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KELVIN SHIP WAVES

Abstract

One of the many successes of the theory of dispersive waves is the explanation of the distinctive wave patterns formed by ships in relatively deep water. The envelopes of these waves stands at a fixed angle of 19.5 degrees and have a characteristic feathered pattern. Such patterns are nearly always the same and are referred to as Kelvin Ship Waves.

This seminar presents a theoretical approach to the solution to the problem of formatting Kelvin Ship Waves and applies mathematical apparatus – Fourier-Bessel integral and method of stationary phase.

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1 Introduction

Ever since waves were studied, water waves have served the natural scientist as a model for wave theory in general, although they are much more complicated than acoustical or optical waves. As surface waves, they are bound to the common surface of two media, while the ordinary acoustic and optical waves are three-dimensional waves.

We shall discuss in this seminar waves with different symmetrical characteristics, such as *plane*, *circular*, *ship* waves.

One of the many successes of the theory of dispersive waves is the explanation of the distinctive wave patterns formed by ships, waterfowls, or fishing lines in relatively deep water. The envelopes of these waves stands at a fixed angle of 19.5 degrees and have a characteristic feathered pattern. Such patterns are nearly always the same and are referred to as Kelvin Ship Waves after Lord Kelvin (William Thomson) who was a leading 19th century physicist and applied mathematician.

2 Plane Gravity Waves in Deep Water

We assume the wave as a completely periodic phenomenon and express the time dependence in the form $e^{-i\omega t}$. Waves of a more general time dependence can be obtained by superposition of partial waves having different circular frequencies.

We further assume that the wave motion is generated *out of the state of the rest*, say, by a gust or a mechanical disturbance. Since the fluid can be considered as *in viscid* and *incompressible* ($\nabla \cdot \vec{v} = 0$), and since we shall also consider only the *potential field of gravity*, it follows from the conservation law ($\nabla \times \vec{v} = 0$) that the motion possesses a *velocity potential* ($\vec{v} = -\nabla\Phi$). Under these conditions Φ must satisfy potential Laplace equation

$$\nabla^2\Phi = 0 \tag{1}$$

In the case of a plane wave Φ only depends on *two* spatial coordinates x and y, where x is the direction in which the wave progresses and y the depth coordinate. The problem is independent of the third spatial coordinate z which is horizontal and orthogonal to the direction of propagation. Velocity potential Φ is thus a two dimensional potential as far as the space coordinates are concerned.

Let y be positive downward; since we wish to obtain the train of waves advancing in the positive x-direction, the x-dependence of f must be of a trigonometric form, or of the form of an imaginary exponential when written in the more convenient way. This leads to the following possibilities:

$$\Phi = Ae^{i(kx-\omega t)}e^{-ky} \qquad \text{or} \qquad \Phi = Be^{i(kx-\omega t)}e^{+ky} \tag{2}$$

k is the wave number ($k = \frac{2\pi}{\lambda}$) and ω circular frequency ($\omega = \frac{2\pi}{\tau}$), where λ is the wave length and τ the period.

We firstly assume the water is *infinitely deep*, that is, the y coordinate of the ground $y = h$ should be very large compared to λ. At the ground,

$$ky = 2\pi \frac{h}{\lambda} \to \infty. \tag{3}$$

This shows that the second option for potential (2) is not usable since it would yield infinite velocity amplitudes at the ground, but the first option satisfies all conditions that have been imposed so far.

The representation (2) contains three parameters A, ω, and k. A determines amplitude of the wave at $y = 0$; both A and ω depend on the particular form of the excitation. While these two quantities can be chosen freely, the wave number k must be determined in its ratio to ω, ($\frac{\omega}{k} = \frac{\lambda}{\tau} = V$), where V is the *velocity of propagation* of the waves.

For the determination of k we must utilize the *condition for the free surface* $p = 0$; the atmospheric pressure is taken as zero. This condition introduces a dynamic element into our theory while our argument so far has been wholly of kinematical nature.

The pressure p and the potential Φ are connected by Euler's equations, which we shall use in the integrated form of Bernoulli's equation. We shall, however, neglect the quadratic term $(\nabla\Phi)^2$, since we consider the amplitude factor A as a small quantity. The abridged form of Bernoulli's equation we use is

$$-\frac{\partial \Phi}{\partial t} + \frac{1}{\rho}(p + U) = const. \tag{4}$$

Here U is the *gravity potential per unit of volume* taken at the surface. Since y is counted positive downward, we have in general $U = -\rho g y$. Let the *equation of the surface profile* be $y = \eta(x, t)$. A positive η means a depression, a negative η an elevation, of the surface.

