RANDOM NETWORKS

Jan Porekar

Mentor: prof. Rudi Podgornik

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Abstract

Complex networks describe a variety of systems found in nature and society. Traditionally these systems have been modelled as random graphs, a relatively primitive and brutal approach. These traditional models do not produce topological and structural properties featured in real network examples. In recent years many new models have been developed, to correctly describe the scale – free structure of real networks. In this papers the real world networks (WWW, Internet, ecological, cellular, etc.) are presented along with various theoretical models, that explain the emergence of the most important structural properties as average path length, clustering coefficient and power – tail degree distributions. In the end of the paper the basics of network evolution process is explained.

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1 Introduction

The phenomena of self-organisation found in nature is a constant source of fascination and inspiration to both physics and biological sciences. Physics has found the basics principles of the world and is therefore capable to explain and model different phenomena both deterministically and statistically. Many phenomena are understood and modelled merely as a statistical formations of smaller units for which some very basic rules apply: properties of condensed matter are determined by underlying properties of atoms, magnetism emerges from the collective behaviour of millions of spins, etc.

Physical models explain phenomena by connecting basic units or elements in accordance to their interdependence and interaction. The success lies in the simplicity of the interactions among the elements. Because physics studies phenomena in physical space, in most cases the strength of the interaction is uniquely defined only by physical distance. (Forces among objects are defined by their distances and relative velocities as well, which are only time derives of the distance)

On the other hand we are at loss in describing systems where physical distance is irrelevant, systems where interaction and its strength is determined by factors beyond the distance in physical space or there is ambiguity whether two components interact. For many complex systems with nontrivial network topology such ambiguity is present and many of these systems are described as complex networks. In recent years it has been found that tools of statistical mechanics offer ideal framework to describe these interwoven systems as well.

The few examples briefly described above represent only o fraction of cases found in real world. Recently a lot of methods and models for determination of network topologies have been developed. Statistical mechanics proved itself to be an ideal tool for describing specific properties of complex networks, through which network topologies are determined.

2 History of Network Modelling

Traditionally the study of complex networks has been the territory of mathematics, especially the graph theory. Initially the graph theory focused on regular graphs, with no apparent design principles were described as random graphs, proposed as the simplest and most straightforward realisation of a complex network.

The pioneer of the theory was Leonhard Euler, who studied first regular graphs in 18^{th} century. In the 20^{th} century the theory became much more statistically and algorithmically oriented.

Later in 1950's graph theory was used to describe large networks, with no particular distributions of nodes and link, whose organisation principles were not easily definable. These networks were first studied by Paul Erdos and Alfred Reyni and were called "random graphs", due to their generating method: we start with N nodes and connect every pair of them with probability p. Obtained graph has on average $p \cdot (N(N-1))/2$ edges distributed randomly. The degree distribution of such graph is Poisson with peak at $P(\langle k \rangle)$. This model has guided our thinking for decades after it has been presented.

The topology of real large networks (i.e. Internet, WWW, telephone networks, ecological networks) substantially differs from the topology of random graphs produced by the simple Erdos-Reyni (ER) model, therefore new methods, tools and models needed to be developed. In past years we witnessed dramatic advances in this direction. The computerisation of data acquisition has led to the emergence of large databases on the topology of various real networks. Wide availability of computer power allows to investigate networks containing millions of nodes, exploring questions that could not be answered before as well as the slow

but noticeable breakdown between different science disciplines allows scientists to access different databases, allowing to uncover the generic properties of large networks.
Networks found in nature show degree distribution that greatly differs from the Poisson degree distribution of random graphs. Because of existence of a few vertices with high degree, the distribution of real networks has a power-law tail $P(k) \sim k^{-\gamma}$, which indicates scale free properties.

