Univerza *v Ljubljani*
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Seminar
MINORITY GAME

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Abstract

In this paper we discuss the application of methods of statistical physics in modeling inductive reasoning, namely the Minority Game. We present its history, proceed to a description of a simple adaptive multi-agent model of financial markets, define the quantities of interest. Next the physical properties of the basic and various extensions to the Minority Game are briefly discussed, we then proceed by defining quantities of interest and analytically solving the basic model. In the last section we apply the before-mentioned model to real-world financial markets and briefly discuss the required assumptions and consequences.

Contents

1 Introduction

A perennial question in social sciences is how do humans reason in situations that are ill-defined or complicated? According to modern psychology humans are only moderately good at deductive logic. But are superb at recognizing, seeing and matching patterns - behaviour that clearly confers evolutionary benefits. When faced with a complicated problem, patterns are first searched for and looked at. The problem is then simplified and worked on by using these patterns to construct internal models, hypotheses and such. When feedback from the environment comes in, we either strengthen or weaken our beliefs or in extreme cases discard our hypotheses and replace them with new ones[1]. Such behaviour is inductive. In this paper we shall show how we can advance from a relatively simple model of inductive reasoning to complex adaptive models which under certain assumptions reproduce the behaviour of traders participating in a real-world market.

2 The 'El Farol Bar' Problem

William Brian Arthur conceived the 'El Farol Bar' Problem in his 1994 paper on Inductive Reasoning and Bounded Rationality. Frustrated by the premise of *perfect rationality* in modern economics, which states that agents are equipped with rational minds, know everything and understand it with implicitly infinite capacity for information, he posed a problem where the rationality of the agents was *bounded*.

Modelling bounded rational behaviour is difficult, because the degree and type of imperfection have to be specified since there are so many ways of being imperfect. Because it is perfect by definition, deductive rationality is much easier to model and has thus become the cornerstone of modern economics.

Arthur came up with the 'El Farol Bar' problem to illustrate the question of inductive reasoning and how it could be modeled. N people decide independently each week whether to go to a bar that offers live music on a certain night. Since space is limited, the evening is enjoyable only if the bar is not too crowded, that is if fewer than xN of the possible N are present, where $x \in (0, 1)$. There is no way to tell the numbers coming for sure in advance, therefore a person goes if he expects fewer than xN to show up, or stay home if he expects more than xN to go[2]. Choices are unaffected by previous visits, there is no communication among the agents and the only information available to them is the attendance numbers of those who came in past weeks.

Since there is no obvious model that all agents can use to forecast attendance and base their decisions on, no deductive solution is possible. Not knowing which model other agents might choose, a reference agent cannot choose his in a well-defined way. The agents are thrown into a world of induction. Any common expectations get broken up - if all believe *few* will go, *all* will go - thus invalidating that belief. Similarly, if all believe *most* will go, *nobody* will go, invalidating that belief. Expectations are forced to differ.

By giving the agents a limited number of strategies - certain rules of thumb, and instead of deciding which of them is supperior to all the others prior to playing, agents would evaluate them during the game itself, according to their performance and adjust their behaviour accordingly. The key is to give agents certain predefined strategies and thus enormously reduce the complexity of their decision-making problem.

Given the recent attendance of the bar:

...44 78 56 15 23 67 84 34 42 76 40 56 22 35

And particular hypotheses or strategies of predicting the next week's number to be:

[1]An example would be studying your opponents chess game, predicting his game by recalling their play in past games.

[2]In the original formulation of this problem Arthur used $N = 100$ and $x = 0.6$ or $xN = 60$ seats in the bar.

- the same as last week's [35]
- mirror image around 50 of last week's [65]
- the Ultimate Answer to the Ultimate Question of Life, The Universe, and Everything [42]
- a rounded average of the last four weeks [49]
- trend of last 8 weeks, bounded by (0,100) [29]
- the same as 2 weeks ago (a two period cycle detector) [22]
- etc...

As seen above, the rationality of the agents is very limited; the given strategies have no merit other than being diversified. The beauty of the model is that even with bounded rationality, no coordination among the agents, the bar attendance evolves to the optimal value, as is seen on Figure 2.1.

The model itself is Darwinian in nature, the agents are denied a priori information. As compensation they are equipped with more alternatives and selection between competing predictors is performed afterwards in the strategy space, according to the actual outcome of gain or loss in each round.

While almost any set of strategies will allow the equilibrium to be reached, fluctuations require more elaborate modelling. While one may think that the seemingly random fluctuations around the equilibrium are of no importance, they in fact hide such important information as whether markets are efficient or not, the nature of the interactions between agents with different aims and trading horizons, whether and why volatilities[3] are excessive or not and lastly whether market equilibria exist.

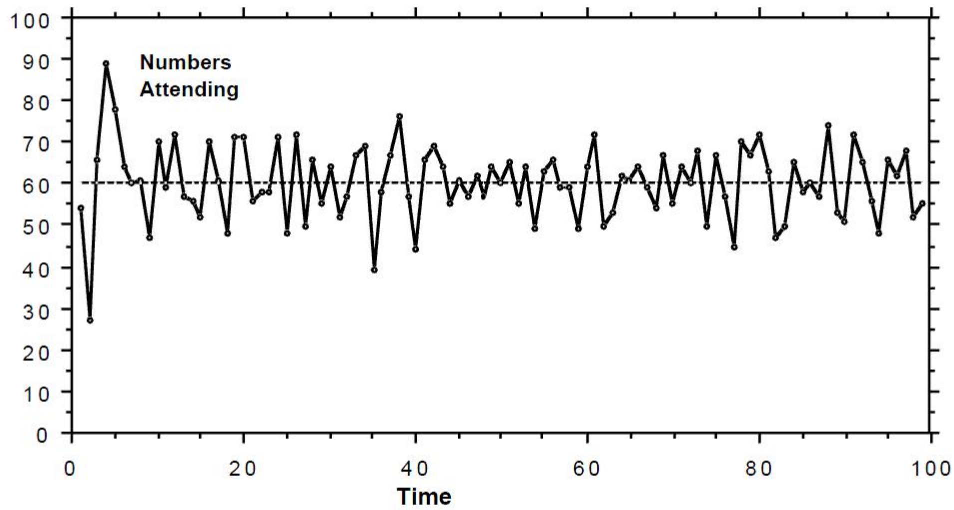

Figure 2.1: Bar attendance in the first 100 weeks. [1]

[3] a/n; economic term for deviations.