The constant in (4) is independent of the space coordinates, but in general dependent on time. The only function of time that in our case does not upset the periodicity and the uniform advancement of the wave is $const = F(t) = 0$.

Under these circumstances (4) assumes the simple form for $p = 0$

$$\frac{\partial \Phi}{\partial t} = -g\eta. \tag{5}$$

The function η must have again the form of a progressive wave, like the one we have set up for the velocity potential Φ. Thus

$$\eta = a\, e^{i(kx - \omega t)}. \tag{6}$$

The constant a introduced here is in general complex since it includes amplitude and phase of the surface function; also, a has the character of a small quantity as A.

We substitute the suggestion for potential Φ and η in (5), cancel the common exponential factor $e^{-i\omega t}$, expand $e^{-k\eta}$ in powers of $k\eta$ and neglect the products $A\eta, A\eta^2$, etc. as small quantities of higher order; so we obtain

$$i\omega A = ga. \tag{7}$$

This is a relation between A and a, but it is not the relation between k and ω we require. The latter is obtained by introducing a further *kinematic* condition: we stipulate that the motion of the surface must coincide at any time with the motion of those fluid particles that happen to be at the surface at this time. That such a condition must be satisfied is rather obvious; we specify, however, that the components of the two motions taken in the normal direction n of the surface should be equal, since motion of the fluid particles in the tangential plane does not change the shape of the surface, hence is immaterial for our problem. On denoting the velocity of the surface with $\vec{\mathbf{V}}$ and the particle velocity with $\vec{\mathbf{v}}$, our condition at the surface reads

$$V_n = v_n. \tag{8}$$

The component v_n, expressed by the velocity potential Φ, is $v_n = -\frac{\partial \Phi}{\partial n}$. However, if A is sufficiently small (the wave sufficiently flat), we can replace, with disregard of terms of second order, $\frac{\partial \Phi}{\partial n}$ by $\frac{\partial \Phi}{\partial y}$.
$\vec{\mathbf{V}}$ is treated correspondingly, we replace V_n by $\frac{\partial \eta}{\partial t}$, that is by the sinking speed of the surface. With these simplification (8) reads

$$\frac{\partial \eta}{\partial t} = -\frac{\partial \Phi}{\partial y}.$$

Substituting for η and Φ, we obtain, again after cancellation of the exponential on both sides,

$$-i\omega a = kA \tag{9}$$

Now the comparison of (7) and (9) yields at once

$$\boxed{\frac{A}{a} = \frac{g}{i\omega} = -\frac{i\omega}{k}} \tag{10}$$

Our conclusions from (10) are:

1. There is a phase difference of $\pi/2$ between the a wave and the A wave. If A is chosen a real quantity, which is permissible, a becomes purely imaginary; or, employing real representation, we can write for Φ in accordance with (2)

$$\Phi = A\cos(kx - \omega t)e^{-ky}, \tag{11}$$

and surfice profile becomes now

$$\eta = -\frac{\omega}{g} A \sin(kx - \omega t) \qquad \text{or} \qquad \eta = -\frac{k}{\omega} A \sin(kx - \omega t) \tag{12}$$

2. The relationship between k and ω given by Eq. (19) is $\omega^2 = gk$. Introducing here the phase velocity of propagation, we obtain

$$V^2 = \frac{\omega^2}{k^2} = \frac{g}{k} = \frac{g\lambda}{2\pi}, \qquad \boxed{V = \sqrt{\frac{g\lambda}{2\pi}}}.$$

The velocity of propagation depends on the wave length; long waves travel faster than smaller ones.

When the propagation velocity of a wave depends on the wave length, we speak of dispersion, using the expression borrowed from optics. The dispersion in a medium is normal when longer (red) waves have larger velocities (smaller index of refraction) than shorter (violet) waves. The behaviour of gravity waves in deep water thus corresponds to the case of normal dispersion in optics.

