seen that the remaining derivative is

$$\frac{\partial \boldsymbol{n}}{\partial s} = \tau \, \boldsymbol{b} - \kappa \, \boldsymbol{t}. \tag{5}$$

If the curvature κ and the torsion τ are known for all s, equations (1), (4), and (5) can be used to find $(\boldsymbol{n}(s), \boldsymbol{b}(s), \boldsymbol{t}(s))$ and later on $\boldsymbol{R}(s)$.

Our next step is to generalise [1] the $(\boldsymbol{n}, \boldsymbol{b}, \boldsymbol{t})$ basis by writing $(\boldsymbol{d_1}, \boldsymbol{d_2}, \boldsymbol{t})$ instead, where $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$ are still two perpendicular unit vectors, both perpendicular to tangent vector $\boldsymbol{t}$, which is the same as before. Equations (1), (4), and (5) can be rewritten for the new basis in a more compact form as

$$\frac{\partial \boldsymbol{d_1}}{\partial s} = \boldsymbol{\kappa} \times \boldsymbol{d_1}, \tag{6}$$

$$\frac{\partial \boldsymbol{d_2}}{\partial s} = \boldsymbol{\kappa} \times \boldsymbol{d_2}, \tag{7}$$

$$\frac{\partial \boldsymbol{t}}{\partial s} = \boldsymbol{\kappa} \times \boldsymbol{t}, \tag{8}$$

where we defined the *twist vector* $\boldsymbol{\kappa} = \kappa_1 \, \boldsymbol{d_1} + \kappa_2 \, \boldsymbol{d_2} + \kappa_3 \, \boldsymbol{t}$. Now we have two curvatures, κ_1 and κ_2, describing how, at every s, the rod is bent in the two directions of $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$ while κ_3 measures how the rod is twisted.

Let us explain the difference between those two basis. The orientation of $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$ around $\boldsymbol{t}$ is fixed to the rod and follows the twisting of the rod while the orientation of $\boldsymbol{n}$ and $\boldsymbol{b}$ around $\boldsymbol{t}$ was determined by the plane containing a circle that described the local curvature of the rod. So the $(\boldsymbol{n}, \boldsymbol{b}, \boldsymbol{t})$ basis measures the mathematical twisting of the curve which represents the rod. Contrarly, $(\boldsymbol{d_1}, \boldsymbol{d_2}, \boldsymbol{t})$ basis measures physical twisting of the rod material.

With these quantities we can describe the shape of a rod. Vector $\boldsymbol{R}(s)$ follows the *centerline* of the rod. Vectors $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$ are attached to the material forming the rod. We will now write the Kirchhoff equations. Two of them are dynamic equations for every slice of the rod with the cross-section shape S which is effected by internal elastic stresses and boundary constraints but there is no external force field like gravity. We also exclude the interaction between remote slices. Let $\boldsymbol{F}(s)$ and $\boldsymbol{M}(s)$ be the total force and total torque that act on the surface S of a rod slice at s by its neighbour slice situated at $s + ds$. The conservation of linear momentum is written as

$$\boldsymbol{F}(s + ds) - \boldsymbol{F}(s) = \rho \, S \, ds \, \frac{\partial^2 \boldsymbol{R}}{\partial t^2} \tag{9}$$

which for the static case gives

$$\frac{\partial \boldsymbol{F}}{\partial s} = 0. \tag{10}$$

Similarly, the conservation of angular momentum of the slice for the static case gives

$$\frac{\partial \boldsymbol{M}}{\partial s} + \boldsymbol{t} \times \boldsymbol{F} = 0. \tag{11}$$

These are two equations for the set of Kirchhoff equations. The third one is the constitutive relation [2] which describes the reaction of the rod on the applied deformation :

$$\boldsymbol{M} = E\, I_1\, \kappa_1\, \boldsymbol{d_1} + E\, I_2\, \kappa_2\, \boldsymbol{d_2} + \mu\, J\, \kappa_3\, \boldsymbol{t}, \tag{12}$$

where E is Young modulus, μ is one of the two Lamé constants, and I_1, I_2, J are functions of the shape S. For a circular cross-section of radius R which is our case, one has

$$I_1 = I_2 = \frac{J}{2} = \frac{\pi R^4}{4}. \tag{13}$$

The quantities $E\, I_1$ and $E\, I_2$ are bending stiffnesses which measure how much torque is needed to bend the rod in the two principal directions while $\mu\, J$ is torsional stiffness which measures how strong the applied torque must be to twist the rod.

In order to reduce the number of independent constants, we rewrite the Kirchhoff equations (10), (11) and (12) in the scaled form, taking in account the circular cross-section. The result is

$$\frac{\partial \boldsymbol{F}}{\partial s} = 0, \tag{14}$$

$$\frac{\partial \boldsymbol{M}}{\partial s} + \boldsymbol{t} \times \boldsymbol{F} = 0, \tag{15}$$

$$\boldsymbol{M} = \kappa_1\, \boldsymbol{d_1} + \kappa_2\, \boldsymbol{d_2} + p\, \kappa_3\, \boldsymbol{t}, \tag{16}$$

where $p = \frac{\mu J}{E I_1} = \frac{2\mu}{E}$ is the only parameter that depends on rod material. It can also be expressed in terms of Poisson ratio as $p = \frac{1}{1+\sigma}$ from where we see that $p \in [\frac{2}{3}, 1]$.