3 General Concepts of Network Modelling

In the recent years 3 different concepts for defining the network topologies have been proposed:

- Small - world : large network, but still small diameter or average path length
- Clustering: large clustering coefficient
- Degree distribution: power-tail

3.1 Small world

Despite the large network size, it commonly happens that there is relatively short distance among any pair of nodes. Path length is defined by minimum number of edges needed to pass from first point to the other (in case of weighted edges, the path length is defined by minimal sum of weights). This phenomena is called the small world effect and can be observed in society and nature: all chemicals inside a living cell are at average 3 reactions away from each other, there is a path of acquaintances between most pairs of people in USA with typical length of about six and the actors in Hollywood are on average within three costars from each other.
Despite the information shown above, the small world concept is not an indication of a special organising principle. The random graphs presented by Erdos and Reyni are the simplest network models to feature small world properties, since the typical distance among any two points in a random graph scales as $\ln(N)$, where N is a number of nodes in a network.

3.2 Clustering

In many real examples of networks or graphs fully connected subgraphs emerge. Such structures are called cliques. A typical example of such feature are circles of friends or acquaintances in social networks where every member of a clique knows every other member. This inherent tendency of clustering is quantified by the clustering coefficient (Watts and Strogatz 1998) an is defined for a single node in the network:

$$C_i = \frac{2E_i}{k_i(k_i - 1)} \tag{0.1}$$

E_i is the number of all edges that actually exist among all first neighbour of selected node. If all the neighbours were connected, there would be $k_i(k_i - 1)/2$ edges among them. The ratio between the actual number of edges E_i and maximum number of edges is the clustering coefficient of a node (0.1).

The clustering coefficient of all the network is the average of all individual Ci's:

$$C = \frac{1}{N}\sum_{i=1}^{N} C_i \tag{0.2}$$

For random graphs the clustering coefficient is equal to graph generating connection probability ($C = p$), since the probability of first neighbours being connected is constant for all nodes.
In real networks the clustering coefficient is much larger than in case of random graphs of equal size (equal number of nodes and edges).

3.3 Degree distribution

The number of edges a node has is called node degree. The spread of node degrees is characterised by a distribution function *P(k*), which gives the probability that randomly selected node has exactly k edges. Since in the random graph the edges are placed randomly, the majority of nodes have approximately the same degree, close to the average $\langle k \rangle$ of the network. The degree distribution of a random graph is a Poisson distribution $P(k) = \frac{e^{-\langle k \rangle}\lambda^k}{k!}$ with a peak at $P(\langle k \rangle)$. On the other hand the empirical results for most large networks show distribution that significantly deviates from Poisson distribution. This degree distribution has a power-law tail:

$$P(k) \sim k^{-\gamma} \tag{0.3}$$

Such network are called scale free. While some real networks still display an exponential tail, often the functional form of *P(k*) still deviates from Poisson distribution expected for a random graph.
In equation (0.3) scaling exponent γ is introduced. γ is an important parameter in defining the topology of a given network. In case of a directed network, two different parameters are introduced (γ_{in}, γ_{out}), for characterising the incoming and outgoing links distribution.

4 Real Networks: Empirical results

The study of most complex networks has been initiated by a desire to understand various real systems.
Complex systems that have been studied are:

1. **World Wide Web (WWW)**: Nodes are web pages and link are hyperlinks. The network is directed, but in some researches is made undirected Some of the researches are made on site level: All the pages in a site are merged into a supernode.
2. **Internet**: topology is studied at two different levels: at the router level the nodes are routers and edges are physical connections between them; at the interdomain level each domain, containing hundreds of routers, is represented as a single node. This is an undirected network.
3. **Cellular networks**: metabolisms of different species from all three domains of life are studied and organised into networks in which the substrates (ATP, ADP, H2O) are nodes and edges represent the predominaly directed chemical reactions in which these substrates can participate.

