University *of Ljubljana*
Faculty *of Mathematics and Physics*

Seminar 4. letnik

Onsager theory of hydrodynamic turbulence

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Abstract

The goal of this seminar is a brief summary of Onsager's published and unpublished contributions to hydrodynamic turbulence and an account of their place in the field as the subject has evolved through the years, but main focus will be on two-dimensional fluids, point-vortices, Saturns "stable" *Great red spot* explained using term of negative absolute temperature and sophisticated method using non-point-vortex model but rather local distribution function, which is more realistic behavior of vortices.

Contents

1 Introduction

Lars Onsager, born in Oslo, Norway, a science genius, won a 1968 Nobel Laureate in Chemistry. He made deep contributions to many ares of physics and chemistry are widely appreciated. His huge contribution to the thermodynamics of irreversible processes and a key result, the reciprocal relation for linear transport coefficient. Among the other celebrated contributions are his work on liquid helium, including quantization of circulation, his semiclassical theory of the de Haas-van Alpehn effect in metals [1], his entropic theory of transition to nematic order for rod-shaped colloids, and the reaction field in his theory of dielectrics.

Probably less known among physicists is Onsager's interest in hydrodynamic turbulence. He published two papers on the subject of fully developed turbulence (see [1],[7]). 1945, Onsager predicted an energy spectrum for velocity fluctuations that rolls off as the $-5/3$ power of the wave number. The published abstract appeared a few years after, but entirely independently of, the now-famous trilogy of papers by Kolmogorov, proposing his similarity theory of turbulence. His only full-length article on the subject in 1949 introduced two ideas - *negative-temperature equilibria for two-dimensional ideal fluids and energy dissipation anomaly for singular Euler solutions* - that stimulated much later work. Reamarkably, his private notes of the 1940s contain the essential elements of at least four major results that appeared decades later in the literature: a mean-field Poisson-Boltzmann equation and other thermodynamic relations for point vortices; a relation similar to Kolmogorov's 4/5 law connecting singularities and dissipation ...

[1] often abbreviated dHvA, was descovered 1930; dHvA effect is quantum mechanical effect in the magnetic moment of a pure metal crystal oscillate as the intensity of an applied magnetic field B is increased. Other quantities also oscillate, such as the resistivity (Shubnikov–de Haas effect), specific heat, and sound attenuation. This effect is due to Landau quantization of electron energy in an applied magnetic field. A strong magnetic field — typically several teslas — and a low temperature are required to cause a material to exhibit the dHvA effect.

2 Brief mathematical review of hydrodynamics

Lets first begin with behavior of ideal[2] and inviscid $\nu := \eta/\rho \equiv 0$ fluid. From Cauchy and Pascal law (stress tensor: $p_{ik}(x_l) \neq p_{ik}(x_l, t)$ in the absence of a shear forces)

$$\rho \ddot{u}_i = \frac{\partial p_{ik}}{\partial x_k} + \rho f_i^{ex}, \qquad p_{ik} = -p\delta_{ik} \tag{1}$$

$$\rho \frac{D\mathbf{v}}{Dt} \equiv \rho \left(\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} \right) = -\nabla p + \rho \mathbf{f} \tag{2}$$

where u_i is deformation vector, $f_i^{ex} \equiv f_i$ external forces and p hydrostatic preasure. This equation is know as *Euler equation*. If we make assumptions that external forces are conservative $\mathbf{f} = -\boldsymbol{\nabla}\phi$, current flow is isotropic and *vorticity* $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v}$ using identity

$$\begin{aligned} \frac{1}{2}\boldsymbol{\nabla} v^2 &= \mathbf{v} \times (\boldsymbol{\nabla} \times \mathbf{v}) + \mathbf{v} \cdot \boldsymbol{\nabla}\, \mathbf{v} \\ 0 &= \frac{D\rho}{Dt} + \rho\, \boldsymbol{\nabla} \cdot \mathbf{v} \end{aligned}$$

we get *Helmholtz equation of vorticity*

$$\frac{D\boldsymbol{\omega}}{Dt} - \frac{\boldsymbol{\omega}}{\rho}\frac{D\rho}{Dt} = \rho \frac{D}{Dt}\left(\frac{\boldsymbol{\omega}}{\rho}\right) = \boldsymbol{\omega} \cdot \boldsymbol{\nabla}\, \mathbf{v} \tag{3}$$