To derive the final form of equations (14) to (16), we project (15) along $\boldsymbol{t}$ and get

$$\boldsymbol{M}' \cdot \boldsymbol{t} = (\boldsymbol{M} \cdot \boldsymbol{t})' - \boldsymbol{M} \cdot \boldsymbol{t}' = 0. \tag{17}$$

Substituting $\boldsymbol{M}$ from (16) and $\boldsymbol{t}'$ from (8) we get

$$\kappa_3' = 0. \tag{18}$$

We perform similar projections of (15) along $\boldsymbol{d_1}$ and $\boldsymbol{d_2}$, substitute $\boldsymbol{M}$, $\boldsymbol{d_1}'$ and $\boldsymbol{d_2}'$ from (16), (6) and (7) respectively and get

$$\kappa_1' + (p-1)\, \kappa_2\, \kappa_3 = 0, \tag{19}$$

$$\kappa_2' + (1-p)\, \kappa_1\, \kappa_3 = 0, \tag{20}$$

In order to find the shape of the rod, i.e. to find $\boldsymbol{R}(s)$ one has to solve equations (18) to (20) first to obtain the curvature and torsion $\kappa_{1,2,3}$. Then equations (6) to (8) are used to find $\boldsymbol{d_1}(s)$, $\boldsymbol{d_2}(s)$ and $\boldsymbol{t}(s)$. $\boldsymbol{R}(s)$ is obtained by integrating the tangent vector $\boldsymbol{t}$ along the arc length. In the next section we will stress the equivalence of the set of Kirchhoff equations with Euler equations describing the motion of a spinning top.

3 Rod statics = top spinning

Let us recall from classical mechanics the equations of motion for a spinning symmetrical top with its axis fixed at one point:

$$\dot{\omega}_3 = 0, \tag{21}$$

$$\dot{\omega}_1 + (p-1)\,\omega_2\,\omega_3 = 0, \tag{22}$$

$$\dot{\omega}_2 + (1-p)\,\omega_1\,\omega_3 = 0. \tag{23}$$

This set is equivalent to set (18) to (20). We only have to find the correspondence between physical quantities that describe each problem. This is done in the following table:

	Rod		Top
$\boldsymbol{t}$	unit vector tangent to rod	$\boldsymbol{t}$	unit vector from fixed point to the centre of mass
$\boldsymbol{d_1}$, $\boldsymbol{d_2}$	perp. unit vectors attached to the rod, both perp. to $\boldsymbol{t}$	$\boldsymbol{d_1}$, $\boldsymbol{d_2}$	the same, attached to the top
s	arc length	t	time
$\boldsymbol{F}$	force in the rod	$-\boldsymbol{F_g}$	force equal and opposite to gravity
$\boldsymbol{M}$	torque in the rod	$\boldsymbol{L}$	angular momentum
$\boldsymbol{\kappa}$	twist vector	$\boldsymbol{\omega}$	angular velocity
$E\,I_{1,2}$	bending stiffness	$I_{1,2}$	moment of inertia in direction orthogonal to $\boldsymbol{t}$
$\mu\,J$	torsional stiffness	I_3	moment of inertia along $\boldsymbol{t}$
p	stiffness ratio	p	moment of inertia ratio

We deal with a rod which has a circular cross-section and this corresponds to a symmetrical top ($I_1 = I_2$). The correspondence means that to every possible motion of the top we can find a particular static shape of the rod. The path of the free end of the top in space and time corresponds to a certain rod shape. A particular top motion has three first integrals, that is the angular velocity along the axis of rotation ω_3, vertical component of the angular momentum L_z and the total kinetic plus potential energy. The corresponding first integrals for a rod are the twist density κ_3, vertical component of the torque applied at its end M_z and its total bending plus torsional energy. Formally, there is a fourth first integral, that is the force opposite to gravity $-F_g$ for the top and vertical component of the force F_z for the rod.

It should be noted that in our particular case where we deal with symmetrical rods and tops, the set of Kirchhoff or Euler equations can be solved analytically.

The described correspondence can be used to classify numerous types of rod shapes by looking at adequate top motions which yields top a systematic classification.

4 Classification of shapes

We will not try to solve the Kirchhoff equations which in general lead to elliptic functions. We will only observe various solutions [1]. They can be divided in different classes.

4.1 Helical filaments

First we have a look at helical filaments where curvature κ defined in (1) and torsion τ defined in (4) are constant. Constant curvature means that the circular arc that locally best fits the filament has a constant radius. Constant torsion means that the plane which contains this circle rotates with a constant rate in space when travelling along the arc length s. The third quantity is the third component of the twist vector, κ_3, which measures how fibres that form the rod are twisted along the rod. This can be observed on the pictures thanks to the colour pattern on the rod surface.

Corresponding top motions are described by depicting the path of the free end of the top axis (top extremity) on the sphere where the fixed end is in the sphere centre. In case of helix filaments, the top extremity describes a circle. Figure 1 shows a helix with pure torsion, that is, $\kappa_3 = \tau$.

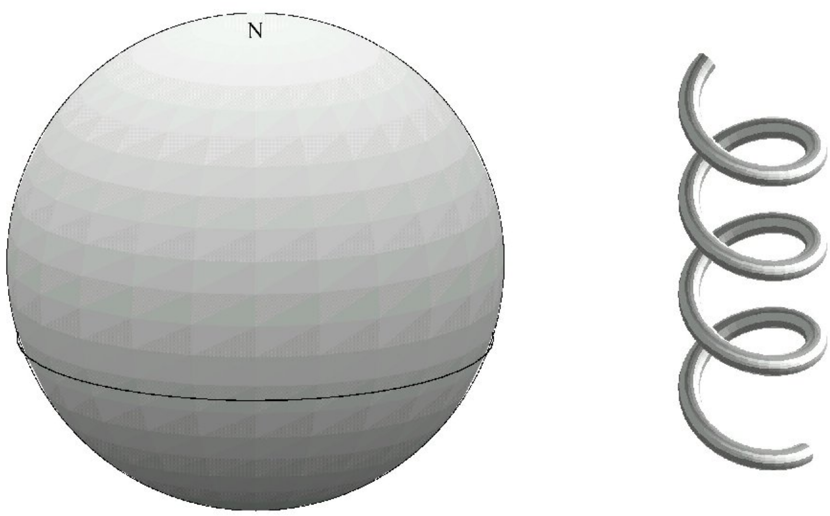

Figure 1

Next two figures show an over- and undertwisted helix which are distinguished by the sign of handedness of the helix itself (τ) and the handedness of the pattern on the rod surface ($\kappa_3 - \tau$). In the first case (figure 2), the signs are the same while for the undertwisted helix (figure 3), the signs are opposite :

