Instability of Fluids

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April 19, 2000

Abstract

First we will look on a transition from a laminar to a turbulent flow arround a cylinder (von Kármán instability). In the next step we will introduce a linear stability analysis (method of normal modes). Having this tool, we will consider a Bénard convection. It will be only a sketch of real analysis. At the end some other instabilities will be presented. We shall pay more attention on experimental data than to rigorous mathematical description.

Contents

1 Introduction

The most important equation in hydrodinamics is Navier-Stokes[1] equation, wrote down in the middle of the nineteenth century. It says that the rate of change of velocity at all points in a fluid is proportional to the sum of the inertial forces.

$$\begin{aligned} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} &= -\nabla p + \nu \nabla^2 \mathbf{v} \\ \nabla \cdot \mathbf{v} &= 0 \end{aligned} \tag{1}$$

It is often difficult to solve this equation. Therefore much of the work on fluid mechanics revolves arround appropriate simplifications without losing the essential features of that flow.

2 Transition from a Laminar Flow to a Turbolent one

The process by which a laminar flow changes to a turbulent one is called *transition*. Instability of a laminar flow does not immediately lead to turbulence, which is a severly nonlinear and chaotic stage characterized by macroscopic mixing of fluid particles. After the initial breakdown of laminar flow because of amplification of small disturbances, the flow goes through a complex sequence of changes, finally resulting in the chaotic state we call *turbulence*. The sequence of events that lead to turbulence is greatly dependent on boundary geometry and shape of obstacle.

2.1 An Example - von Kármán Instability

Let us observe a flow past a cylinder with external flow velocity U_∞. The cylinder will deflect the flow. We would expect the disturbance of the flow to be increasingly pronounced as the flow gets faster, or as the cylinder gets bigger, or as the viscosity changes. Therefore we introduce a dimensionless number - Reynolds[2] number

$$R_e = \frac{U_\infty d}{\nu},$$

[1]Claude Louis Marie Henri Navier, February 10, 1785 – August 21, 1836
George Gabriel Stokes, August 13, 1819 – February 1, 1903

[2]Osborne Reynolds, August 23, 1842 - February 21, 1919

based on the cylinder diameter, the fluid viscosity, and the upstream velocity. Now we can observe what happens when increasing the Reynolds number.

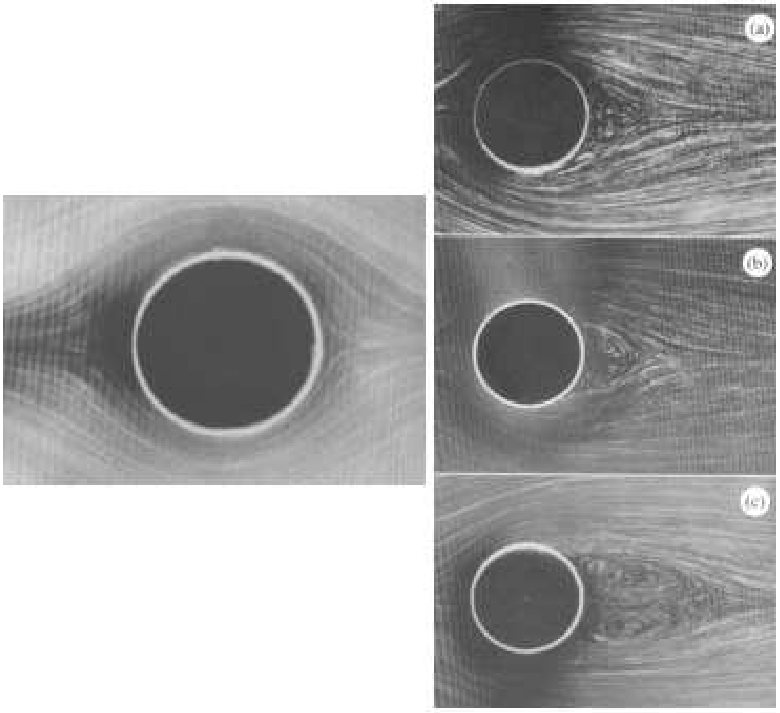

Figure 1: On left uniform flow past a cylinder at $R = 0.16$. On right flow at (a) $R = 9.6$, (b) $R = 13.1$ and (c) $R = 26$. [2]

For $R_e < 4$ nothing much happens at all. The streamlines bend arround the cylinder and then become parallel again (Fig. 1, left).