This equation is in general too complicated, however, for a start let us look at incompressible fluid and liquids, where the third dimension is negligible - for $2D$ fluid flow, we get

$$\frac{D\boldsymbol{\omega}}{Dt} = \left(\frac{\partial \boldsymbol{\omega}}{\partial t} + \mathbf{v} \cdot \boldsymbol{\nabla}\, \boldsymbol{\omega} \right) = 0 \tag{4}$$

Interpretation of this equation is that vorticity, when moving with liquid, is conserved - this is rigorously true only in $2D$, which tells Kelvin theorem. We define *circulation*

$$\Gamma := \oint_{C(t)} \mathbf{v} \cdot d\mathbf{r} \tag{5}$$

where $C(t)$ represents loop moving together with liquid. Using parametrization with natural parameter s, $C(t): \quad \mathbf{r}(s,t), s \in [0,1], \quad \mathbf{r}(0,t) = \mathbf{r}(1,t)$, than

$$\Gamma(t) = \oint_{C(t)} \mathbf{v} \cdot d\mathbf{r} = \int (\boldsymbol{\nabla} \times \mathbf{v}) \cdot d\mathbf{S} = \int \boldsymbol{\omega} \cdot d\mathbf{S} = \text{const.} \tag{6}$$

this means, if in the irrotational fluid, fluid did not had vorticity at the beginning $\boldsymbol{\nabla} \times \mathbf{v}(\mathbf{r}, t = 0) = 0$ - this property conserved also at later time - this is known as *potential flow*[3]. Kelvin theorem is known as *vorticity theorem conservation*. Let us define another quantity e.i. *complex potential* ($2D$)

$$w(z) = \phi(z) + i\psi(z), \quad z = x + iy \tag{7}$$

[2]In the deriving the equations of motion we have taken no account of processes of energy dissipation, which may occure in a moving fluid in consequence of internal friction (viscosity) in the fluid and heat exchange between different parts of it - motions of fluids in which thermal conductivity and viscosity are *unimportant*.

[3]potential flow beacause $\mathbf{v} = \boldsymbol{\nabla}\phi$, where ϕ is velocity potential.

where ϕ *velocity potential*, ψ *stream function*. Stream function ψ, is defined by

$$\begin{aligned} v_x &:= \frac{\partial\psi}{\partial y}\left(=\frac{\partial\phi}{\partial x}\right) \quad \text{and}\mathit{explain} \quad v_y := -\frac{\partial\psi}{\partial x}\left(=\frac{\partial\phi}{\partial y}\right) \\ \boldsymbol{\nabla}\cdot\mathbf{v} &= \frac{\partial v_x}{\partial x}+\frac{\partial v_y}{\partial y}=\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial^2\psi}{\partial y\partial x} \end{aligned} \tag{8}$$

Velocity of vortices in two dimensions (x, y) using *Biot-Savart's law* (see [2],[10],[12]) in cilindrical geometry

$$\begin{aligned} v_\theta &= \frac{\Gamma}{2\pi} \qquad v_r = 0 \\ w(z) &= -i\frac{\Gamma}{2\pi}\ln z = \frac{\Gamma}{2\pi}\theta + i\frac{-\Gamma}{2\pi}\ln r = \phi + i\psi \end{aligned} \tag{9}$$

Let vortices be at $z_1 = (x_1, y_2), \dots, z_n = (x_n, y_n), \dots$, than $2D$ velocity field at this vortices distribution has a form

$$w(z) = -\frac{i}{2\pi}\sum_i \Gamma_i \ln(z - z_i) = \sum_i w_i(z) \tag{10}$$

Try to use same principle but now for N vortices, each has its own circulation $\Gamma_i \equiv \kappa_i$. Velocity field distribution $v(z) := \frac{dw(z)}{z} = v_x - iv_y$ and let us this Hamiltonian function

$$\mathcal{H} = -\frac{1}{2\pi}\sum_{i\neq j}\kappa_i\kappa_j \ln|z_i - z_j| \tag{11}$$

and can be written as

$$\begin{aligned} \kappa_i\frac{dx_i}{dt} &= \frac{\partial\mathcal{H}}{\partial y_i} \qquad \kappa_i\frac{dy_i}{dt} = -\frac{\partial\mathcal{H}}{\partial x_i} \\ \frac{d\mathcal{H}}{dt} &= \frac{\partial\mathcal{H}}{\partial x_i}\frac{dx_i}{dt}+\frac{\partial\mathcal{H}}{\partial y_i}\frac{dy_i}{dt} = 0 \end{aligned}$$