3 Minority Game

The basic Minority Game was formulated by physicists Damien Challet and Yi-Cheng Zhang in 1997. By taking a more pragmatic approach they avoided the main obstacle of the El Farol problem - namely the definition of agents' strategies. With N agents, the attendance can take N possible values each week in the past M weeks. This makes N^M possible combinations of information about the past. If strategies predicting attendance are based on past history, there are N possible predictions for each combination of information and thus N^{N^M} possible predictions for any combination. Hence in the El Farol problem there are N^{N^M} possible predictor strategies, which is prohibitive since it scales with population N and presents a heavy task when searching for the best strategy.

The first simplification they made was that agents don't have to predict the exact attendance size, but only whether it is worthwhile to go or not, thus the number of strategies simplifies to 2^{N^M}. Forcing agents to consider the precise past attendance sizes is somewhat redundant, thus only information enconding the past M correct choices suffices. By doing this the dependence on population is eliminated from the strategy space and only 2^{2^M} strategies remain.

Secondly, convergence to the bar's optimal comfort level is unimportant, the real challenge is to reduce fluctuations around the equilibrium. Hence the model is symmetrized by assuming that the bar can contain half of the players.

This definition of the Minority Game is in essence a binary symmetric version of the El Farol bar problem and as the choices remain unspecified, it becomes a generic model of a complex adaptive system.

3.1 The original definition

The basic Minority Game models market interactions between a population of N agents competing in repeated games, where N is an odd integer. At each time step (round) of the game, every agent has to choose between one of two possible actions, either *"buy"* or *"sell"*. These two bids are represented by "-1" and "1".[4] This is represented by assuming that each agent $i = 1, ..., N$, at time t, can either do the bid $a_i(t) = +1$ or the opposite bid $a_i(t) = -1$. The minority choice wins the round at that time step and all the winning agents are rewarded.

At the start of the game, S *strategies* are randomly drawn by the agents from a strategy pool. These strategies help the agents in their decision making throughout the game. There are no a priori best strategies, i.e., strategies that are better or preferred. We represent them in the form of a table where each strategy has two columns, a "history" and "prediction" column. Each row of the history column[5] contains in it a string of M bits, which represent the *history* of past winning choices in the previous M time steps. The parameter M is known as the *brain size* or *memory* of the agents. Table 1 contains an example strategy for $M = 3$. The number of all possible signals for a Minority Game with memory M is 2^M and thus the total number of possible strategies in the strategy pool is 2^{2^M}.[6] Additionally the past history of the game is itself an evolving time function and is denoted by $\mu(t)$.

All the S strategies have to predict at every round of the game, and points are given to those strategies (no matter whether they had been selected by the agent to make real actions) that give correct predictions. The scores of all strategies available to an agent are accumulated and are known as the *virtual points*, *virtual scores* or the *cumulated payoffs* of the strategies. In the basic Minority Game they start at zero. At every round of the game, agents make their decisions according to the strategy with

[4] a/n; in older Minority Game literature "0" and "1" were used. Note that this does not significantly change the results, merely the derivation.

[5] also known as signal or information.

[6] It should be noted, that even for a relatively small brain size, $M = 5$, the total number of all possible strategies is huge $(4,294,967,296)$.

the highest virtual score at that time step. If one or more strategies have the same highest score, one of these strategies is randomly employed by the agent. Agents themselves who make the winning decisions are also awarded points, these are called the *real points* (distinguishing them from the *virtual points* of the strategies).

History	Prediction
000	0
001	1
010	0
011	1
100	0
101	0
110	0
111	1

Table 1: A strategy example for $M = 3$ using "0" and "1" as actions. [Author]

The collective sum of actions from all agents at time step t is defined as *attendance* $A(t)$. If we denote the prediction of strategy s of agent i under the information $\mu(t)$ as $a^{\mu(t)}_{i,s}$ at time t, which can be either "-1" or "1", then each strategy can be represented by a 2^M-dimensional vector $\vec{a_{i,s}}$ where all entries are either "-1" or "1". The attendance $A(t)$ and individual gain of agent i is then defined as

$$A(t) = \sum_{i=1}^{N} a^{\mu(t)}_{i,s_i(t)} = \sum_{i=1}^{N} a_i(t), \quad and \quad g_i(t) = -a_i(t)A(t) \tag{3.1}$$

where $s_i(t)$ denotes the best strategy of agent i at time t,

$$s_i(t) = \underset{s}{argmax} U_{i,s}(t) \tag{3.2}$$

and $a_i(t)$ denotes the real actions or *bids* of agents

$$a_i(t) = a^{\mu(t)}_{i,s_i(t)} \tag{3.3}$$

Along with $A(t)$, the cumulative virtual score or payoff $U_{i,s}$ of a strategy s of agent i is updated as follows

$$U_{i,s}(t+1) = U_{i,s}(t) - a^{\mu(t)}_{i,s} A(t) \tag{3.4}$$

The negative sign in Eq. (3.4) corresponds to the minority nature of the game, i.e., when $a^{\mu(t)}_{i,s}$ and $A(t)$ are of opposite signs, points are added to the strategy, this is also known as a *linear payoff scheme*[7].

Thus, agents are considered to be adaptive since they can choose between their s strategies and the relative preference of using strategies is changing with time and is adaptive to the market outcomes. Agents are also considered to be inductive, since their decisions are based on the best choice they know, with their limited number of strategies, but not the global best choice given by all possible strategies, i.e., the one with the highest virtual score within the entire strategy space.

As the total number of agents N in the game is an odd integer, the minority side can always be determined and the number of winners is always less than than the number of losers, implying the

[7] The other possibility is the so-called *step payoff scheme* where the last term in Eq. (3.4) becomes $-sign[a^{\mu(t)}_{i,s}(t)A(t)]$ ("-1", "1" as actions) or $-sign[(2a^{\mu(t)}_{i,s}(t)A(t)]$ (with "0", "1" for actions).