4. **Ecological networks or food webs**: the nodes are species and the edges represent predator-prey relationships among them. Food webs are directed networks.
5. **Protein folding**: Different states of single protein are represented by different nodes. Conformations are linked if they can be obtained from each other by an elementary move. This is an undirected network.
6. **Citation networks**: Published articles are represented by nodes and a directed edge represents a reference to a previously published article. This is an undirected network.
7. **Co authorship networks**: Collaboration network exists of scientists represented by nodes and two nodes are connected if two scientists have written an article together.
8. **Movie actor collaboration networks**: In this network the nodes are actors and two nodes have a common edge if two actors have acted in a movie together. This is an undirected network.
9. **The web of human sexual contacts**: Many sexually transmitted diseases spread on a network of sexual relationships. This is an undirected network.
10. **Phone-call networks**: A large directed graph can be constructed using telephone numbers as nodes and completed phone calls as edges, directed from caller to receiver.
11. **Networks in linguistics**: The complexity of human language offers several possibilities to define and study complex networks. One way of building a network is to describe words as nodes and connect them with edges if they appear one word form each other inside sentences of the literature of certain language. This is an undirected network. The other way to construct a network is to link words bases on their meaning: words are represented as nodes and are linked by an edge id they are known to be synonyms. This is an undirected network as well.
12. **Power networks**: Power grid is described as an undirected network where nodes are generators, transformers and substations and the edges are high-voltage transmission lines.
13. **Neural networks**: Nerve systems of different animal species are studied. An undirected network nodes are neurons joined together by an edge if connected by either synapse or gap-junction.

Studies of complex systems stated above were performed by different scientists on different datasets of different network sizes, ranging from small networks with only few hundred nodes (ecological networks) to large networks with as many as 10^9 nodes like WWW. Studied networks are of both directed and undirected type. In researches the average path length among the nodes of a graph, clustering coefficient and degree distribution were measured and compared to the same properties of random graphs. For a estimation of clustering coefficient the directed networks need to be turned into undirected, since coefficient can only be calculated for undirected webs.
All the real networks mentioned in this section feature short average path lengths, large clustering coefficients and many of them have power-tail degree distribution and are scale free (WWW, cellular networks, Internet, some social networks and the citation networks). However, others like the power grid or the neural network appear to feature exponential or a coherent mixture of scale-free and exponential degree distributions. As it is shown in the paper below these networks are far from being random like ER random graphs, these systems are best described by evolving networks and can therefore develop both power - law and exponential degree distributions or a mixture of them. While the power - law regime appears to be robust, sublinear preferential attachment, aging effects, growth constraints lead to crossovers to exponential decay.

5 Theoretical models

Network is in mathematical terms represented by a graph, which is a pair of two sets $G = \{P,E\}$, where P is a set of N nodes and E is a set of edges (links, lines) that connect two elements of P. In this section different models will be presented. This models will help us understand complex systems found in nature, how they emerge and their development in time.

5.1 Random Graphs

Random graphs were first presented by P. Erdos and E. Reyni. They defined random graph as N nodes connected with n edges which are chosen randomly from $N(N-1)/2$ possible edges.

In total there are precisely $C^{n}_{\frac{N(N-1)}{2}}$ different graphs with N nodes and n edges possible, from which any graph is equiprobale.

An alternative definition is the binomial model: We start with N nodes and connect each pair of the nodes with probability p. Consequently the total number of edges is a random variable with the expectation value $E(n) = p\frac{N(N-1)}{2}$. If G_0 is a graph with N nodes and n edges, the probability of obtaining it by this graph construction process is binomial $P(G_0) = p^n(1-p)^{\frac{N(N-1)}{2}-n}$.

The construction of a graph is often called the evolution process. Starting with a set of N isolated vertices, the graph develops by the successive addition of random edges. The graphs obtained at different stages of this process correspond to larger and larger connection probabilities p, eventually obtaining a fully connected graph $p \to 1 \Rightarrow n = N(N-1)/2)$.

5.1.1 Degree distribution

In a random graph with connection probability p the degree ki of a node i follows a binomial distribution with parameters p and $N-1$:

$$P(k_i = k) = C^k_{N-1} p^k (1-p)^{N-1-k} \tag{1.1}$$

This probability represents the number of ways in which k edges can be drawn from a certain node: the probability of k edges is p^k , the probability of the absence of additional edges is $(1-p)^{N-1-k}$, and there are C^k_{N-1} equivalent ways of selecting k endpoints to these edges. If i and j are different nodes $P(k_i=k)$ in $P(k_j=k)$ are close to be independent random variables.