For $\nu := \eta/\rho \neq 0, \boldsymbol{\nabla}\cdot\mathbf{v} \neq 0$ (η dynamic viscosity, ρ fluid density) we get *Navier-Stokes equation*

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\mathbf{f}^{ex} - \boldsymbol{\nabla}p + \eta\nabla^2\mathbf{v} + (\zeta + \frac{1}{3}\eta)\boldsymbol{\nabla}\boldsymbol{\nabla}\cdot\mathbf{v} \tag{12}$$

$\zeta := \lambda + \frac{2}{3}\eta = \text{const.}$, λ *Lamé coefficient* and η *dynamic viscosity*. For the final, we define *enstrophy* for incompressible fluid $\boldsymbol{\nabla}\cdot\mathbf{v} = 0$ and $\mathbf{f} = -\boldsymbol{\nabla}\Phi$

$$\frac{\partial}{\partial t}\int_V \left(\frac{1}{2}\rho v^2 + \rho\Phi\right) d^3\mathbf{r} = -\eta\int_V (\boldsymbol{\nabla}\times\mathbf{v})^2\, d^3\mathbf{r} \geq 0 \tag{13}$$

When $\mathbf{f} = \mathbf{0}$ it can be seen that viscosity $\eta \geq 0$. However, in incompressible fluid with vortices there is alaways energy dissipation!

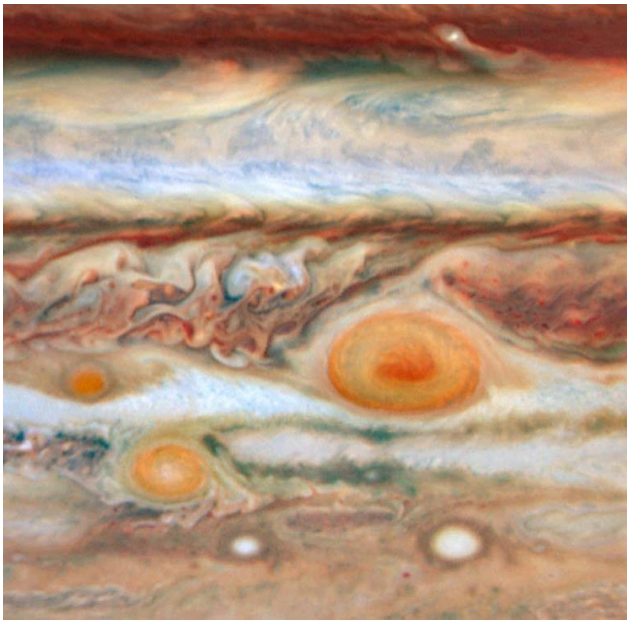

Figure 1: *The large, oval-shaped mark on the clouds is the Great Red Spot - believed to be an intense atmospheric disturbance. An anticyclonic vortex in the upper atmosphere of Jupiter has existed at least since it was observed in* 1610 *by Galileo with one of the first telescopes - at its widest, it is about three times the diameter of the Earth. Similar large-scale, long-lived vortices exist in the atmospheres of the other gas giant planets of our solar system e.i. "icy" giant Neptune, Saturn etc (Voyager 2, NASA)*

3 Onsager theory of turbulence

3.1 Onsager's theory of point-vortex equilibria

In the published paper Onsager (see [7]) discussed a simple Hamiltonian particle model of $2D$ ideal fluid flow, the point-vortex model of Helmholtz and Kirchhoff, describing this motion for a system of N vortices in a plane, or of straight and paralel line vortices in three dimensions (see [14]). If the planar coordinates of the ith vortex are $\mathbf{r}_i = (x_i, y_i)$ and if that vortex carries a *net circulation* κ_i, *the equations of motion* are

$$\kappa_i \frac{dx_i}{dt} = \frac{\partial H}{\partial y_i} \qquad \kappa_i \frac{dy_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{14}$$

$$H = -\frac{1}{2\pi} \sum_{i<j} \kappa_i \kappa_j \, ln(r_{ij}/L) \tag{15}$$

where H is the fluid kinetic energy, r_{ij} is the distance between the ith and jth vortex and L is an arbitrary length scale. Where there are no bounderies, H has the form (15).