Minority Game to be a *negative-sum game*[8]. Due to the minority nature of the game and the symetricity of both actions, the time average[9] of attendance $A(t)$ always[10] has a value of 0. We denote σ^2 to be the variance of attendance, also known as *volatility*, given by

$$\sigma^2 = \langle A^2 \rangle - \langle A \rangle^2 = \langle A^2 \rangle \tag{3.5}$$

As an example we can consider two extreme cases of game outcomes. In the first one, there is only one agent choosing one side while everyone else chooses the opposite side. There is only one winner and $N-1$ losers which is considered to be highly inefficient in the sense of resource allocation, since supply and demand are highly unbalanced. In the second case, $(N-1)/2$ agents pick one side whereas $(N+1)/2$ agents pick the opposite side. There are $(N-1)/2$ winners, supply and demand are maximally balanced. Therefore it becomes important to minimize fluctuations of attendance to benefit the agents as a whole.

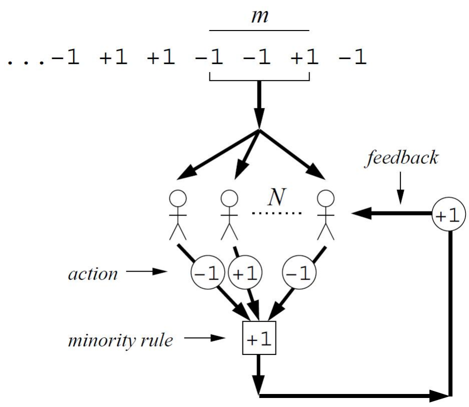

Figure 3.1: A schematic overview of the inductive learning model implemented in the Minority Game. [5]

3.2 Physical properties

Nekaj o fizikalnih lastnostih.

There are several parameters in the basic Minority Game the list includes N, M and S which correspond to the population size, the memory of the agents (sometimes replaced by the complexity of the total amount of possible information $P = 2^M$) and the number of strategies that each agent is allowed to hold. The predictions $a^{\mu}_{i,s}$ of strategies fixed for every agent throughout the game and are in a physical sense considered to be quenched disorders of the system. The game is also a highly frustrated model because not all the agents can be satisfied simultaneously.

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[8] Aggregate gains and losses of participants are less than zero, i.e., $\sum_i -a_i(t)A(t) = -A^2(t)$, contrast to a *zero-sum game (constant sum)* where they always sum to zero (a constant).

[9] A quick note on averages: The temporal average of a given time-dependent quantity $R(t)$ is defined $\langle R \rangle = \lim_{T\to\infty} \frac{1}{T} \sum_{t=1}^{T} R(t)$. This can be decomposed into conditional averages on histories $\langle R^\mu \rangle = \lim_{T\to\infty} \frac{2^M}{T} \sum_{t=1}^{T} R(t)\delta_{\mu(t),\mu}$, *with* $\langle \delta_{\mu(t),\mu} \rangle = 1/2^M$, this is the temporal average of the quantity $R(t)$ subject to the condition that the actual history $\mu(t)$ was μ. Finally, averages over histories μ of a quantity R^μ are defined as $\overline{R} = \frac{1}{2^M} \sum_{\mu=1}^{2^M} R^\mu$.

[10] In the case of "0" and "1" as actions, the time average of $A(t)$ becomes $\langle A \rangle = N/2$.

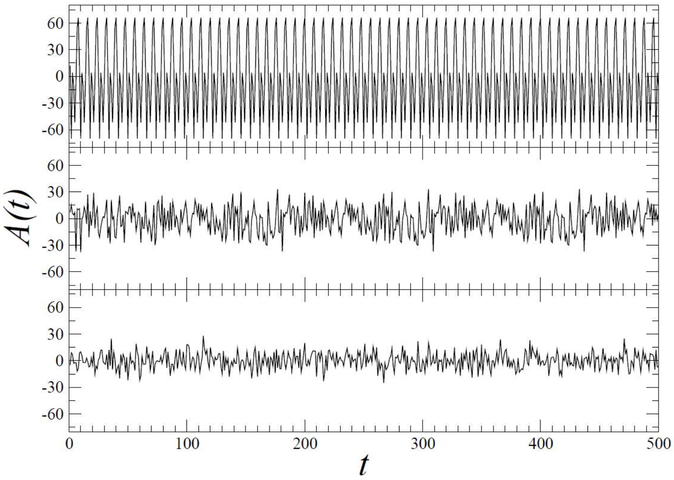

Figure 3.2: Time evolution of attendance $A(t)$ for the basic Minority Game with a linear payoff scheme. Parameters $N = 301$ and $S = 2$. Panels correspond to $M = 2, 7, 15$ from top to bottom. [5]

The macroscopic behaviour of the system does not depend independently on the parameters N and M, but instead depends on the ratio

$$\alpha = \frac{2^M}{N} = \frac{P}{N} \tag{3.6}$$

which also serves as the most important control parameter in the game. The volatility σ^2/N for different values of N and M depend only on the ratio α, a plot of σ^2/N versus α is shown in Fig. 3.3.

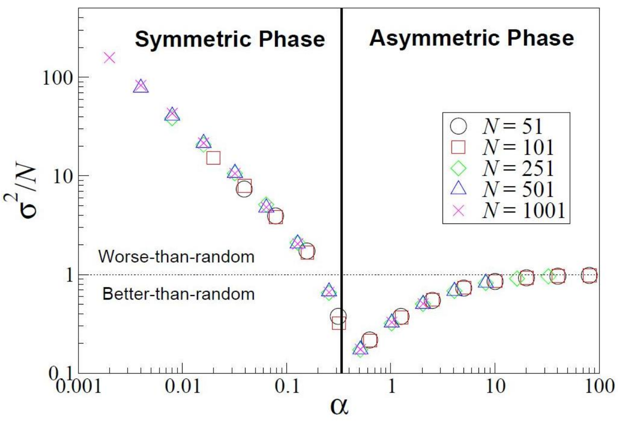

Figure 3.3: Simulation results of volatility σ^2/N as a function of the control parameter α for games with $S = 2$ strategies for each agent averaged over 100 samples. A linear payoff scheme has been used. The solid line shows the critical value of $\alpha = \alpha_c \approx 0.3374$. [6]