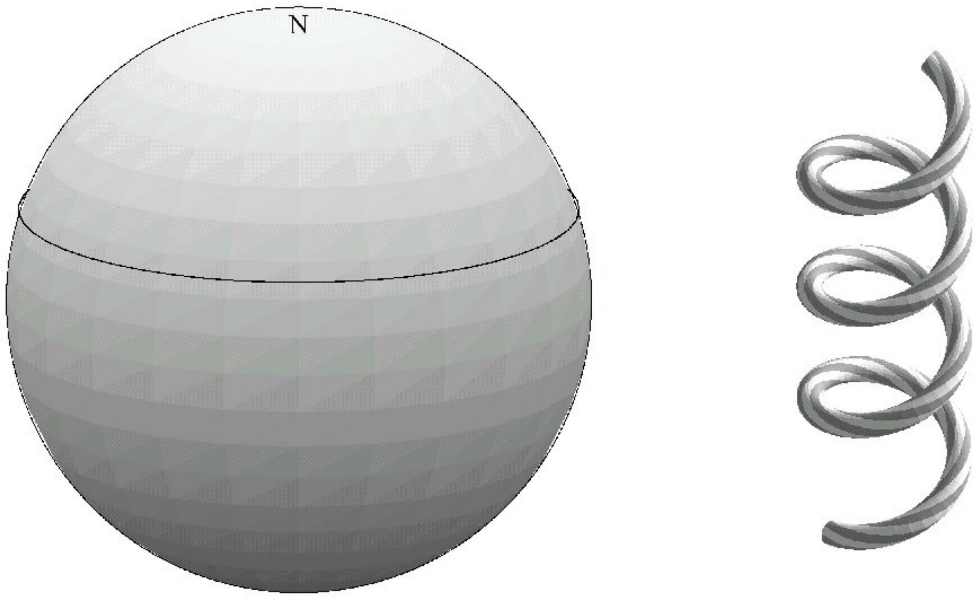

Figure 2

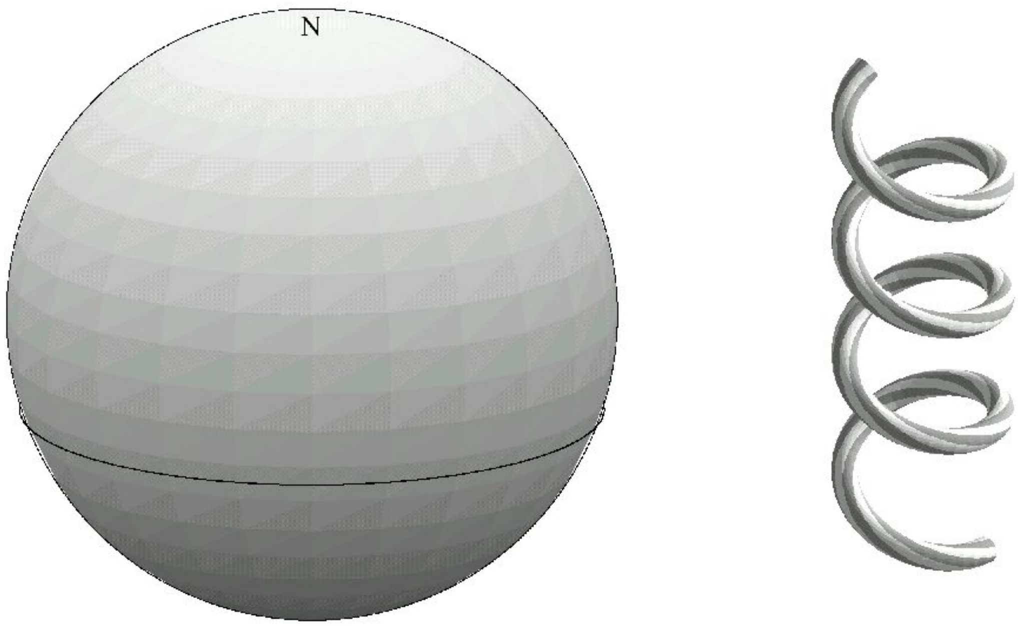

Figure 3

If $\tau = 0$, the helix turns into a ring. For the top, this means that the top axis rotates in the horizontal plane :

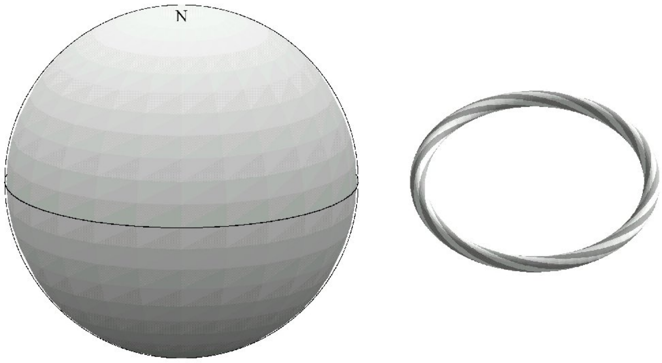

Figure 4

Straight rod solutions correspond to a sleeping top, this is one whose extremity is at rest. A rod subject to extensive tension ($F_3 > 0$) means a top with its axis pointing upwards :