Modern source for the point-vortex model is *Marchioro* and *Pulvirenti* (1994) in which it is provided that the model describes the motion of concentrated blobs of vorticity, evolving according to the $2D$ incompressible Euler equations, as long as the distance between the blobs is much greater than their diameters[4]. Another result in the opposite direction states

[4]In fact, these equations of motion have been formally derived from quatum many-body equations for parellel line vortices in superfluids.

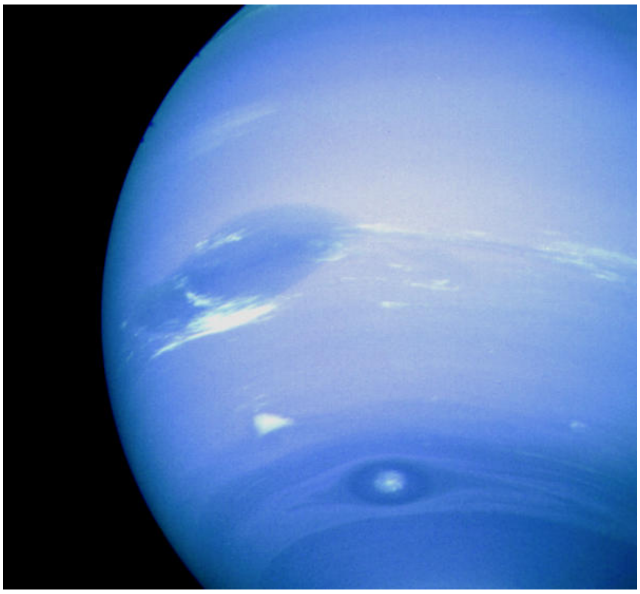

Figure 2: *Great Dark Spot of Neptune is thought to be a hole, similar to the hole in ozone layer on Earth, however unlike Jupiter's Great Red Spot, Dark Spot* can dissapeare - *this was first noticed with disappearance in* 1989, *in* 1994 *was replaced by a similar "spot". Neptune, made up chiefly of hydrogen, helium, water and silicates - does not have a solid surface. Neptune clouds consist mainly of frozen methane. Deep down inside Neptune, the planet "might" have a solid surface (astronomers try to explain dissipative behavior of a Neptune spot)(Voyager 2, NASA).*

that a smooth solution of the $2D$ Euler equations $\omega(\mathbf{r}, t)$ can be approximated as $N \to \infty$, over any finite time interval, by a sum $\omega_N(\mathbf{r}, t) = \sum_{i=1}^{N} \kappa_i \delta(\mathbf{r} - \mathbf{r}_i(t))$, where $\kappa_i = \pm 1/N$ and $\mathbf{r}_i(t)$, $i = 1, \ldots, N$ are the solutions (14).

Theoretical proposal for explanation for a commonly observed feature of nearly two-dimensional flows was: *the spontaneous appearance of large-scale, long-lived vortices.* For examples are the large, lingering storms in the atmospheres of the gas giants of the outer solar system, such as the Great Red Spot of Jupiter (see [16],[17]). Large vortices are also readily seen downstream of flow obstacles. Jupiter is heavier than any other planet. Its mass is 318 times larger than that of Earth. Although Jupiter has a large mass, it has relatively low density ($\sim 1.33 g/cm^3$) - about 1/4 of Earth density. Astronomers believe that the planet consists primarly of hydrogen ($\sim 86\%$) and helium ($\sim 14\%$) (the lighest elements), and tiny amounts of methane, ammonia, phosphine, water, acetylene, ethane, germanium, and carbon monoxide. Interesting is also that chemicals have formed colorful layers of clouds at different heights (as can be seen from *Figure 1*). White clouds, which are the highest in the zones, are made of crystals of frozen ammonia, darker (lower) clouds are of other chemicals occur in the belts - the lowest level are blue clouds. As it can be seen, Great Red Spot colour differ much from neighboring due to small amounts of sulfur and phosphorus in the ammonia crystals.
Astronomers began to use telescopes to observe these features in the late 1600's, the features have, however, *changed* size and brightness but have kept the *same patterns*.