The dotted in line in Fig. 3.3 corresponds to the coin-toss limit (random choice), in which agents play by making random decisions at every round of the game. The value of volatility in the coin-toss limit can be obtained through a binomial distribution of agents' actions, with probability 0.5 of choosing either action, then $\sigma^2/N = 0.5(1-0.5)\cdot 4 = 1$. When α is small, the volatility of the game is larger than the coin-toss limit which implies the collective behaviour of agents is worse than random choice. This is known as the *worse-than-random* regime. When α increases, the volatility decreases and enters a region where agents are performing better than choosing randomly, this is known as the *better-than-random* regime. The volatility then reaches a minimum value α_c which is smaller than the coin-toss limit. When α further increases, the volatility increases again and approaches the coin-toss limit.

The phase transition point α_c has a numerical value of $\alpha_c = 0.3374$ for $S = 2$ strategies. For $\alpha < \alpha_c$ the volatility σ^2 is proportional to N^2. Beyond the transition point $\alpha > \alpha_c$ the volatility σ^2 is proportional to N. In the case of more than two strategies S, the phase transition point approximately scales as

$$\alpha_c(S) \approx \alpha_c(S=2) + \frac{S-2}{2} \tag{3.7}$$

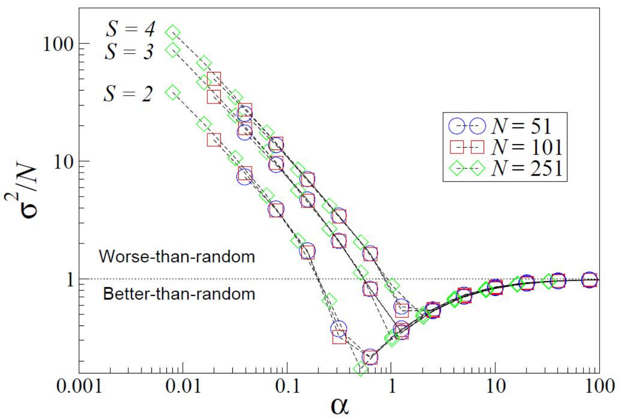

Figure 3.4: Simulation of σ^2/N against α for games with $S = 2, 3, 4$ strategies for each agent averaged over 100 samples with a linear payoff scheme. Volatility increases with the number of strategies S per agent. [6]

Distributions of winning probabilities for a particular action after different history strings also completely differ in both regimes. By taking a look at $P(1|\mu)$ defined as the conditional probability of action "1" turning out to be the minority group after the history μ. The histogram in Fig. 3.5(a) which is almost uniformly flat implies that below α_c there is no extractable information from a history string of length M, since both actions have an equal probability of winning (0.5). Beyond the phase transition $\alpha > \alpha_c$, Fig. 3.5(b), there is an unequal winning probability of the two actions. The phase $\alpha < \alpha_c$ is also known as the *unpredictable* or the *symmetric* phase, as agents are incapable of predicting the winning actions from the past M-bit history. Contrast to this the phase $\alpha > \alpha_c$ is called the *predictable* or *asymmetric* phase since the winning probabilities are asymmetric.

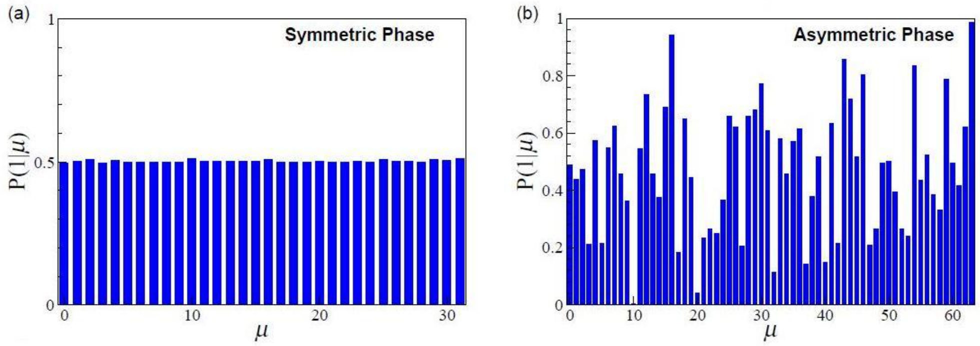

Figure 3.5: Histogram of probabilities $P(1|\mu)$ of winning action to be "1" given the history string μ. Plots are decimal representations of binary strings of information. $N = 101$ agents and $S = 2$ strategies with a linear payoff scheme. a) symmetric phase with $M = 5$ ($\alpha = 0.316$) b) asymmetric phase with $M = 6$ ($\alpha = 0.634$) [6]

As a measure of non-uniformity of the winning probabilities the *predictability* H is defined accordingly

$$H = \frac{1}{2^M} \sum_{\mu=1}^{2^M} \langle A|\mu \rangle^2 \tag{3.8}$$

In the symmetric phase $\langle A|\mu \rangle = 0$ for all μ as the actions of "-1" and "1" are equally likely to appear after the history string μ. Hence $H = 0$ in the symmetric phase. In the asymmetric phase $\langle A|\mu \rangle \neq 0$ for all μ as there is a bias towards one of the both actions, i.e., "-1" or "1" are not equally likely after μ. Thus H begins to increase in the asymmetric phase.