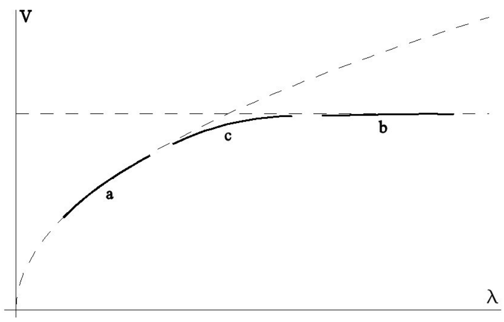
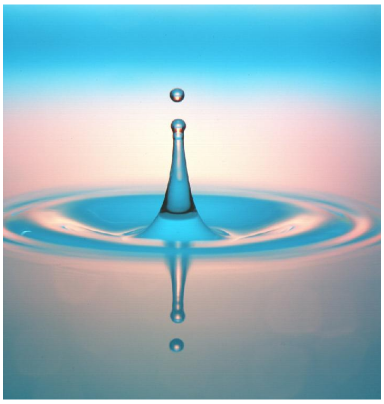


Figure 1: The phase velocity V as a function of the wave length λ.

Fig. **??** should make this clearer. V is represented by the upper half of an ordinary parabola which has the V axis as a tangent at $\lambda = 0$. Only the middle part a of the parabola has been drawn as a solid line, this being the region for which our assumption are actually valid. For, if the wave length λ keeps increasing, it finally becomes of the same order of magnitude as the depth of the water h, and our assumption (3) is no longer valid. On the other hand, if one goes to very small values of λ, gravity is no longer the decisive dynamic parameter, but surface tension takes the lead; this brings about a fundamental change of the dispersion law.

3 Circular Waves

The waves that are produced when a stone is thrown into the water form a series of concentric crests and troughs; their amplitudes are not constant, nor are the distances



between the crests. What one observes is a sharp decrease of amplitude and an increase of the distance between two subsequent crests, which seems to follow a peculiar law. This problem that appears so simple requires for its solution a considerable mathematical apparatus: we not only need Bessel functions and Fourier integrals, but we should have to use the method of steepest descent if we were to treat it in full accuracy.

For our analysis we shall replace the stone that hits the water surface by a standard disturbance: at $r = 0$ a cylindrical piston of radius r_0 is immersed in the water to a distance a from surface and suddenly withdrawn at the time $t = 0$. If we again denote the surface depression by η, the initial state is given by

$$\eta = \begin{cases} a & r < r_0 \\ 0 & r > r_0 \end{cases} \tag{1}$$

In preparation for the problem of a single disturbance we first consider the much simpler case of a periodic excitation.

3.1 The Periodic case. Introduction of Bessel Function

The excitation we have in mind works in a similar way as the device that produces waves in a swimming pool: a straight board subjected to a periodic motion excites plane progressive gravity waves advancing normal to the board.

Introducing cylindrical polar coordinates r, φ, y (y positive downward), we write the condition (2.1) for the velocity potential Φ in the form

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0$$

If we at first assume circular symmetry, Φ does not depend on φ and exponentially decreases with y. The solution of radial part of differential equation is $J_0(kr)$; regular at $r = 0$.

The solution for velocity potential

$$\Phi = AJ_0(kr)e^{-ky}e^{-i\omega t}$$

satisfies the condition for deep water, but it should also satisfy the conditions for the surface $y = 0$. The analytical form of the surface depression is assumed in correspondence to the form of Φ:

$$\eta = aJ_0(kr)e^{-i\omega t}.$$

Our expressions for Φ and η leads to correspondence between coefficients A and a from where we again obtain the normal dispersion law $\omega = \sqrt{gk}$.

The final expression for the potential becomes

$$\Phi(r, y, t) = \frac{ag}{i\omega} J_0(kr)e^{-ky-i\omega t} = \frac{g}{i\omega}\,\eta(r, t)\,e^{-ky}, \tag{2}$$

written also as a relationship between velocity potential and surface profile. We can see there is a $\pi/2$ phase difference between them.