Expected number of nodes with degree k is:

$$E(X_k) = NP(k_i = k) = NC^k_{N-1} p^k (1-p)^{N-1-k} = \lambda_k \tag{1.2}$$

If estimated that degrees of different nodes are indipendent, we use the theorem from the Bollobas' probability theory used on graphs, which states that if expected values of moments of a random variable form a power series then the probability distribution of random variable limits to Poisson distibution (see B. Bollobas, 2001, Random Graphs (second Edition)).

$$P(X_k = r) = e^{-\lambda_k} \frac{\lambda_k^r}{r!} \tag{1.3}$$

with expected value λ_k . Poisson distribution falls very quickly, expected deviation is $\sigma_k = \sqrt{\lambda_k}$. With a bit of simplification we could say that (1.3) implies that X_k doesn't diverge much from the approximate result $X_k = NP(k_i = k)$, valid only if the nodes are independent. Thus with a good approximation the degree distribution of random graph is a binomial distribution which for large N limits to Poisson distribution:

$$P(k) \approx e^{-pN} \frac{(pN)^k}{k!} = e^{-\langle k \rangle} \frac{\langle k \rangle^k}{k!} \tag{1.4}$$

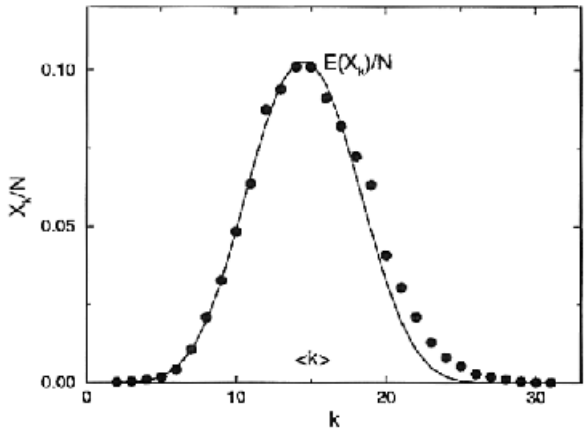

Figure: the degree distribution that results from numerical simulation of random graph. A graph with 10,000 nodes and connection probability p=0.0015 was created. The plot compares the experimental values (X_k / N) with the expectation value of Poisson distribution (1.3). The deviation is very small.

5.1.2 Connectedness and Diameter

The diameter of a graph is the maximal distance between any pair of its nodes. The diameter of a disconnected graph, which is made up of several isolated chunks, is infinite. Sometimes diameter is defined as the largest diameter of graph's components, but there are cases in which this definition can be misleading.

Random graphs tend to have small diameters if connection probability *p* is not too small. This is due to the fact that random graph is likely to be spreading: with a large probability the number of nodes at distance *l* from a given node is not much smaller than $\langle k \rangle^l$ - in this case the graph would need to be a tree. Equating $\langle k \rangle^l$ with *N* it is obvious that diameter is proportional with $\frac{\ln(N)}{\ln(\langle k \rangle)}$, thus it depends logarithmically on the number of nodes.

Diameter of random graphs has been studied by many authors. A general conclusion is that for most values of the connection probability *p*, almost all graphs have precisely the same diameter. This means when we consider all graphs with N nodes and connection probability p, the range in which the diameter values vary is very small, concentrated around:

$$d = \frac{\ln(N)}{\ln(pN)} = \frac{\ln(N)}{\ln(\langle k \rangle)} \tag{1.5}$$

The summarisation of important results:

- If $\langle k \rangle = pN < 1$: Graph is composed of isolated trees. Diameter of a graph equals the largest diameter of its subtrees.
- If $\langle k \rangle = pN > 1$: A giant cluster appears. The diameter of a graph equals the diameter of a giant cluster.
- If $\langle k \rangle = pN \geq \ln(N)$: the graph is fully connected. Its diameter is concentrated on a few values around $\frac{\ln(N)}{\ln(\langle k \rangle)}$.