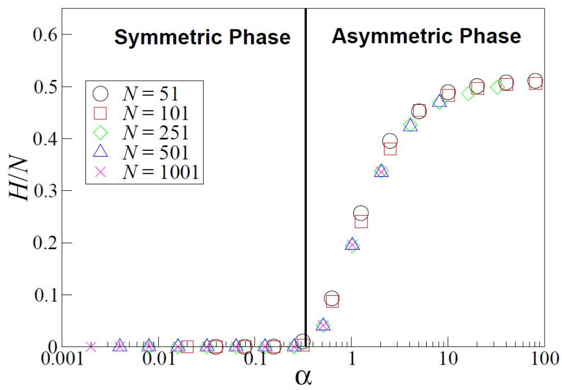

Figure 3.6: Predictability H as a function of the control parameter α, for games with $S = 2$ strategies averaged over 100 samples with a linear payoff scheme. [6]

It is also possible to introduce probabilistic fashion to the decision-making process of agents in the Minority Game. This model is known as the Thermal Minority Game and we shall return to it later. It introduces *temperature* which can be implemented in the original game by changing the way agents employ their strategy s. Instead of choosing the best strategy for sure, agents employ it with probabilities $\pi_{i,s}$ given by

$$Prob\{s_i(t) = s\} = \pi_{i,s} = \frac{e^{\Gamma U_{i,s}(t)}}{\sum_{s'} e^{\Gamma U_{i,s'}(t)}} \tag{3.9}$$

where $s_i(t)$ denotes the employed strategy by agent i at time step t. Similarly to physical systems Γcorresponds to the *inverse temperature* of individual agents. It is also possible to interpret it as the *learning rate* of the system. Namely the dynamics of scores take a time of approximately $1/\Gamma$ to learn a difference in the cumulated payoffs of strategies. The final (steady) state of the system and volatility are both independent of Γ in the asymmetric phase. In the symmetric phase the final state of the system is dependent on Γ and the volatility of the system increases with increasing Γ. This property is somewhat different to ordinary physical systems where fluctuations always increase with increasing temperature.

In addition to dependence on Γ in the symmetric phase, the final state of the system is dependent on the initial conditions. For games with the same set of strategies among agents, the final state of the system depends on the initial bias of virtual scores of strategies. For the case of $S = 2$, the volatility of the system is smaller if larger differences are assigned to the initial virtual scores (a larger initial bias) of the two strategies. A system where the final state is dependent on the initial state of variables corresponds to the spin glass phase, or the replica symmetry breaking phenomenon in physical systems. The symmetric phase corresponds thus corresponds to the behaviour of broken replica symmetry. On the other hand, in the asymmetric phase the final state of the system and volatility do not depend on initial conditions. The asymmetric phase thus corresponds to the replica symmetry phase in physical systems.

3.3 Variants of the Minority Game

3.3.1 The Thermal Minority Game

In addition to the stochasticity in choosing strategies as in Eq. (3.9) several other modifications are made to the basic Minority Game in this variant. The strategy becomes a vector in the 2^M-dimensional space $\mathbf{R}^P$ denoted as $\vec{a}_{i,s}$, with $||\vec{a}_{i,s}|| = \sqrt{2^M}$. The strategy space is thus the surface of a 2^M-dimensional hyperspace and the components of the strategies are continuous contrast to the discrete strategies of the basic model.

The information processed by the strategies is a random vector $\vec{\eta}(t)$, with unit length in $\mathbf{R}^P$. The action of an agent or the bid of a strategy is no longer an integer and is obtained through the inner product of strategies and information $\vec{a}_{i,s} \cdot \vec{\eta}(t)$. The attendance is then

$$A(t) = \sum_{i=1}^{N} \vec{a}_{i,s_i(t)} \cdot \vec{\eta}(t) \tag{3.10}$$

The cumulated payoff of a strategy is updated by

$$U_{i,s}(t+1) = U_{i,s}(t) - A(t)(\vec{a}_{i,s} \cdot \vec{\eta}(t)) \tag{3.11}$$

TMG is a continuous formulation of the basic Minority Game in which the game is no longer discrete and binary. The response of a strategy in TMG is defined as the inner product, all components of the strategy have to predict at each round. This is a difference to the original formulation where at each round, only one of the 2^M predictions is effective.

3.3.2 The Minority Game without information

In this simplification of the Minority Game, no information is given to the agents and thus they have no strategies. Agents choose between the choice "+1" or "-1" according to the following equation

$$Prob\{a_i(t) = \pm 1\} = \frac{e^{\pm U_i(t)}}{e^{U_i(t)} + e^{-U_i(t)}} \tag{3.12}$$

and the virtual payoff is updated by

$$U_i(t+1) = U_i(t) - \frac{\Gamma}{N} A(t) \tag{3.13}$$

$U_i(t)$ can be considered as the virtual score for agent i to make a decision of "+1" and $-U_i(t)$ correspondingly the virtual score for agent i to make a decision of "-1". If the virtual score is positive, then past experience of an agent shows that it is more successful to take action $a_i(t) = +1$ and vice versa.

3.3.3 The Grand-canonical Minority Game

This is a subclass of Minority Games where the number of agents who actively participate in the market is variable. Instead of summing over all N agents, the attendance $A(t)$ is effectively the sum of the actions of active agents at time step t. Agents can be active or inactive at any time, depending on their potential profitability from the market. This variant of the original model borrows the name "grand-canonical" from the grand-canonical ensemble in statistical physics where the number of particles in the observing system is variable. This is made possible by allowing that an agent whose highest virtual score of strategies falls below some preset treshold ϵt refrains from participating in that round of the game. ϵ is a positive constant usually referred to as the *interest rate* and t is the number of rounds or time steps from the start of the game. This is equivalent to giving agents an *inactive strategy* and the virtual score of this strategy is ϵt. Economically, it corresponds to circumstances of gaining an interest of ϵ at each time step by keeping the capital in the form of cash, so agents would only participate in the market if the gain from investments is greater than the interest rate. Alternatively instead of using the virtual score of strategies, the real score of agents is compared with the interest rate.

The Grand-canonical Minority Game also reproduces the stylized facts of financial markets, while preserving the two-phase structure of the predictable and unpredictable phase. The stylized facts reproduced include fat-tail price return distributions and volatility clustering when the system is close to the critical state. The model serves as a tool in understanding how macroscopic features are produced from microscopic dynamics of individual agents, which also support the conjecture of self-organized criticality of financial markets[11].