3.2 Single Disturbance. The Fourier-Bessel Integral

The content of the Fourier integral theorem is this: an arbitrary function $F(x)$, provided it is not too irregular, can be represented by superposition of Bessel functions in the form

$$F(r) = \int_0^\infty kdkJ_0(kr)\int_{-\infty}^{+\infty} \xi d\xi F(\xi)J_0(k\xi), \qquad 0 < r < +\infty. \tag{3}$$

We apply this theorem to the initial state (1). Our function $F(r)$ is then given by

$$F(r) = \eta_{t=0} = a\int_0^\infty kdkJ_0(kr)\int_0^{r_0} \xi d\xi J_0(k\xi).$$

For the evaluation of the inner integral the relationship for Bessel functions $\int_0^{\rho_0} \rho J_0(\rho)d\rho = \rho_0 J_1(\rho_0)$ is used. Instead of the double integral we have

$$\eta_{t=0} = ar_0\int_0^\infty J_0(kr)J_1(kr_0)dk. \tag{4}$$

Now we can show that η and Φ at any later instant $t > 0$ have the following forms:

$$\eta = ar_0\int_0^\infty J_0(kr)J_1(kr_0)\exp(-i\sqrt{gk}\ t)dk, \tag{5}$$

$$\Phi = -iagr_0\int_0^\infty J_0(kr)J_1(kr_0)\exp(-ky - i\sqrt{gk}\ t)\frac{dk}{\sqrt{gk}}. \tag{6}$$

For a proof, we observe first that (5) takes the value of (4) if $t = 0$. Now the expression (5) and (6) are obtained by applying the same "operator"

$$r_0\int_0^\infty J_1(kr_0)dk\cdots$$

to the periodic solutions Φ and η from (2), with due regard to the dispersion law. Since Φ and η from (2) satisfy the differential equation of our problem and the boundary condition for $y = 0$ and $y = \infty$, the same must be true for our expressions (5) and (6). Furthermore they also satisfy the initial condition for $t = 0$ and thus are the required solutions.

3.3 Integration with Respect to k. The Method of Stationary Phase

The quantities of physical interest in circular waves are all connected with the surface functions η to which we therefore limit the following discussion. To evaluate η we first replace J_0 by its integral representation $J_0(\rho) = \frac{1}{\pi}\int_0^\pi e^{i\rho\cos\alpha}d\alpha$ and reverse the order of integration, obtaining the double integral

$$\eta = \frac{ar_0}{\pi}\int_0^\pi d\alpha \underbrace{\int_0^\infty dk J_1(kr_0)\exp(ikr\cos\alpha - i\sqrt{gk}\; t)}_{K(\alpha)}. \tag{7}$$

We denote the inner integral by $K(\alpha)$, put $2\tau = t\sqrt{gr_0}/r$ and introduce a new integration variable by $p = \sqrt{kr_0}$; with these notations we have

$$K(\alpha) = \frac{2}{r_0}\int_0^\infty pdpJ_1(p^2)\exp[i(p^2\cos\alpha - 2p\tau)r/r_0].$$

Here r/r_0 is by assumption a very large number and then the phase of the exponential function is some very large angle. The real (or imaginary) part of exponential function is cosine (or sine) of this angle. This is likely to be plus or minus. Now if we change the integration variable p for small amount Δp, the change in factor in front of ratio r/r_0 might be small, but not when multiplied by this large ratio. These small changes in p, generally, make enormous changes in phase, and our cosine or sine will oscillate exceedingly rapidly between plus and minus values. We can abbreviate the factor in front of ratio r/r_0 by

$$f(p) = p^2\cos\alpha - 2p\tau.$$

In general, the positive and negative oscillations of the integral will add to zero except at those places where the phase does not change, $\Delta f(p) = f'(p)\,\Delta p = 0$. In our case this will happen when

$$f'(p) = 0; \qquad p = p_0 = \frac{\tau}{\cos\alpha}.$$

This observation leads to a method of estimating the value of the integral known as the *method of the stationary phase.* Since the phase of the quickly oscillating exponential function becomes stationary at p_0, one limits the integration by elementary means.
It was particularly Lord Kelvin who applied this method expertly to the many problems of hydrodynamics and optics. This approximation, known also as Wentzel-Kramer-Brillouin (WKB) method, is very useful for estimating solutions to classical wave and Schrödinger equations.