Another way to characterise the spread of a random graph is to calculate the average distance among all the pairs of nodes. The average path length is expected to scale with N the same way the diameter does:

$$l_{G(N,p)} \sim \frac{\ln(N)}{\ln(\langle k \rangle)} \tag{1.6}$$

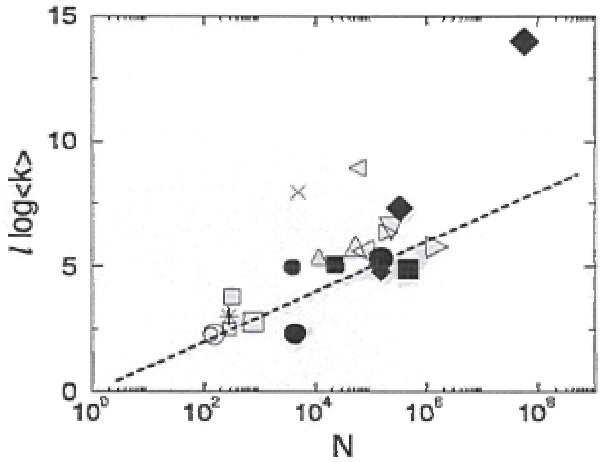

Figure: Comparison between the average path length of real networks from chapter 2 (Real Networks: Empirical Results) and the prediction (1.6) of a random graph theory (dashed line).

5.1.3 Clustering coefficient

Complex networks exhibit a large degree of clustering. If we consider a node in a random graph and its first neighbours, the probability that two of these neighbours are connected is equal with the probability that two randomly selected nodes are connected. Consequently the clustering coefficient of a random graph is

$$C_{G(N,p)} = p = \frac{\langle k \rangle}{N} \tag{1.7}$$

If the ratio $C_{rand} / \langle k \rangle$ is plotted as a function of *N* for random graphs of different sizes on a log-log scale plot they will align along a straight line of slope -1. When the ratios of the clustering coefficient of real networks is plotted against the prediction of clustering coefficient of random graphs, it is obvious that real networks do not follow the prediction of random graphs. The fraction $C / \langle k \rangle$ does not decrease as N^{-1}, instead, it appears to be independent of N.

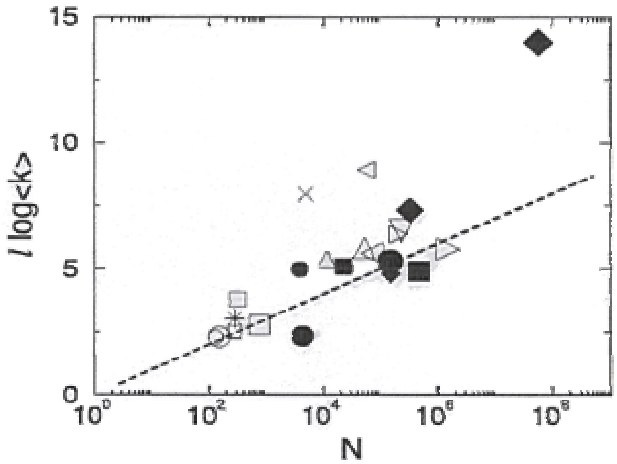

Figure. Comparison between the clustering coefficient of real networks and random graphs. All networks from Table 1 are included in the figure

5.2 Scale free model

Many large networks are scale - free: their degree distribution follows a power - law for large *k*. Even for those real networks for which *P(k)* has an exponential tail, the degree distribution significantly deviates from a Poisson. Random graph theory and the WS model are unable to reproduce this feature.

What is the mechanism responsible for the emergence of scale - free networks?

A shift from modelling network topology to modelling the network assembly and evolution is required to get insight into mechanisms responsible to create scale – free networks.