Figure 5

while a rod subject to compressive tension ($F_3 < 0$) means a top whose axis points downwards :

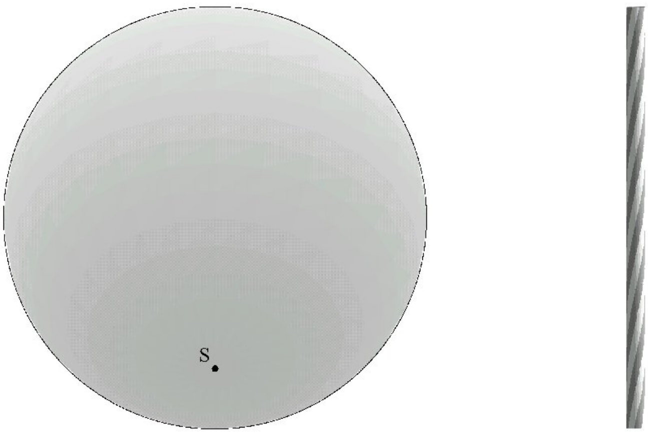

Figure 6

4.2 Planar filaments

Planar shapes were examined already by Euler. We exclude straight and circular shapes that were already cited within helical shapes. In general, a rod with a planar shape corresponds to a top which behaves like a plane pendulum, hung at the fixed point. If we look at the phase portrait of a plane pendulum (figure 7) we see that there exist oscillating (=closed) orbits and revolving (=open) orbits where the pendulum turns around the fixed point.

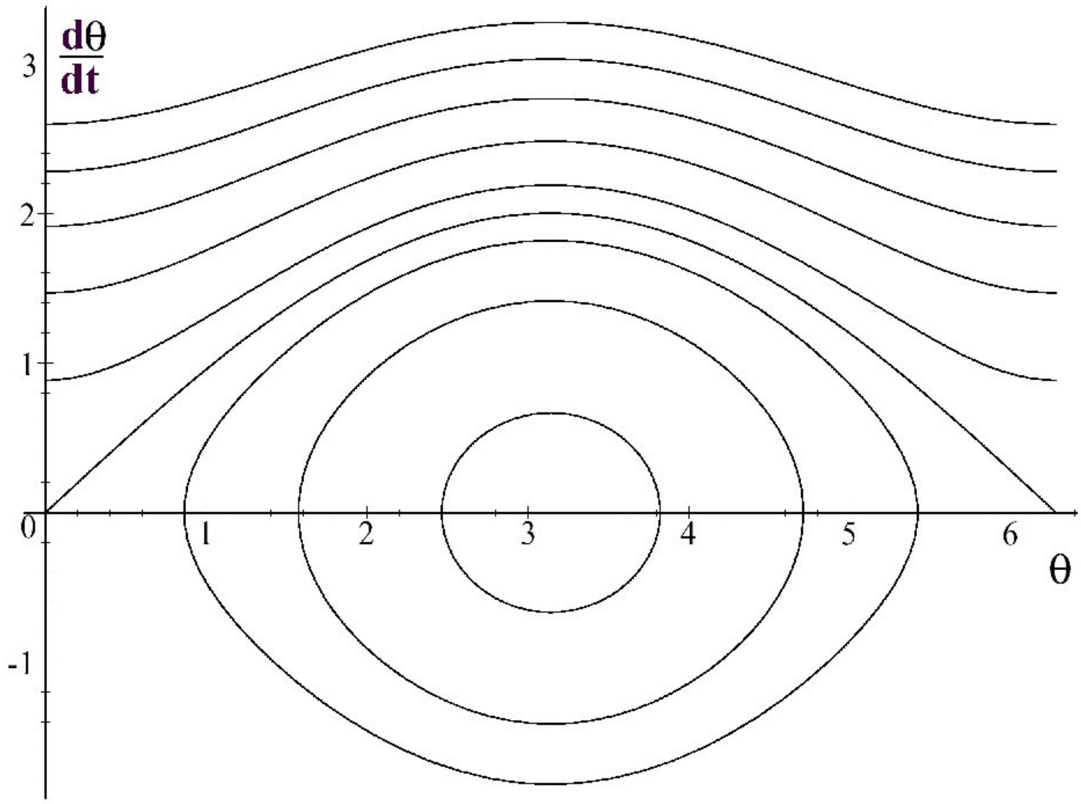

Figure 7

The corresponding equation describing such a pendulum is

$$\ddot{\theta} + \omega_0^2 \sin\theta = 0. \tag{24}$$

Orbits and thus rod shapes can be ordered regarding the energy of the pendulum. First we look at some planar filaments corresponding to low energy oscillating orbits of the pendulum. The energy increases from a) to e) :