Now the factor $f(p)$ can be written in the form

$$f(p) = \cos\alpha[(p - p_0)^2 - p_0^2]$$

The integral for $K(\alpha)$ then becomes

$$K(\alpha) = \frac{2}{r_0}\exp(-ip_o^2\cos\alpha\frac{r}{r_0})\int_{p_0-\epsilon}^{p_0+\epsilon} J_1(p^2)\exp[i\cos\alpha(p-p_0)^2 r/r_0)]p\,dp,$$

where the range of integration is now $\epsilon-$ neighbourhood of p_0. The factors $J_1(p^2)$ and p in the integrand are slowly variable compared with the exponential function and may be replaced by $J_1(p_0^2)$ and p_0. After substitution $s = q(p - p_0)$ and $q^2 = \frac{r}{r_0}\cos\alpha$ we obtain

$$K(\alpha) = \frac{2}{r_0}\exp(-ip_o^2\cos\alpha\frac{r}{r_0})J_1(p_0^2)\frac{p_0}{q}\int_{-q\epsilon}^{+q\epsilon} e^{is^2}ds. \tag{8}$$

The final evaluation is achieved by means of formula $\int_{-\infty}^{+\infty} e^{is^2}ds = e^{i(\pi/4)}\sqrt{\pi}$. The left member of this equation is our integral in (8) which limits become $\pm\infty$ if r/r_0 is made to approach infinity at constant ϵ and $\cos\alpha \neq 0$. Introducing this result in $K(\alpha)$ we obtain

$$K(\alpha) = \frac{2\sqrt{\pi}}{r_0}\exp(-ip_o^2\cos\alpha\frac{r}{r_0} + i\pi/4)J_1(p_0^2)\frac{p_0}{q}.$$

Now the value of p_0 is very small, provided $\frac{gr_0t^2}{4r^2} \ll 1$ (but we must exclude a small finite neighbourhood of $\alpha = \pi$). The Bessel function J_1 is, for small p_0, sufficiently well approximated by the first term of expansion $J_1(p_0^2) \cong \frac{p_0^2}{2}$. Taking the value of q we obtain finally for $K(\alpha)$

$$K(\alpha) = \sqrt{\frac{\pi}{\cos^7\alpha}\frac{r_0}{r^2}}\left(\frac{gr_0t^2}{4r^2}\right)^{3/2}\exp(-igt^2/4r\cos\alpha + i\pi/4). \tag{9}$$

This is the result we need to continue with Eq. (7).

3.4 Integration with Respect to α. Discussion of a Limiting Case

Returning to (7) we should restrict the integration interval to $0 < \alpha < \pi/2$ because $K(\alpha) = 0$ for $\pi/2 < \alpha < \pi$. On introducing the volume displaced by the initial impulse, $V_0 = \pi r_0^2 a$, (7) is transformed into

$$\eta = \frac{V_0}{r^2}\left(\frac{gt^2}{4\pi r}\right)^{3/2}\int_0^{\pi/2}\frac{d\alpha}{\cos^{7/2}\alpha}\exp(-i\frac{gt^2}{4r}\cos\alpha + i\pi/4). \tag{10}$$

We shall finally have to make $r_0 \to 0$. In order to obtain a finite effect in the limit, V_0 must be kept constant, that is to say, the depth of immersion a must approach ∞ in a definite way. We shall come back to this eventually.

The representation of η by (10) depends on the variable $u = \frac{gt^2}{4r}$. Our aim is to determine the asymptotic behaviour of η if $u \to \infty$. Now for large u the exponent in (10) becomes once more a rapidly varying function of α, so that the method of the stationary phase can again be applied. Similarly as before, we denote the factor of u in the exponent of (10) by $f(\alpha)$ and have

$$f(\alpha) = \frac{1}{\cos\alpha}, \qquad f'(\alpha) = \frac{\sin\alpha}{\cos^2\alpha}.$$

The critical α value is thus $\alpha = \alpha_0 = 0$ and consequently

$$\eta = \frac{V_0}{r^2}\left(\frac{u}{\pi}\right)^{3/2} \exp(-iu + i\pi/4) \int_0^{\epsilon} \exp(-iu\alpha^2/2) d\alpha.$$

We need the value of this integral for large u. Keeping ϵ constant, we apply the same arguments that led us to the result (9) in previous section. In the end we finally obtain

$$\eta = \frac{V_0}{\sqrt{2}\pi r^2} u e^{-iu}.$$

The real part of this result is the surface equation in which we are interested; we have

$$\boxed{\eta = \frac{V_0}{\sqrt{2}\pi r^2} u \cos u} \qquad u = \frac{gt^2}{4r} \tag{11}$$

η becomes infinite for $r = 0$, which is quite understandable since the depth of immersion a has become infinite in the limiting process $r_0 \to 0$ for fixed V_0. The amplitudes of the crests decrease according to u/r^2 or r^{-3}, the crests follow each other at the distance

$$\Delta r = \frac{8\pi}{g} \frac{r^2}{t^2},$$

as it is easily seen, if the phase of neighbouring crests $u = 2\pi n$, $u = 2\pi(n+1)$ are compared for constant t. Hence the wave length is no longer a constant as in our previous examples of wave motion, but increases at constant t with r^2 and decreases at constant r with t^2. Fig. **??** is a diagram of the surface profile η. Its appearance agrees well with that of a water surface disturbed by the fall of a small object like a stone or a raindrop.