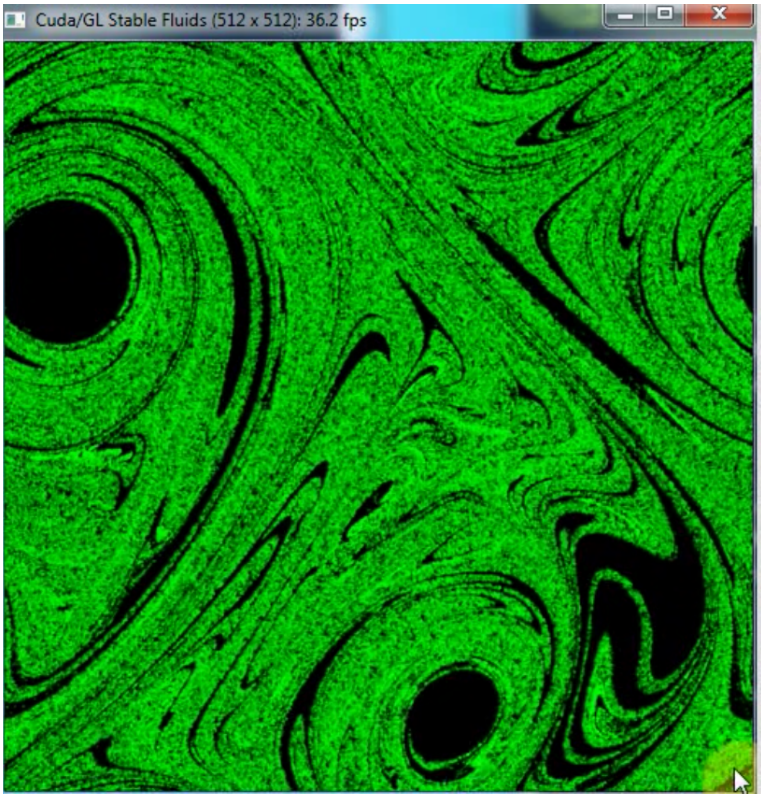

Figure 3: CUDA simulation, using Jos Stam's FFT algorithm, solving Navier-Stokes equation in two-dimensional flow (Nvidia, Open GL Fluid).

Explanation of the phenomenon, Onsager proposed an application of *Gibbsian equilibrium statistical mechanics* to the point-vortex model (see [5]). His theory assumed that the generation of the large-scale vortices was a consequence of the inviscid Euler equations, which form a Hamiltonian system conserving total kinetic energy. This is rigorously true in $2D$ due to conservation of enstrophy. In particular, no sustained forcing is required to maintain the vortex in this theory as long as the dissipation by viscosity is weak [5] (see [15]). When we compare our idealized model with reality, we have one profound difference: the distributions of vorticity which occure in the actual flow of normal fluids are continuous. As a statistical model in two-dimensions it is ambiguous: what set of discrete vortices will best approximate a countinuous distribution of vorticity[6]?

Burgers in his articles attempts to applay statistical-mechanical maximum entropy ideas to turbulent flow (for non-equilibrium conditions see [9]). The ingenious step in Onsager's theory was his realization that *point vortices would yield states of negative absolute temperature, at sufficiently high energy, and that this result could explain the sponeaneous appearance of large-scale vortices in two-dimensional flows* (see [3], [4], [7]) [7].

[5]Onsager also assumed the validity of the point-vortex approximation, though with reservations.

[6]Onsager assumed: "...the point-vortex dynamics is ergodic in phase space over the surface of constant energy, so that a microcanonical distribution is achieved at long times ... We inquire about the ergodic motion of the system.

[7]Onsager, in his notes, doesn not explain physical concept of negative-absolute temperature

Now how to explain negative-absolute temperature? The essential requirements for a thermodynamical system to be capable of negative temperature

1. the elements of the thermodynamical system must be in thermodynamical equilibrium among themselves in order for the system to be described by a temperature at all.

2. there must be an upper limit to the possible energy of the allowed states of the system - negative temperatures are to be achieved with a finite energy.

3. the system must be thermally isolated from all systems which do not satisfy both upper requirements - internal thermal equilibrium time among the elements of the system must be short compared to the time during which appreciable energy is lost to or gained from other systems.

Let us assume that we have a spin-system (most of the system do not satisfy condition(2.), however spin-system can satisfy all three of the conditions)[8] where, we add more and more energy, temperature starts off positive, approaches positive infinity as maximum entropy is approached, with half of all spins up. After that, the temperature becomes negative infinite, coming down in magnitude toward zero, but always negative, as the energy increases toward maximum. When the system has negative temperature, it is hotter than when it is has positive temperature. If you take two copies of the system, one with positive and one with negative temperature, and put them in thermal contact, heat will flow from the negative-temperature system into the positive-temperature system. Some systems does not have a property that entropy increases monotonically with energy, however are cases when energy is added to the system configuration acctually decreases for some energies (in some energy region).