3.4 Analytic approach

Several analytic approaches have been employed to solve the Minority Game. Most of them are based on models on the models of the basic Minority Game with little modifications or simplifications. Here we shall focus on the special case of $S = 2$ which allows us to employ the tools of statistical physics and still contains the richness of the basic model. We denote the first strategy of an agent i to be "+1" while the second one to be "-1" whereas the best strategy available to the agent at time step t is now expressed as $s_i(t) = \pm 1$. The bid or action of an agent i at time t can then be expressed as

$$a_i(t) = a^{\mu(t)}_{i,s_i(t)} = \omega_i^{\mu(t)} + s_i(t)\xi_i^{\mu(t)} \tag{3.14}$$

[11] It states, that the financial market is always close to or attracted to the critical state.

where $\omega^{\mu(t)} = \frac{a_{i,+}^{\mu(t)} + a_{i,-}^{\mu(t)}}{2}$ and $\xi^{\mu(t)} = \frac{a_{i,+}^{\mu(t)} - a_{i,-}^{\mu(t)}}{2}$. ω^{μ} and ξ^{μ} are quenched disorders and are fixed at the beginning of the game with values ω_i^{μ}, $\xi_i^{\mu} = 0, \pm 1$ for all μ. $s_i(t)$ is a dynamic variable and becomes explicit in the action of agents, corresponding to spin in the Ising model. The attendance can thus be expressed as a function of spin $s_i(t)$ given by

$$A(t) = \Omega^{\mu(t)} + \sum_{i=1}^{N} \xi_i^{\mu(t)} s_i(t) \tag{3.15}$$

where we introduced $\Omega^{\mu} = \sum_i \omega_i^{\mu}$.

Other than spin $s_i(t)$ the virtual scores of the two strategies are also dynamic and we denote the difference of the virtual scores of the two strategies of agent i to be $Y_i(t)$ given by

$$Y_i(t) = \frac{\Gamma}{2}(U_{i,+}(t) - U_{i,-}(t)) \tag{3.16}$$

This $Y_i(t)$ determines the relative probabilities of using the two strategies "inverse temperature" Γ and is updated by

$$Y_i(t+1) = Y_i(t) - \frac{\Gamma}{N} \xi_i^{\mu(t)} A(t) \tag{3.17}$$

which is given by the update of the individual scores $U_{i,+}(t)$ and $U_{i,-}(t)$ in Eq. (3.4) with a factor of $1/N$ in the last term. Thus the probabilities Eq. (3.9)

$$Prob\{s_i(t) = \pm 1\} = \pi_{i,\pm} = \frac{1 \pm \tanh Y_i(t)}{2} \tag{3.18}$$

From this equation the time average of $s_i(t)$ at equilibrium, denoted as m_i[12], can be calculated

$$m_i = \langle s_i \rangle = \langle \tanh(Y_i) \rangle \tag{3.19}$$

The system will be stationary with $\langle Y_i \rangle \sim v_i t$ which corresponds a stationary state solution of the set of m_i. With v_i expressed from Eq. (3.4) as

$$v_i = -\overline{\Omega \xi_i} - \sum_{j=1}^{N} \overline{\xi_i \xi_j} m_j \tag{3.20}$$

with $\overline{\cdots}$ denoting the average over μ. For $v_i \neq 0$, $\langle Y_i \rangle$ diverges to $\pm\infty$ giving $m_i = \pm 1$, corresponding to frozen agents who always use the same strategy. For $v_i = 0$, $\langle Y_i \rangle$ remains finite even after a long time and $|m_i| < 1$, corresponding to fickle agents who always switch their active strategy even in the stationary state of the game. We can identify $-\overline{\Omega \xi_i} - \sum_{j \neq i} \overline{\xi_i \xi_j} m_j$ to be an external field and coupling between agents i and j, while $\sum_i \overline{\xi_i^2} m_i$ represents self-interaction of agent i. For an agent to be frozen, the magnitude of the external field has to be greater than the self-interaction. Thus in order to have fickle agents in the stationary state, the self-interaction term is crucial.

Eq. (3.7) and the corresponding conditions of frozen and fickle agents are equivalent to the minimization of predictability H

$$H = \frac{1}{2^M} \sum_{\mu=1}^{2^M} [\Omega^{\mu} + \sum_{i=1}^{N} \xi^{\mu} m_i]^2 \tag{3.21}$$

[12]This can be identified as a soft-spin, opposed to a (hard) spin $s_i = \pm 1$, and is a real number $m_i \in [-1, 1]$.

Since m_i is bounded in the range of [-1,+1], H attains its minimum at $dH/dm_i = 0$, giving $\overline{\Omega\xi_i} + \sum_j \overline{\xi_i\xi_j} m_j = 0$ (fickle agents) or at the boundary of the range, giving $m_i = \pm 1$ (frozen agents). Thus H can be identified as the Hamiltonian, where the stationary state of the system is the ground state which minimizes the Hamiltonian.

The volatility $\sigma^2 = \langle A^2 \rangle$ is expressed from Eq. (3.2) as

$$\sigma^2 = H + \sum_{i=1}^{N} \overline{\xi_i^2}(1 - m_i^2) + \sum_{i \neq j} \overline{\xi_i\xi_j} \langle (\tanh Y_i - m_i)(\tanh Y_i - m_i) \rangle \tag{3.22}$$

The last term involves the fluctuations around the average behaviour of the agents, i.e. off-diagonal correlations accross different agents $i \neq j$ and is related to the dynamics of the system.

4 Application to the analysis of financial markets

4.1 Price dynamics

In order to connect the Minority Game with financial markets, a major ingredient must be added - price dynamics.

We assume that the price $p(t)$ on a single asset market is driven by the difference between the number of shares or assets being bought and sold. The behaviour of an agent is again restricted to two possible actions - "buy" $a_i(t) = 1$ and "sell" $a_i(t) = -1$. Then the attendance $A(t) = \sum_i a_i(t)$ is simply the difference between demand and supply, i.e., *excess demand*.

The simplest way to link price return[13] $r(t)$ and excess demand $A(t)$ is to assume a linear dependence

$$r(t) = \log p(t) - \log p(t-1) = \frac{A(t)}{\lambda} \tag{4.1}$$

where λ is called the *liquidity* or *market depth* and is used to control the sensitivity of price on attendance. The above equation can be justified in limit order markets. These are markets that allow agents to submit *limit orders*[14], which are requests to buy or sell a given quantity of an asset at a given price. Each order will be executed only if there is an opposite matching request. Orders waiting to be executed are stored in the *order book*. Fig. 4.1 shows a schematic order book.