As R_e is increased above 4, we can find two little vortices of circulating fluid (called eddies) immediately behind the cylinder. The eddies circulate in opposite directions. The wake is completely laminar. As R_e is increased, but still smaller than 40, the eddies get longer (Fig. 1, a.,b., and c.).

Beyond a Reynolds number of about 40, something dramatic starts to happen to the wake. It becomes unstable and develops a slow oscillation. The oscillating rolls up into two staggered rows of vortices with opposite sense of rotation. This pattern is called a *Kármán*[3] *vortex street* (Fig. 2). The vortices are carried along with the flow, but more slowly than the average speed of the flow. They slowly dissipate their energy through viscous drag and vanish further downstream.

[3]Theodore von Kármán, May 11, 1881 – May 6, 1963

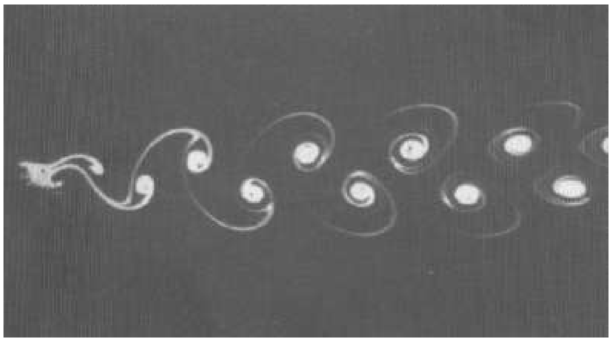

Figure 2: Kármán vortex street behind a cylinder at $R = 105$. [2]

In the range $40 < R_e < 80$ the vortex street does not interact with the pair of attached vortices. But beyond 80 the vortex street forms closer to the cylinder and attached eddies themselves begin to oscilate. Finally the attached eddies periodically break off alternately from the cylinder. As vortices of opposite circulations are shed off from the two sides, the circulation arround the cylinder changes sign, resulting in an oscillating lift. If the frequency of vortex shedding is close to some resonant frequency of the body, then an appreciable lateral vibration has been observed (the "singing" of telefon and transmission lines).

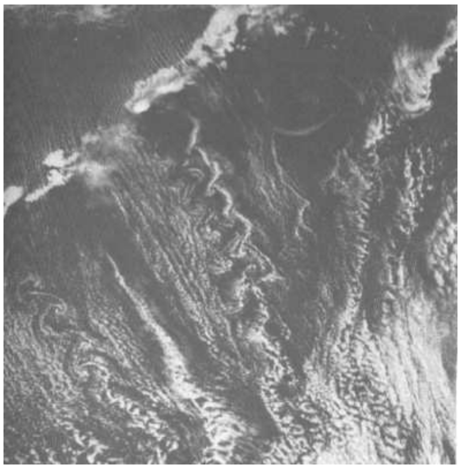

Figure 3: Kármán vortex street downstream of mountains peaks in atmosphere. North is upward, and the wind is blowing toward the southeast. [3]

Below $R_e = 200$ the vortices in the wake are laminar. Above this number, the vortex street becomes unstable and irregular. Finally above a very high R_e number $\approx 5\,000$ the wake may be described as completely turbulent.

3 Seeking the Solution

We will examine stability of certain fluid flows with respect to infinitesimal disturbances. We shall introduce perturbation on a particular flow, and determine whether the equations of motion demand that the perturbations should grow or decay with time.

In this analysis the problem is linearized by neglecting terms quadratic in the perturbation variables and their derivatives. This linear method therefore only examines the initial behavior of the disturbances. The loss of stability does not in itself constitute a transition to turbulence.