How can be this realized in the real world or it is just theoretical science fiction? Best to explain is using a spin system in a magnetic field. Atoms "must" have other degrees of freedom in addition to spin, making (usually) total energy of the sistem unbounded upward due to translation degree of freedom. Sometimes it is useful to define "spin-temperature" of a collection of atoms but only one condition is met, that is, the coupling between atomic spins and the other degrees of freedom is sufficiently weak, and the coupling between atomic spins sufficiently strong, that the timescale for energy to flow from the spins into other degrees of freedom is very large compared to the timescale for thermalization of the spins among themselves. Using this condition make sense to talk about temperature of spins separately from the temperature of the atoms as a whole - in strong magnetic fields we can met described condition. Interesting is also that only certain degrees of freedom of a particle can have negative absolute temperature.

For the existence of negative absolute temperature is the same as that published by the Onsager (1949*d*) some two years prior to their introduction by Purchell for nuclear-spin system (see [1]). Electron (nuclear) spin can be promoted to negative absolute temperatures by using suitable radio frequency tehniques[9].

[8] there is no upper limit to the possible kinetic energy of a gas molecule

[9] Various experiments and applications (RF amplifier) in the calorimetry of negative temperatures can be found (see [3],[4])

The crucial feature of the point-vortex system which permits this conclusion is the fact that the *total phase-space volume is finite*. Since x and y components of the vortices are canonically conjugate variables, the total phase-space volume is $\Phi(\infty) = A^N$, where A is the area of the flow domain and

$$\Phi(E) = \int \prod_{i=1}^{N} d^2\mathbf{r}_i \, \theta(E - H(\mathbf{r}_1, \ldots, \mathbf{r}_N)) \tag{16}$$

where θ is the Heviside step function $\theta(x > 0) = 1$ and $\theta(x < 0) = 0$. $\Phi(E)$ is a non-negative increasing function of energy E, with constant limits $\Phi(-\infty) = 0$ and $\Phi(\infty) = A^N$

$$\Omega(E) = \Phi'(E) = \int \prod_{i=1}^{N} d^2\mathbf{r}_i \, \delta(E - H(\mathbf{r}_1, \ldots, \mathbf{r}_N)) \tag{17}$$

is a non-negative function going to zero at both extremes, $\Omega(\pm\infty) = 0$. Thus the function must achieve a maximum value at some finite E_m, where $\Omega'(E_m) = 0$ (I use this with reservation!!). For $E > E_m$, $\Omega'(E_m) < 0$. On the other hand, by Boltzmann's principle, the *thermodynamic entropy* is $S(E) = k_B \, ln\Omega(E)$ [10] and the inverse temperature $1/\Theta = dS/dE < 0$ for $E > E_m$. Onsager further pointed out that negative temperatures will lead to the formation of large-scale vortices by clustering of smaller ones. More precise: in the former case when $1/\Theta > 0$, vortices of oposite sign will tend to approach each other. However, if $1/\Theta < 0$, then vortices of the same sign will tend to cluster - preferably the strongest ones - so as to use up energy at the least possible cost in terms of degree of freedom. It stands to reason that the large compound vortices formed in this matter will remain as the only conspicuous features of the motion; because the weaker vortices, free to roam practically at random, will yield rather erratic and disorganized contribution to the flow.

The statistical tendency of vortices of the same sign to cluser in the negative-temperature regime is clear from a description by a canonical distribution
$\propto e^{-\beta H}$ with $\beta = 1/k_B\Theta$. Negative β corresponds to reversing the sign of the interaction, making like "charges" statisticaly attract and opposite "charges" repel.

3.2 Point-vortex model

Montgomery returned to Onsager's theory and worked out a predictive equation for the large-scale vortex solutions (see [8]) conjectured by Onsager[11].
A brief review of the Joyce-Montgomery considerations, in the language of the $2D$ point-vortex system. Consider a neutral system, which we describe as consisting of N vortices of circulation $+1/N$ and N vortices of circulation $-1/N$; for this system, there are two vortex densities,

$$\rho_\pm(\mathbf{r}) = \frac{1}{N} \sum_{i=1}^{N} \delta(\mathbf{r} - \mathbf{r}_i^\pm) \tag{18}$$

[10] the entropy of a system in which all states, of number Ω, are equally likely; such as system is one in which the volume, number of molecules, and internal energy are fixed - the microcanonical ensemble.