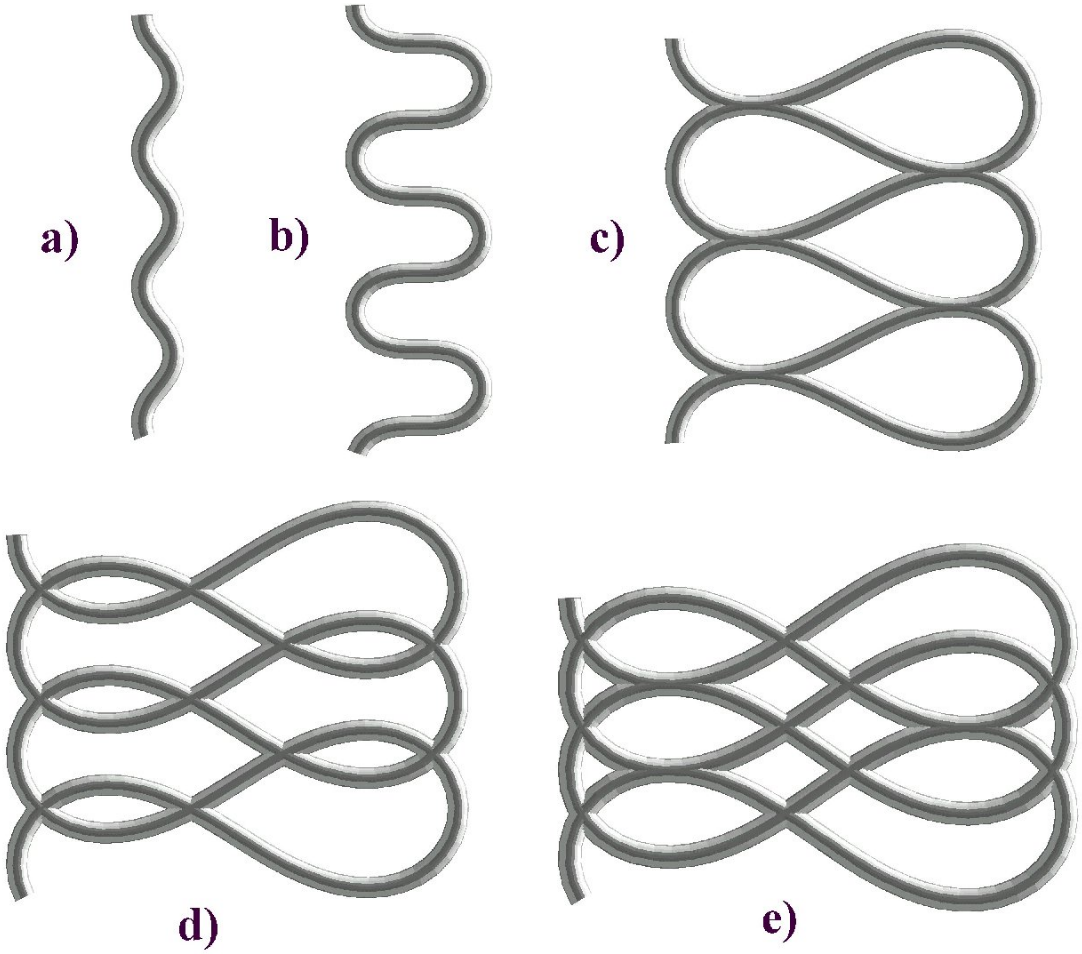

Figure 8

When energy increases further on, rod shapes become more and more interesting. With increasig energy, the distance between loops also increases :

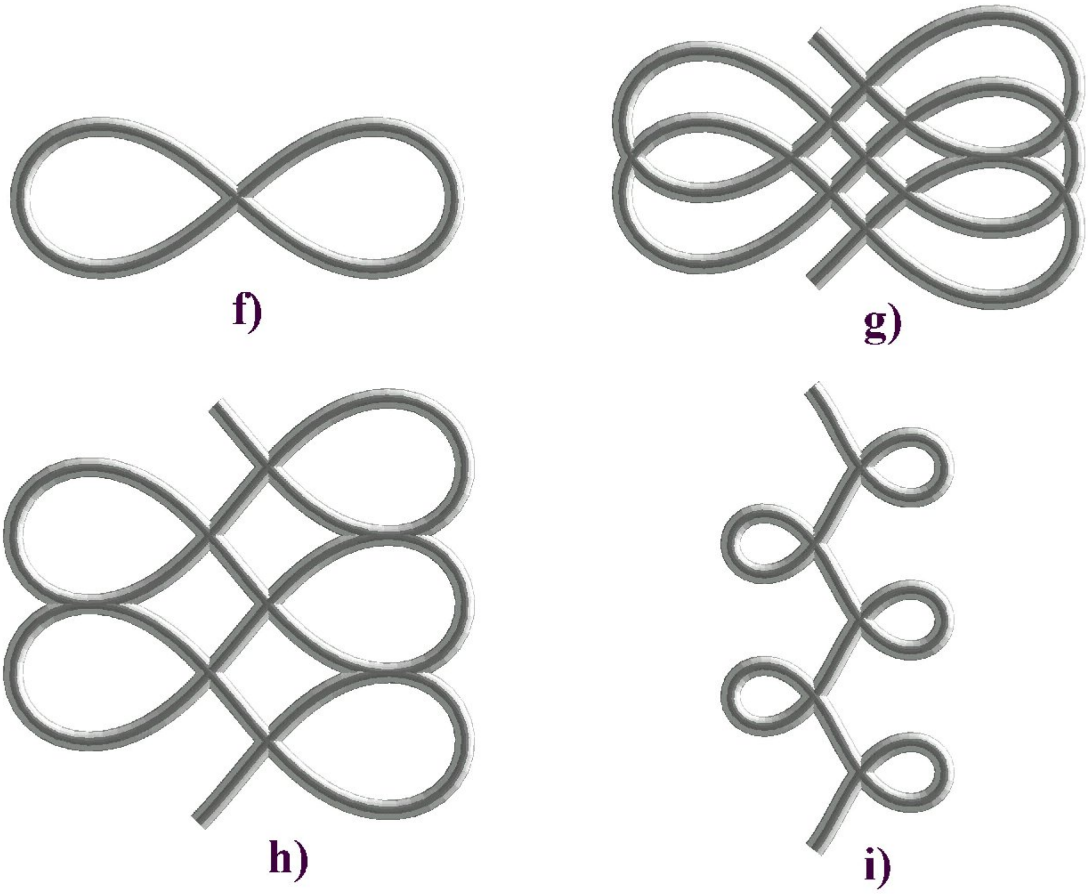

Figure 9

The limit case is reached when the pendulum is at the turning point from oscillating to revolving orbit :