While the goal of the other models (random graphs and small - world models) is to construct a graph with correct topological features, modelling scale - free networks puts the emphasis on capturing the network dynamics. The assumption behind evolving or dynamic networks is that if we capture correctly the processes that assembled the networks that we see today, then we will obtain their topology correctly as well. Dynamics takes the driving role, topology being only a by product of this modelling philosophy.

5.2.1 Evolving Networks: Common generic mechanisms

The former network models (random networks and small worlds) assume that we start with a fixed number *N* of nodes that are then randomly connected or rewired, without modifying N, the actual number of nodes. In contrast, most real world networks describe open systems, which grow by the continuous addition of new nodes. Starting from a small number of nodes, the number of nodes increases through the lifetime of the network by the subsequent addition of new nodes.

Second, network models discussed so far, do not assume any dependence of connecting or rewiring probability on the node degree, or assume random placement or rewiring of edges.

Most real networks, however, exhibit preferential attachment, such that the likelihood of connecting to a node depends on the node's degree. (For example, a webpage will more likely include hyperlinks to popular documents with already high degree, because such highly connected documents are easy to find, etc).
These two ingredients, *growth* and *preferential attachment*, inspired the introduction of the scale - free (SF) model that has a power - law degree distribution. The algorithm of the SF model is the following:

1. ***Growth***: Starting with a small number (m_0) of nodes, at every timestep we add a new node with $m \le m_0$ edges that link the new node to *m* different nodes already present in the system.
2. ***Preferential attachmen****t*: When choosing the nodes to which the new node connects, we assume that the probability Π that a new node will be connected to node *i* depends on the degree k_i of node *i*, such that

$$\Pi(k_i) = \frac{k_i}{\sum_j k_i} \tag{2.1}$$

After *t* timesteps this algorithm results in a network with $N = t + m_0$ nodes and *mt* edges. Numerical simulations indicated that this network evolves into a scale - free state with the probability that a node has *k* edges following a power - law with an exponent $\gamma_{SF} = 3$, where the scaling exponent is independent of *m*, the only parameter in the model.

5.2.2 Average path length

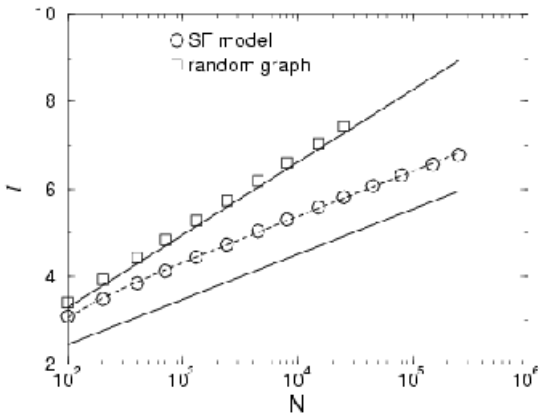

Figure shows the average path length of a network with average degree $\langle k \rangle = 4$ generated by the SF model as a function of the network size, N , compared with the average path length of a random graph with the same size and average degree. The figure indicates that the average path length is smaller in the SF network than in a random graph for any *N*, indicating that the heterogeneous scale - free topology is more efficient in bringing the nodes close than the homogeneous topology of random graphs. Average path length of the SF network increases approximately logarithmically with N . Dashed line follows an empirical generalised logarithmic fit*:*

$$l = A\log(N - B) + C \tag{2.2}$$

The solid lines represent the average path length calculated from generalised scale-free forced graphs estimation

$$l \approx \left(\ln\left[N/z_1\right] \Big/ \ln\left[z_2/z_1\right] \right) + 1 \qquad (2.3)$$

where z1 and z2 is the number of first and second neighbours. While this fit is good for a random graph it underestimates the average path length of the SF network, as it does with the real networks.

Apart from the empirical fit there is no theoretical expression that would give a good approximation for the path length in the scale - free model. The failure of (2.3) is due to the fact that the topology of the network generated by the SF model is different from the topology of a random network with power - law degree distribution. The dynamical process that generates the network introduces nontrivial correlations that affect all topological properties.