[11] Onsager carried these considerations no further in his 1949 paper nor in any subsequent published work. Von Neumann (1949) took note of the point-vortex model (see [18],[11]) and Onsager's statistical-mechanical theory; this led von N. to speculate about the limits of Kolmogorov's reasoning in the $3D$ and to recognize the profound consequances of enstrophy conservation in two dimensions.

where $\mathbf{r}_i^{\pm}$, $i = 1, \ldots, N$, are the positions of the N vortices of circulation $\pm 1/N$, respectively. *Vorticity field* is represented by $\omega(\mathbf{r}) := \rho_+(\mathbf{r}) - \rho_-(\mathbf{r})$. Joyce and Montgomery (1973) *derived formula for the entropy* (per particle) of a given field of vortex densities

$$S = -\int d^2\mathbf{r}\rho_+(\mathbf{r}) \ln \rho_+(\mathbf{r}) - \int d^2\mathbf{r}\rho_-(\mathbf{r}) \ln \rho_-(\mathbf{r}) \tag{19}$$

They next reasoned that the equilibrium distributions should be those which maximized the entropy subject to the constraints of fixed energy

$$E = \frac{1}{2}\int d^2\mathbf{r}'\, G(\mathbf{r}, \mathbf{r}')\omega(\mathbf{r})\omega(\mathbf{r}'), \qquad \int d^2\mathbf{r}\rho_\pm(\mathbf{r}) \equiv 1 \tag{20}$$

From here, we work out the variational equation

$$\rho_\pm(\mathbf{r}) = \exp\left[\mp\beta \int d^2\mathbf{r}'\, G(\mathbf{r}, \mathbf{r}')\omega(\mathbf{r}') - \beta\mu_\pm\right] \tag{21}$$

where β and $\mu_\pm$ are Langrange multipliers to enforce the constraints, having the interpretation of inverse temperature and chemical potentials, respectively. A closed equation is obtained by introducing the *stream function*

$$\begin{aligned} \psi(\mathbf{r}) &= \int d^2\mathbf{r}'\, G(\mathbf{r}, \mathbf{r}')\omega(\mathbf{r}') \\ -\Delta\psi(\mathbf{r}) &:= \omega(\mathbf{r}) = \exp\left[-\beta(\psi(\mathbf{r}) + \mu_+)\right] - \exp\left[\beta(\psi(\mathbf{r}) - \mu_-)\right] \end{aligned} \tag{22}$$

This is the final equation derived by Joyce and Montgomery. Its maximum-entropy solutions give exact, stable, stationary solutions of the $2D$ Euler equations and should describe the macroscopic vortices proposed by Onsager when $\beta < 0$.

3.3 Advance and applications

One issue that Onsager never addressed was the appropriate thermodynamic limit for validity of this statistical theory of large-scale $2D$ vortices. The Debye-Hückel theory[12] is valid in the standard thermodynamic limit in $2D$, for which area $A \to \infty$ with the number of vortices $N \to \infty$ and energy $E \to \infty$ in such a way that $n = N/A$, $e = E/A$ tend to a finite limit. Further, the circulation κ_i are held fixed, independent of N, e.g. $\kappa_i = \pm 1$. The inverse temperature $1/T$ scales as[13] A/N, since $\sum_i \kappa_i^2 \sim O(N)$, and approaches a finite limit in thermodynamic limit. As E/A varies over all real values the temperature T stays positive, and even more, *Campbell and O'Nell* (1991) rigorously proved that the standard thermodynamic limit exists for the point-vortex model, but yields only positive temperatures. To obtain the negative temperature states proposed by Onsager, *one must consider energies that are considerably higher, greater than the critical energy.*

Point-vortex approximation: we have already mentioned that there are rigorous results which show that any smooth $2D$ Euler solution $\omega(\mathbf{r}, t)$ may be approximated arbitrarily well over a finite time interval $0 < t < T$ by a sum of point vortices $\sum_{i=1}^{N} \kappa_i \delta(\mathbf{r} - \mathbf{r}_i(t))$ with

[12] Onsager realize the connection with Debye-Hückel theory of electrolytes; that is enough to know for this seminar.

[13] shown by Edwards and Taylor (1974).