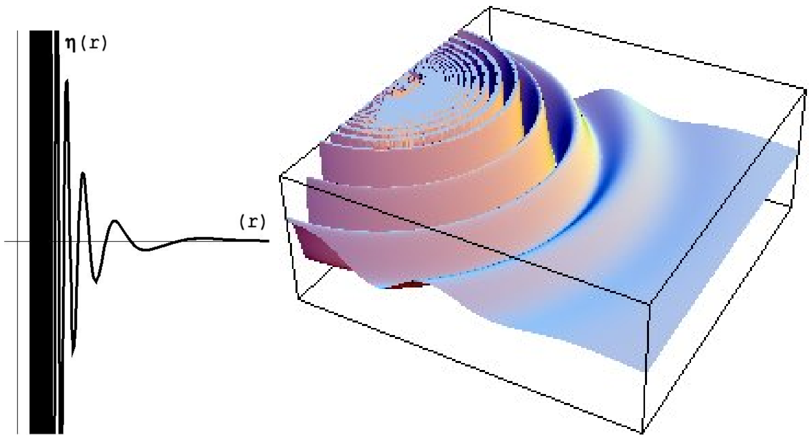


Figure 2: Shape of the water surface t seconds after the ring waves were excited; t must satisfy condition $u \gg 1$.

4 Ship Waves; Kelvin's Limit Angle

The wave pattern that is left behind by a ship at sea consists of a system of waves that envelopes the hull lengthwise and is interwoven with a system of cross waves. The two systems advance with the boat so as to be stationary relative to it. The beauty of this pattern is most impressive when viewed from an airplane or from the top of a high cliff, but the same phenomenon on a more modest scale develops behind a duck swimming in a pond.

In our analysis the object that produces the waves will be considered as a point. The problem can be formulated in the following way: The instantaneous location of the boat is the origin of a system of circular waves; this origin is in uniform rectilinear motion, its velocity being the speed of the boat v. Our task is to find *the result of the superposition of the successive circular waves.* That it will be stationary relative to the boat is evident, but the detailed structure of the wave pattern is surprising enough and can only be unravelled by a careful analysis.

Let be O the location of the boat at the time $t = 0$, and Q its location t seconds earlier so that $QO = vt$. We wish to find the ordinate of the water surface η at the any field point P. It is to be compounded of all ordinates η_t, that were produced at earlier instants t by means of the formula

$$\eta = \beta \int_{-\infty}^{0} \eta_t dt. \tag{1}$$

The factor β must have the dimension of reciprocal time; we put it equal to v/l. For the length l there is no other choice other than the cube root of the initial displacement V_0. Thus we obtain

$$\eta = C \int_{-\infty}^{0} \frac{1}{r_t^2} u_t \exp(-iu_t) dt, \qquad C = \frac{V_0^{2/3} v}{\sqrt{2}\pi} \tag{2}$$

where r_t is the distance QP, that is, the distance between the field point P and the location of the source of disturbance, t seconds ago.

Now let P have the polar coordinates r and ϑ relative to the pole O; r and ϑ are therefore independent of t. If ϑ is an acute angle we obtain $r_t^2 = r^2 + v^2t^2 + 2rvt\cos\vartheta$, where t is negative. We put again $u_t = \frac{g}{4}\frac{t^2}{r_t} = f(t)$. The representation (3.11) which we have applied was computed under the assumption $u \gg 1$. With this condition, also $f(t)$ becomes again a rapidly varying function so that the method of stationary phase can be applied. We then have to find the roots of the equation $f'(t) = 0$.

$$\frac{4}{g} f'(t) = \frac{2t}{r_t} - \frac{t^2}{r_t^3}(v^2 t + rv\cos\vartheta) = \frac{t}{r_t^3}(v^2t^2 + 3rvt\cos\vartheta + 2r^2).$$