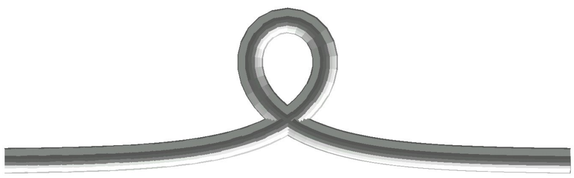

Figure 10

After the limit case, pendulum turns around and the rod changes its behaviour. The energy increases from a) to c) :

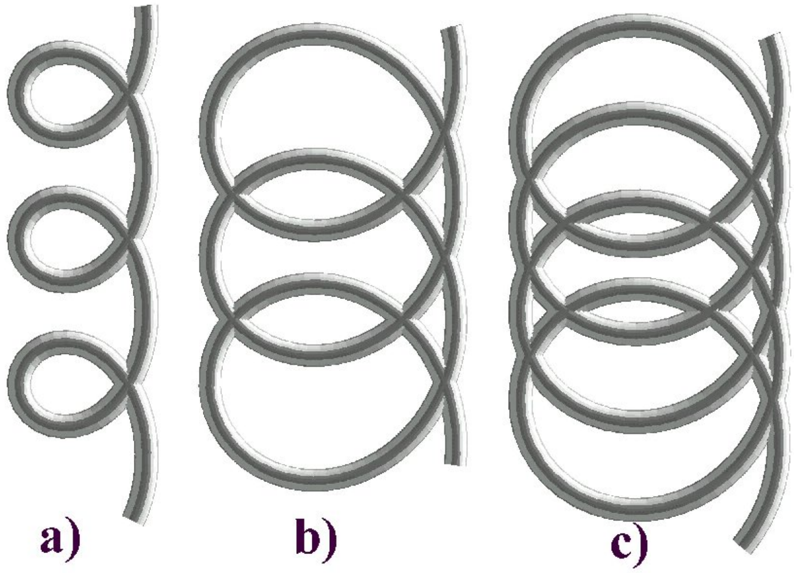

Figure 11

4.3 Generic shapes

In general case, the top precesses and nutates and its path on the sphere is limited between two polar angles. This leads to a rod shape which on average, i.e. on large scales, behaves like a helix. It is obvious that a periodic top orbit corresponds to a periodic filament shape.

A case of great importance in biochemical applications like DNA is the supercoiled helix. It is defined as a curve that looks like a helix on small scales, but its central line forms another helix on large scale. This occurs when the top precesses slowly in almost circular loops :

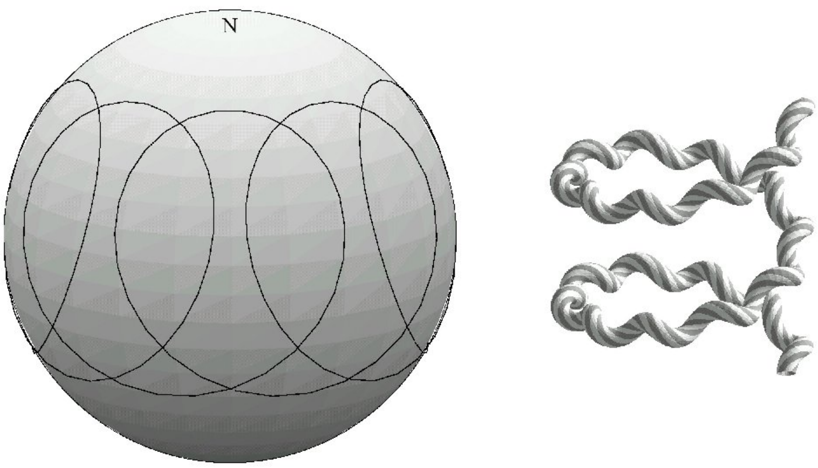

Figure 12

Such a superhelix can also be deformed into a ring, like it was the case for a simple helix :

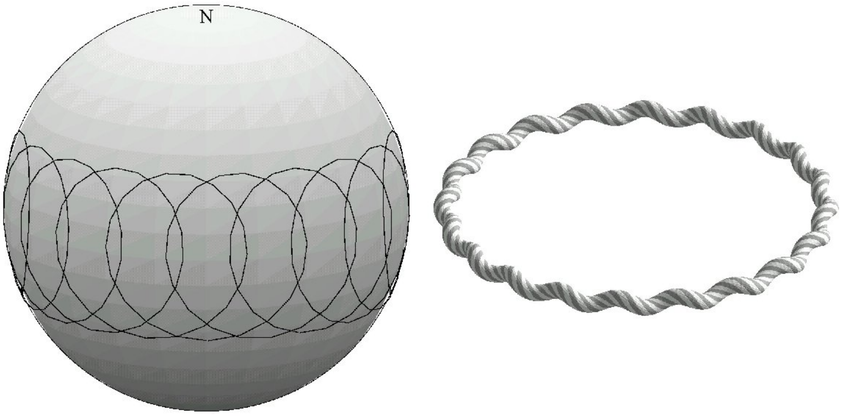

Figure 13

If the top extremity passes the north or the south pole, we have the following configuration :