$\kappa_i \sim c_i/N$, where c_i are constants as $N \to \infty$ (Marchioro and Pulvirenti, 1994)[14]. Onsager had own reservations about the point-vortex approximation - the restrictions imposed by the *incompressibility* of the fluid. Onsager's concerns can be cleary understood by considering the initial condition of an ideal vortex patch, with a constant level of vorticity on a finite area. Because that area is conserved by incompressibility under the $2D$ Euler dynamics, it is not possible for the vorticity to concentrate or to intensify locally for this initial condition. However, this is not true of one were to approximate the patch by a distribution of point vortices at high energies. In that case, the mean-square distance between point vortices could decrease over time and the effective area covered could similarly decrease, leading to more intense, localized vortex structure. Thus one expects discrepancies here between the continuum $2D$ Euler and the point-vortex model for long times.

A great step toward eliminating these defects was taken independently by *Miller* (1991) and *Robert* (1990). They both elaborated an equilibrium statistical-mechanical theory directly for the continuum $2D$ Euler equations, without making the point-vortex approximation. The basic object of both of these theories was a *local distribution function* $n(\mathbf{r}, \sigma)$, the *probability density* that the microscopic vorticity $\omega(\mathbf{r})$ lies between σ and $\sigma + d\sigma$ at the space $\mathbf{r}$. The picture here is that the vorticity field in its evolution mixes to very fine scales so that a small neighborhood of the point $\mathbf{r}$ will contain many values of the vorticity with levels distributed according to $n(\mathbf{r}, \sigma)$, thus n satisfies $\int d\sigma n(\mathbf{r}, \sigma) \equiv 1$ at each point $\mathbf{r}$ in the flow domain. Same as before, the *stream function* satisfies the generalized mean-field equation

$$-\Delta\overline{\psi}(\mathbf{r}) = \frac{1}{Z(\mathbf{r})} \int d\sigma\, \sigma \exp\left[-\overline{\beta}(\sigma\overline{\psi}(\mathbf{r}) - \mu(\sigma))\right] \tag{23}$$

where $Z(\mathbf{r}), \mu(\sigma)$ and $\overline{\beta}$ are Langrange multipliers. This theory is is an application to $2D$ Euler of the method worked out by *Lynden-Bell* (1967) to describe gravitional equilibrium after "violent relaxation" in stellar systems. The *Robert-Miller theory* solves the problems discussed by Onsager - point-vortex assumption. The new theory incorporates infinitly many conservation laws of $2D$ Euler. In fact, the point-vortex model, in the generality considered by Onsager, also has infinitely many conserved quantities, i.e. the total number of vortices of a given circulation. Robert-Miller theory includes *information about the area of the vorticity level sets*, which is lacking in the point-vortex model.
As rekarked by Miller et al., the Joyce-Montgomery mean-field equation is formally recovered in a "dilute-vorticity limit" in which the area of the level sets shrinks to zero keeping the net circulation fixed [15].

Lundgren and Pointin (1976) performed numerical simulations of the point-vortex model (see [1]) with initial conditions corresponding to several local clusters of vortices at some distance from each other. The equilibrium theory predicted their final coalescence into a single large supervortex. Lundgren and Pointin argued theoretically that the vortices will eventially reach the equilibrium, *single-vortex state.*

[14] this is not sufficient to justify equlibrium statistical mechanics at long times because the limits $T \to \infty$ and $N \to \infty$ need not commute.

[15] This corresponds well with conditions suggested by Onsager for validity of the point-vortex model that "vorticity is mostly concentrated in small regions."

4 Conclusions

Onsager observed the spontaneous appearance of large-scale, long lived vortices is a frequent occurence in two-dimensional flow, especially in planetary atmospheres. Onsager's theory, from mathematical point of view, is now largely explored and undestood and also those of its genaralization by Miller and Robert (it has been mentioned at the end). As the empirical confirmation of the equilibrium vortex theories is concerned, it must be addmited that while reasonable agreement has been obtained with a few numberical simulations and laboratory experiments (see [13]), we know of not really convincing verification for flows in nature. Onsager's theoretical view of an "ideal turbulence" described by the inviscid fluid equations is a proper idealization for understanding high Reynolds number flows. All of the mentioned processes are truly predictive devices especially cascade theory of dissipation. However both theories (Onsager and Kolmogorov) are based on experimental observation, where Kolmogorov's theory(cascade theory) (see [19]) is based on important hypotheses combined with dimensional arguments. Theory of hydrodynamic turbulence have many unanswered questions: what are the scales where vorticity is dissipated, or in the other words, what is the size of the smallest eddies that are responsible for dissipating the energy? Some are explained by Kolmogorov, but some are still unanswered.

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