Hence the roots of $f'(t) = 0$ are

$$t_1 = -\frac{2}{3}\frac{r}{v}\left(\cos\vartheta + \sqrt{\cos^2\vartheta - \frac{8}{9}}\right), \qquad t_2 = -\frac{2}{3}\frac{r}{v}\left(\cos\vartheta - \sqrt{\cos^2\vartheta - \frac{8}{9}}\right). \tag{3}$$

Now, in order to fall in our integration interval $-\infty < t < 0$, the roots not only have to be negative, they must also be real. This implies

$$\cos^2 \vartheta > \frac{8}{9}, \qquad |\vartheta| < \vartheta_0,$$

Where ϑ_0 denotes the limiting angle

$$\cos^2 \vartheta_0 = \frac{8}{9} \qquad \text{or} \qquad \tan \vartheta_0 = \frac{1}{\sqrt{8}}; \qquad \boxed{\vartheta_0 = 19^0 28'}. \tag{4}$$

The angle was first determined by Lord Kelvin. For $|\vartheta| > \vartheta_0$ there is no such t value as this would make the phase stationary, that is to say, the whole wave pattern is bounded on either side by a straight line forming the angle ϑ_0 with the direction of the motion of the boat. This is schematically shown in Fig. **??**.

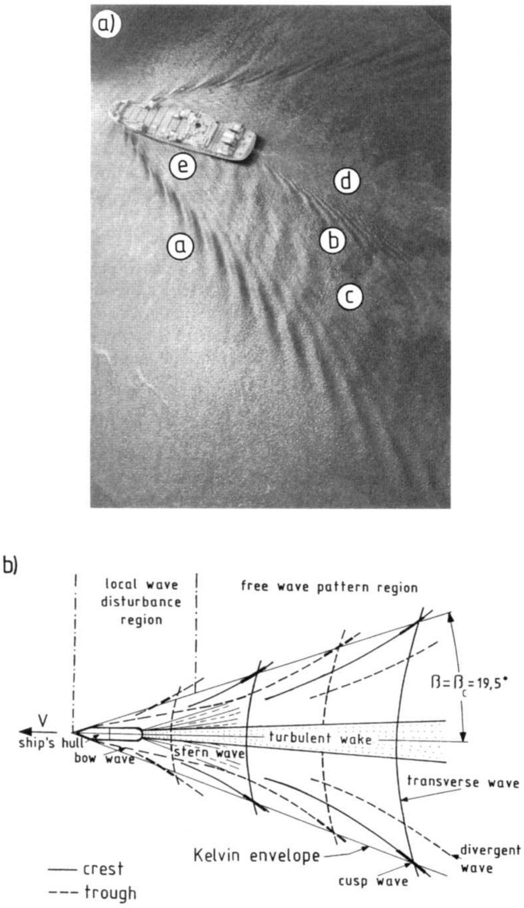


Figure 3: Lengthwise and transverse waves.

The interference pattern itself can be understood on the basis of the integral (2) which essentially reduces to the two contributions of the neighbourhoods of t_1 and t_2. These contributions contain the phase factors $\exp[-if(t_1)]$ and $\exp[-if(t_2)]$. By putting $f(t_1)$ and $f(t_2)$ constant, one obtains the two systems of curves, the lengthwise and transverse waves mentioned before. In Fig. **??** the successive crest of two systems have been drawn.

Let us start with a field position on the limiting line $\vartheta = \vartheta_0$. The two values t_1 and t_2 coincide. The direction of the curves of the uniform phase $f(t_1) = f(t_2) = const$ in this field point is given by a circular arc element. Both curves of constant phase pass through this area in the same forward direction. Considered as one curve they form a *cusp*.

If now ϑ is decreased and if phase $f(t_2)$ remains constant we obtain the field poind at which the transverse wave intersects the course of the boat and the direction of the curve is perpendicular to the course.

If, on the other hand, we choose the t values for decreasing ϑ so that $f(t_1)$ remains constant, the projection into the course of the boat, $-vt_1$, decreases. The slope of the lengthwise wave becomes flatter and should become tangential to the direction of travel at O if our method were still valid in the neighbourhood of O. This, however, is not the case: our method of stationary phase brakes down for a short running time t. Nevertheless the general shape of the lengthwise waves has thus been clarified.