It is also possible to submit *market orders*, which are requests to buy or sell immediately at the best price. For these kinds of orders, the time and volume of transactions is fixed, but the transaction price is not known in advance.

Imagine that N market orders of size 1 arrive simultaneously at time t; out of these $(N + A)/2$ are buy orders and $(N - A)/2$ are sell orders. It is possible to match $(N - |A|)/2$ buy and sell orders and to execute them at the current price. $|A|$ orders of one kind are left unexecuted. If $A > 0$, buy orders remain unexecuted, else they are sell orders. These remaining orders are then matched with the best limit orders of the opposite type present in the order book. By assuming a uniform density λ of limit orders, the price is displaced by A/λ, as all orders between $p(t-1)$ and $p(t) = p(t-1) + A/\lambda$ are executed. This process can go on assuming there are new limit orders that fill the gap between $p(t-1)$ and $p(t)$, restoring a uniform distribution of limit orders.

In a market where the price is fixed by market clearing, a further derivation of Eq. (4.1) is possible. Again, we assume traders actions at time t are restricted to two choices: $a_i(t) = +1$ means agent i invests 1€ in order to buy the asset, while $a_i(t) = -1$ means he sells $1/p(t-1)$ units of assets, where $p(t-1)$ is the price of the last transaction. Then total demand equals $\frac{N+A(t)}{2} \cdot \frac{1}{p(t)}$ units of assets and

[13] a/n; the rate of return on an investment portfolio.

[14] a/n; giving the the trader control over the price at which the trade is executed. However, the order may never be executed or filled.

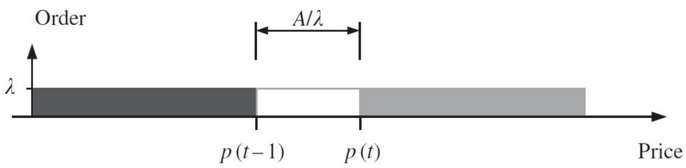

Figure 4.1: Order density against price as a schematic justification of Eq. (4.1). The dark grey block represents buy orders, the light grey one sell orders, and the box the executed sell orders for $A > 0$. [8]

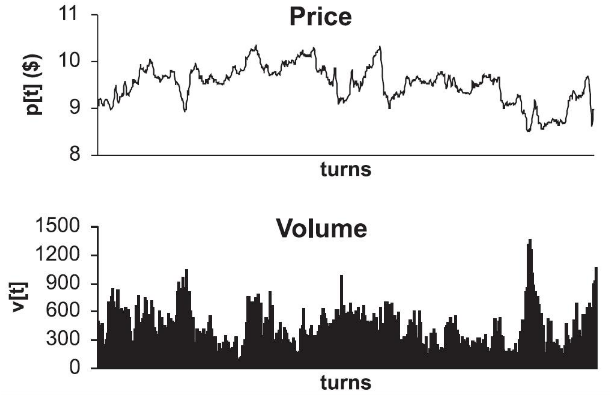

Figure 4.2: Price and volume of transactions for a single run of a market simulation. Parameters $N = 101$ out of which $N_{frozen} = 50$ (frozen agents, following a trend, e.g., only buying or selling), $M = 3$, $S = 2$, $\lambda = 0.07$. [10]

total supply is $\frac{N-A(t)}{2} \cdot \frac{1}{p(t-1)}$ units of assets, with $A(t) = \sum_i a_i(t)$. By matching supply and demand the price $p(t)$ is fixed

$$p(t) = p(t-1)\frac{N + A(t)}{N - A(t)} \tag{4.2}$$

If logarithms of both sides are taken and $A(t) \ll N$, then by keeping the leading order term, Eq. (4.1) is recovered with $\lambda = N/2$.

4.2 Speculative trading

The argument of a Minority-Game-like payoff $\delta U_i(t) = -a_i(t)A(t)$ is that when the majority buys (sells), the price is likely to be high (low), thus it is convenient to sell (buy), i.e., to be on the minority side. Buying an asset at price $p(t)$ becomes profitable if one manages to sell it for a higher price $p(t')$ at a later time $t' > t$. Speculative market payoffs depend on at least two prices which implies they have an inherently temporal nature, i.e., the speculator's actions are correlated in time. Because in the Minority Game all action takes place within a single time step, it cannot model speculative trading outright without making several assumptions regarding traders' expectations first.

The *position* of each agent i is defined by introducing capital $C_i(t) = M_i(t) + p(t-1)S_i(t)$, which is dependent on the amount of money $M_i(t)$, asset $S_i(t)$ owned at time t and on the last market price $p(t-1)$ of the asset. In order to define a market payoff, a round-trip operation of two consecutive trades must be considered, e.g., $a_i(t) = +1$, $a_i(t+1) = -1$ or vice versa. This means agent i at time t

first buys assets for 1€ and sells them at $t+1$ or he first sells $1/p(t-1)$ units of assets (earning 1€) and then buys them back for 1€. These two operations can be expressed as

$$M_i(t+2) = \begin{cases} M_i(t) - 1 + \frac{p(t+1)}{p(t)}; & a_i(t) = +1 \\ M_i(t) + \frac{p(t)}{p(t-1)} - 1; & a_i(t) = -1 \end{cases} \tag{4.3}$$

$$S_i(t+2) = \begin{cases} S_i(t); & a_i(t) = +1 \\ S_i(t) - \frac{1}{p(t-1)} + \frac{1}{p(t+1)}; & a_i(t) = -1 \end{cases} \tag{4.4}$$

by also assuming that supply and demand equalize (cf., Eq. (4.2)), the capital at time $t+2$ can be expressed as

$$C_i(t+2) = C_i(t) + [p(t+1) - p(t-1)]S_i(t) + (\frac{N - a_i(t)A(t)}{N - A(t)})(\frac{2a_i(t)A(t+1)}{N - A(t+1)}) \approx C_i(t) + [p(t+1) - p(t-1)]S_i(t) + \tag{4.5}$$

where the last term is the leading term for $A(t)$, $A(t+1) \ll N$. The second term in the above equation is the change in the capital due to the price change which takes place even without any transactions. The last term is the market payoff resulting from the pair of operations and the model is known as the \$-game.