5.2.3 Clustering coefficient

There is no analytical prediction for the SF model.

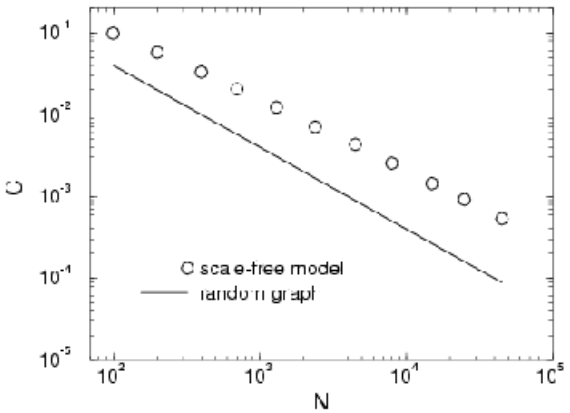

Figure shows the clustering coeffcient of the SF network with average degree $\langle k \rangle = 4$ and different sizes, compared with the clustering coefficient $C_{rand} = \langle k \rangle / N$ of a random graph. The clustering coefficient of the scale - free network is about 5 times higher than that of the random graph, and this factor slowly increases with the number of nodes. The clustering coefficient of the SF model decreases with the network size following approximately a power - law $C = N^{-0.75}$, random graphs exhibit faster decay $C = \langle k \rangle N^{-1}$, but small worlds show no dependence of *C* on *N* .

6 Application: Bose – Einstein condensation

Bianconi and Barabasi show the existence of a close link between evolving networks and an equilibrium Bose gas. Starting with the fitness – gets - richer model (FGR), where the connection probability of a node *i* is proportional to the degree and the fitness η_i of node *i*, the mapping to a Bose gas can be done by assigning an energy ε_i to each node. ε_i is determined by its fitness through the relation

$$\varepsilon_i = -\frac{1}{\beta}\log \eta_i \tag{2.4}$$

where $\beta = 1/T$ plays the role of inverse temperature. An edge between two nodes i and j, having energies ε_i and ε_j , corresponds to two non-interacting particles, one on each energy level Adding a new node to the network corresponds to adding a new energy level ε_l and 2m new particles to the system. Half of these particles are deposited on the level ε_l (since all new edges start from the new node), while the other half are distributed between the energy levels of the endpoints of the new edges, the probability that a particle lands on level *i* is given by:

$$\Pi_i = \frac{e^{-\beta\varepsilon_i} k_i}{\sum e^{-\beta\varepsilon_i} k_i} \tag{2.5}$$

In the $t \to \infty$ limit the occupation number, giving the number of particles with energy ε_l , follows the familiar Bose statistics:

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} - 1} \tag{2.6}$$

Maping to Bose gas predicts 2 distinct phases as a function of energy distribution. In the FGR phase the fittest node has the most of the edges, but it is not the absolute winner, since the ratio of its edges and the absolute number of edges in the network decays to 0 as the system size increases.
For $T < T_{BE}$ the condensation phase occures. The fittest node acquires a finite fraction of edges and maintaines the share over time.