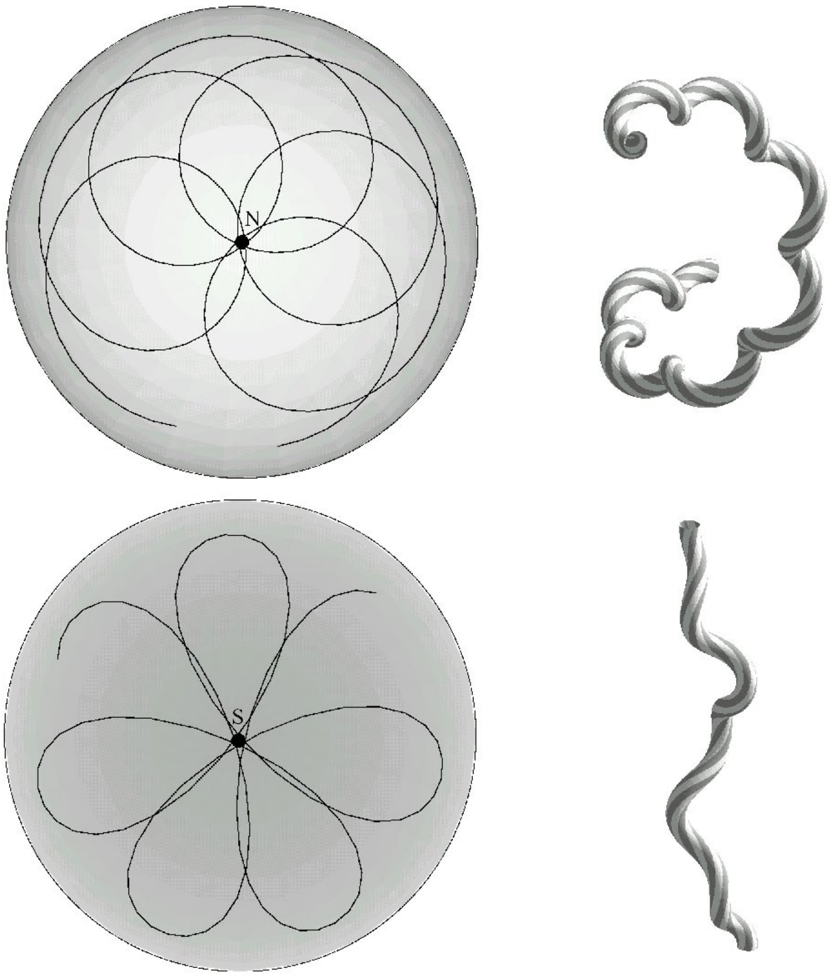

Figure 14

If the amplitude of the small scale pattern is large enough for two large scale turns to overlap, the shape becomes extremely complex, while the corresponding top orbit is quite simple :

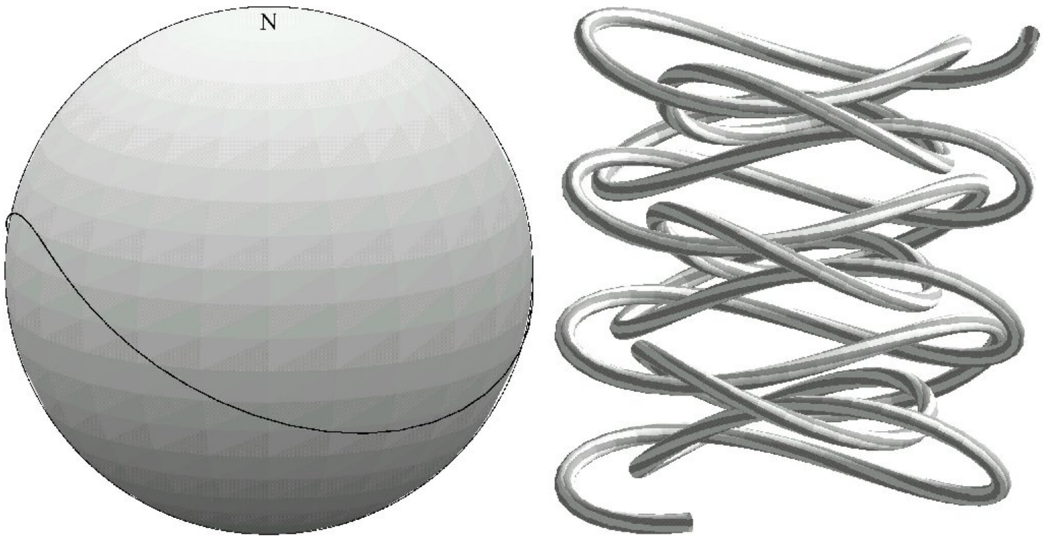

Figure 15

When the top orbit has turn-back points, this yields to a filament with points of vanishing curvature :