$$U_i^{\$}(t) = a_i(t)A(t+1) \tag{4.6}$$

This payoff allows trends to establish and dominate the dynamics, e.g., if the majority of agent play +1 (-1) for a given period of time, $A(t)$ will take to a positive (negative) sign self-reinforcing the choices of agents. This is similar to bubble phases in real markets, where expectations of a trend can be sustained as a self-fulfilling prophecy. These bubbles do not last forever because traders cannot continue buying or selling ad infinitum. Sooner or later they will remain either without money or without assets. Hence Eq. (4.6) alone does not guarantee a stable market model without considering budget constraints. Another problem that poses itself is that of the intermediate action at time $t+1$, because Eq. (4.6) is valid only for agents who take $a_i(t+1) = -a_i(t)$. In the \$-game, the action $a_i(t+1)$ is not fixed once $a_i(t)$ is chosen, but is set on the basis of scores as $a_i(t)$.

The solution to this is to bring in agents' expectations at time step t about quantities at time step $t+1$. By assuming a linear dependence on future excess demand

$$E_i[A(t+1)|t] = -\phi_i A(t) \tag{4.7}$$

where $E_i[...|t]$ is the expectation operator of agent i at time t. The expected payoff of agent i is then

$$E_i[U_i^{\$}(t)|t] = a_i(t)E_i[A(t+1)|t] = -\phi_i a_i(t)A(t) \tag{4.8}$$

For $\phi_i > 0$ agent i expects the future movements $A(t+1)$ to counterbalance those that just occured $A(t)$. This is typical of believing that price fluctuates around some equilibrium value of p_f. A deviation away from it is likely to be followed by a restoring movement in the opposite direction. Hence traders $\phi_i > 0$ are also known as *fundamentalists* or *contrarians*, obviously their market is a Minority Game. On the contrary, *trend followers* or *chartists*, with $\phi_i < 0$ would play a majority game when interacting, according to Eq. (4.8).

If the expectation on which a trading rule is based is wrong, those using it suffer losses, and sooner or later abandon or revise that expectation. Thus expectations (variables ϕ_i) also have their own dynamics on a time scale longer than that of trading activity. Stationary states can only occur when the

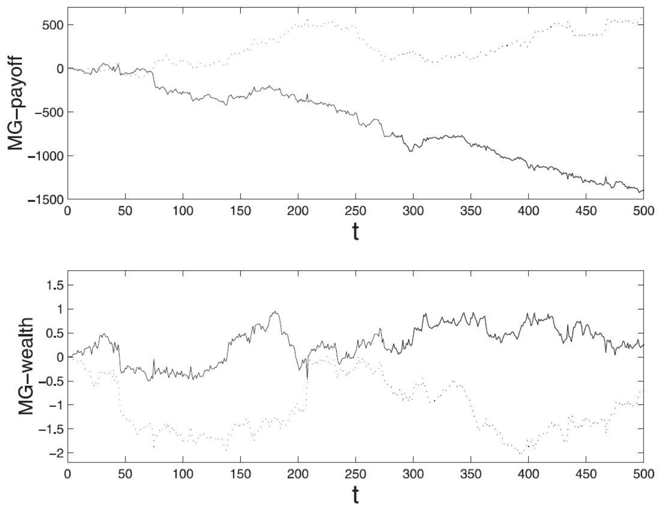

Figure 4.3: Payoff function and wealth of the best (dotted line) and worst (solid line) performing agent using the \$-game payoff function from Eq. (4.6). Parameters $N = 501$, $M = 10$, $S = 10$. [12]

expectations are consistently validated by the dynamics, that is

$$\phi_i \simeq -\frac{\langle A(t+1)A(t)\rangle}{\langle A^2(t)\rangle} \tag{4.9}$$

This implies that both expectations $\phi_i < 0$ and $\phi_i > 0$ are self-reinforcing.

Taking Eqs (4.1) and (4.9) as a serious description of real-world markets, would suggest that markets with a positive (negative) autocorrelation of price returns $\langle r(t)r(t+\tau)\rangle$ should be described by a Majority (Minority) Game on the timescale τ. If the autocorrelation function changes sign with a time lag of τ, it could be said that markets are Minority Games on some timescales and Majority Games at other timescales. Real markets are much more complex than the model presented and the conclusions derived from it. In reality autocorrelation is very small when τ is not in the high frequency range. Empirically it was found that$\langle r(t)r(t+\tau)\rangle < 0$ in foreign exchange and bond markets as long as τ is smaller than a few minutes while for bigger timescales $\langle r(t)r(t+1)\rangle < 0$ does not exceed the noise level. However, even though $\langle r(t)\rangle \neq 0$, this does not necessarily imply that the market is a Majority Game, speculators may effectively play a Minority Game by exploiting the fluctuations of $r(t)$ around its mean $\langle r(t)\rangle$.

5 Concluding remarks

A question that naturally arises next is what has this to do with financial markets? Is the market really a Minority Game? A generic argument which suggests this conclusion is that markets are institutions which allow agents to exchange goods. Through these exchanges, traders can reallocate their resources in order to increase their well-being, selling what they have in excess and buying what they need or prefer. Since trading per se does not create any wealth, the market must be a zero-sum game. By taking out transaction costs and making several assumptions regarding the behaviour of agents, the market becomes a Minority Game, analyzable by the methods and models presented in this paper. The Minority Game then becomes a possible market modeling tool and given the richness of the standard model of inductive reasoning, it provides a natural microscopic explanation for the volatility correlations found in real markets. Thus it offers a broad picture of how markets operate and suggests that they operate close to a phase transition.

I conclude this paper with a quote:

> There are 10^{11} stars in the galaxy. That used to be a huge number. But it's only a hundred billion. It's less than the national deficit! We used to call them astronomical numbers. Now we should call them economical numbers.
> - Richard P. Feynman

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