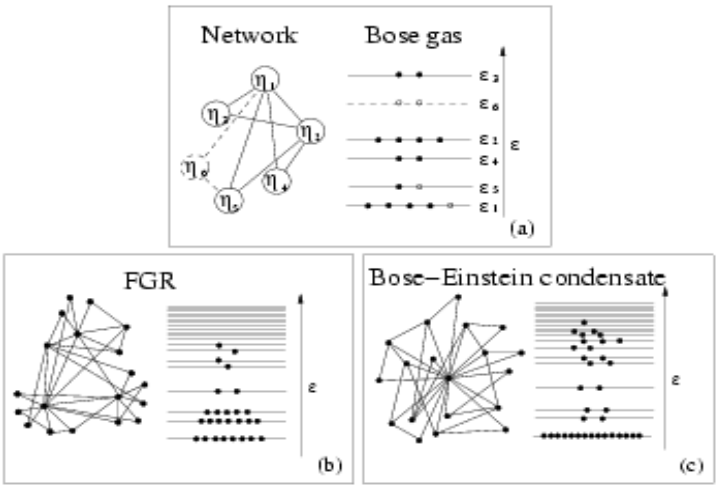

Figure: (a) Mapping between the network model and the Bose gas. (b) In the FGR phase we have a continuous degree distribution, the several high degree nodes linking the low degree nodes together. In the energy diagram this corresponds to a decreasing occupation number with increasing energy. (c) In the Bose-Einstein condensate the fittest node attracts a finite fraction of all edges, corresponding to a highly populated ground level, and sparsely populated higher energies.

7 Conclusion

In the paper above, it has been explained that many complex systems can be described by networks. It has been shown that few network properties exist by which the network topology can be determined. Most important properties are clustering coefficient, average path length and degree distribution. The user was presented real world examples that have been studied in recent few years and theoretical models that produce different network topologies as well. The high interest in scale - free networks might give the impression that all complex networks in nature have power - law degree distributions. It is true that several complex networks of high interest for the scientific community, such as the WWW, cellular networks, Internet, some social networks and the citation network are scale - free. However, others like the power grid or the neural network appear to be exponential, which doesn't mean that they are random. These systems are also best described by evolving networks and can therefore develop both power - law and exponential degree distributions or a mixture of them. While the power - law regime appears to be robust, sublinear preferential attachment, aging effects, growth constraints lead to crossovers to exponential decay. Thus, while evolving networks are rather successful at describing a wide range of networks, the functional form of $P(k)$ cannot be guessed until the microscopic details of the network evolution are fully understood.

8 REFERENCES

1. Aiello W., Chung F., Lu L., 2000, A Random Graph Model for Massive Graphs
2. Aiello W., Chung F., Lu L., 2000, A Random Graph Model for Power Law Graphs
3. Albert R., Jeong H., Barabasi A. L., 1999, Diameter of WWW. Nature 401,130
4. Albert R. ,Jeong H., Barabasi A. L., 2000, Large Scale Organization of Metabolic Networks. Nature 400, 651- 654
5. Albert R., Jeong H., Barabasi A. L., 2000, Scale-free characteristics of random networks: the topology of the WWW. Physica A 281, 69- 77
6. Albert R., Barabasi A. L., 2000, Topology of evolving networks: local events and universiality. (arXiv: cond-mat/0005085)
7. Albert R., Barabasi A. L., 2001, Statistical mechanics of Complex Networks. (arXiv: cond-mat/0106096)
8. Bianconi, G., Barabasi, A. L., 2000, cond-mat/0011224
9. Bollobas B., 2001, Random Graphs (second Edition)
10. Bollobas B., 1978, Extremal Graph Theory
11. Cohen R., Erez K., Ben-Avraham D., Havlin S., 2000, Resilience of the Internet to random breakdowns (arXiv: cond-mat/0007235)
12. Dorogovtsev, S. N. , J. F. F. Mendes, 2001, Evolution of Random Networks
13. Dorogovtsev, S. N. , J. F. F. Mendes, 2000, Structure of Growing Networks: Exacts solution of the BA model (arXiv: cond-mat/0004434)
14. Dorogovtsev, S. N. , J. F. F. Mendes, 2000,WWW and the internet models from 1955 till our days and the “popularity is attractive” principle
15. Kumar R., 2001, Stochastic models for the Web Graph
16. Krapivsky P. L., Render S., 2001, Organization of Growing Random networks (arXiv: cond-mat/0011094v2)
17. Newman, M. E. J., 2000, The Structure of Scientific Collaboration networks (arXiv: cond-mat/0007214)
18. Newman M. E. J., Strogatz S. H., Watts D. J., 2001, Random Graphs with arbitary degree distributions and their applications (arXiv: cond-mat/0007235)
19. Vetsigian K., 2001, Models of complex networks