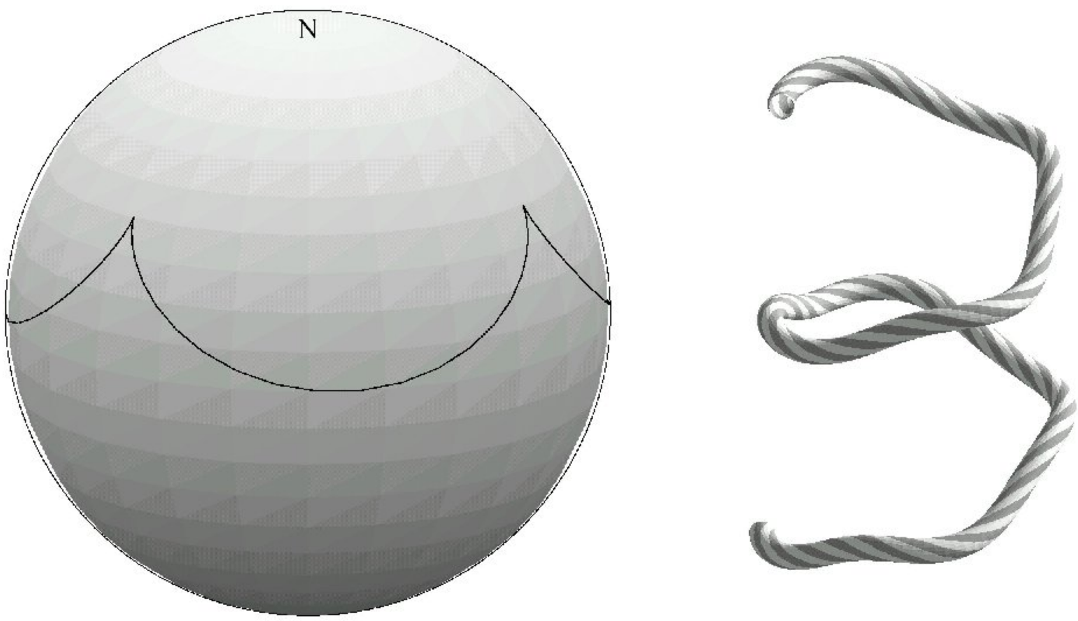

Figure 16

As the last example we have a closed filament :

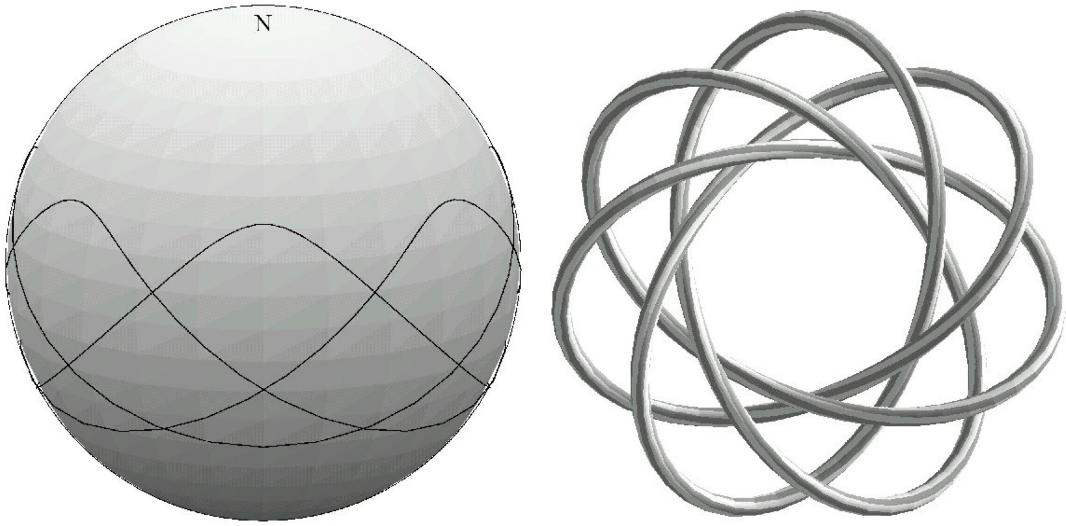

Figure 17

5 Estimation of dimensions for macroscopic filaments

Complex structures of filaments can be observed mostly on microscopic level. We can obtain such shapes with DNA molecules. However, we will make an estimation of possible dimensions for a macroscopic filament. This could be a piece of steel wire, for example.

There is no direct restriction for the radius R of such wire. The only condition is that the material rests in the elastic range of deformation where the bending force is linearly proportional to relative stretch. The estimation can easily be made ([3],[4]) by observing a segment of filament shown in picture 18 :

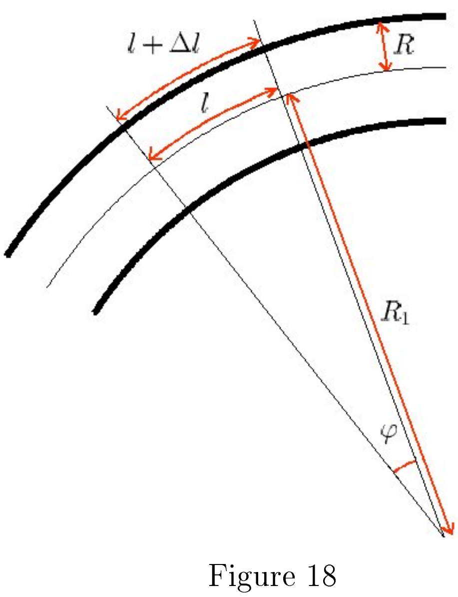

Figure 18

In the case of pure bending, the centerline of the filament is not deformed. Maximal deformation occurs on the outer edge of the filament. It can be calculated using the following geometric relation :

$$\varphi = \frac{l}{R_1} = \frac{l + \Delta l}{R_1 + R}. \tag{25}$$

From here we see that the relative stretch is equal to

$$\frac{\Delta l}{l} = \frac{R}{R_1}. \tag{26}$$

If we consider, as said before, a steel wire, we can find that the maximal relative stretch for the steel is of the order of 0.1% [3]. Beyond this limit, the relation between the stretch and the force which causes the bending is not linear. This means that we have the following relation :

$$\left(\frac{R}{R_1}\right)_{max} = 0.001. \tag{27}$$

Figure 19 shows an example of maximal curvature for a given radius of wire :

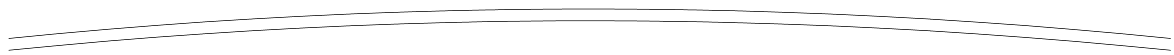

Figure 19

6 Conclusion

In this seminar we showed the relation between two completely different parts of physics which, however, are governed by the same equations and are thus directly comparable. Our intention was not to deal with mathematical properties of these equations but to observe the behaviour of their solutions. In this particular case we saw that the solutions are extremely varied and give very interesting results which are visually attractive. The main goal was to show how the comprehension of a given phenomenon can be reinforced when observed from a supplementary point of view.

7 References

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