Numerical calculation of integral (2) does not produce as observable results as an interpretation made by using the method of stationary phase. The oscillations between Kelvin envelopes are too rapid to produce the characteristic feathered pattern.

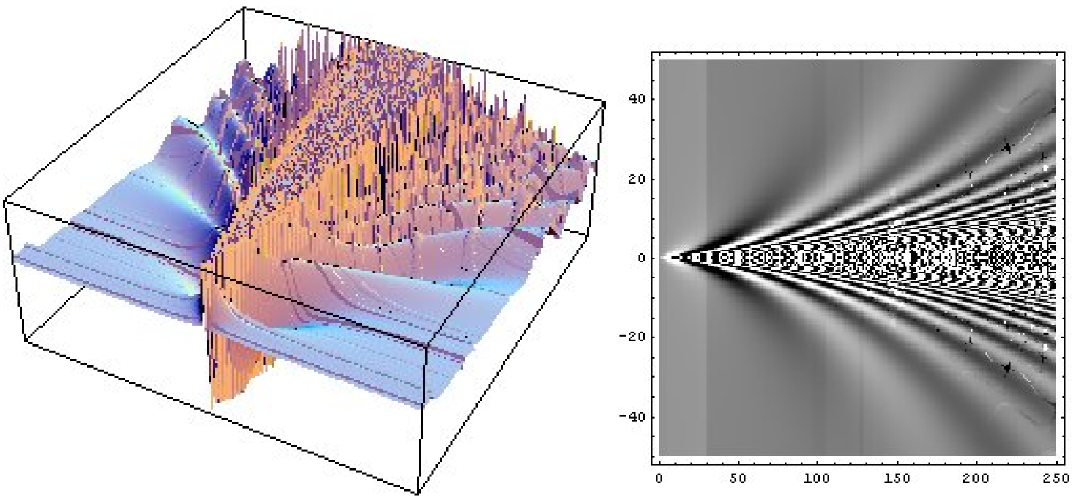


Figure 4: Numerical integration of integral (2) with *Mathematica*

5 The Conclusion

The most noticeable feature of Kelvin ship waves is the fact that they remain in tight groups. This characteristic of ship waves is due to the fact that water waves in water of moderate depth are dispersive, i.e., their wave speed depends on their length or their frequency. The Kelvin waves are actually a wave packet which is travelling with the so-called "group velocity".

This pattern differs from that predicted by the shallow water theory. The shallow water theory is mathematically analogous to the theory of the compressible flow of perfect gases. In shallow water, definite bow waves are formed at supercritical speeds which stand at an angle which depends on the speed of the ship.

We are all familiar with well known photographs of projectiles moving with supersonic speed, first obtained by Austrian philosopher and physicist Ernst Mach. The projectile in Mach's theory is shrunk to a moving point, just as a steamboat before, from which compression waves originate continually. At the time of observation the wave that has originate t seconds ago has now a speed over a spherical surface of radius $r = c\,t$ where c is the sound velocity. In the meantime the projectile has travelled a distance $x = v\,t$. The spherical shell so produced has an enveloping circular cone, the Mach cone. Half its apex angle is called the Mach angle, given by

$$\sin\vartheta_0 = \frac{r}{x} = \frac{c}{v}$$

and is approaching zero with increasing v, in contrast to the limiting angle ϑ in the case of the ship waves.

The reason for this different behaviour is found in the dispersion. The sound waves travel at fixed velocity c without dispersion. The deep water waves follow the dispersion law $V = \sqrt{g/k}$. With the velocity thus depending on the wave length, there exist waves which at any given speed of the boat run along with the boat, while in the Mach phenomenon all waves are overtaken by the projectile. Thus the fact that ϑ_0 is independent of v becomes understandable.

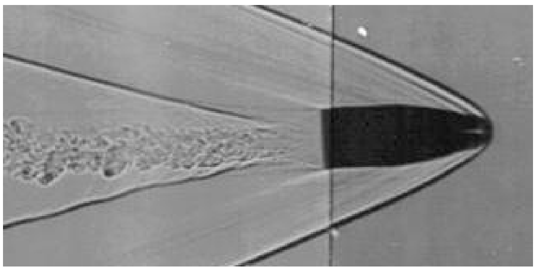


Figure 5: Mach cone of the projectile

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