3.1 Method of Normal Modes

The method of linear stability analysis consists of introducing sinusoidal disturbances on a *basic state* (background, initial state), which is the flow whose stability is being investigated.

For example we have a flow parallel to x-axis and varying along y-axis.

$$\mathbf{U} = (U(y), 0, 0)$$

On this background we superpose a disturbance[4]

$$\mathbf{u}(x,t) = u(y)e^{ikx+imz+\sigma t} \quad , \quad u(y) \in \mathcal{C} \tag{2}$$

The flow field is assumed to be unbounded in the x and z direction, so that the wavenumbers k and m can only be real in order that the dependent variables remain bouded, as $x, z \to \infty$. σ is regarded as complex

$$\sigma = \sigma_r + i\sigma_i.$$

If $\sigma_r > 0$ for any value of the wavenumber, the system in unstable. We say that:

$$\begin{array}{cc} \sigma_r < 0 & \text{stable} \\ \sigma_r > 0 & \text{unstable} \\ \sigma_r = 0 & \text{neutrally stable} \end{array}$$

[4]We should take the real part of the right side to obtain physical quantities.

The method of analysis involving the examination of Fourier components such as equation (2) is called the normal mode. An arbitrary disturbance can be decomposed into a complete set of normal modes. The linearity of the problem implies that the various modes do not interact.

The boundary between stability and instability is called the *marginal state*, for which $\sigma_r = 0$. There can be two types of marginal state depending on σ_i:

- if $\sigma_i = 0$ the marginal state is characterized by a *stationary pattern* of a motion,
- if $\sigma_i \neq 0$ then the instability sets in as oscillations of growing amplitude.

4 Thermal Instability: The Bénard Problem

A layer of fluid heated from below is "top heavy", but does not necessarily undergo a convective motion. This is because the viscosity and thermal diffusivity of fluid try to prevent the appearance of convective motion, and only for large enough temperature gradients is the layer unstable.

The first experiments on this instability were conducted by Henry Bénard in 1900. Stimulated by these experiments, Rayleigh[5] in 1916 derived the theoretical requirement for the development of convective motion in a layer of fluid with two free surfaces. He showed that the instability would occur when the ratio

$$Ra = \frac{\text{bouyancy force}}{\text{viscous force}} = \frac{g\alpha\Gamma d^4}{\kappa\nu} \tag{3}$$

exceeds a certain value. Here g is the acceleration due to gravity, α is the coefficient of thermal expansion, $\Gamma = -dT/dz$ is the vertical temperature gradient of the background state, d is depth of layer, κ is the thermal diffusivity, and ν is kinematic viscosity.

The parameter Ra is called Rayleigh number and we can see this as the ratio of the destabilizing efect of buoyancy force to the stabilizing efect of viscous force.

4.1 Formulation of the Problem

Consider a layer confined between two isothermal walls (Fig. 4), in which the lower wall is maintained at a higher temperature. For flows satisfying certain conditions, we can neglect the density changes in the fluid, except in

[5] John William Strutt Rayleigh, November 12, 1842 – June 30, 1919

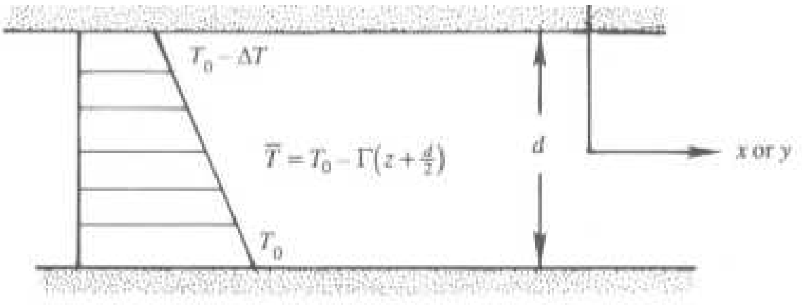

Figure 4: Definition sketch for the Bénard problem. [3]

the gravity term where ϱ is multiplied by g. This approximation treats the properties of the fluid (such as C_p, κ) as constant. And very important, the temperature gradient stays constant and uniform.

Then we take the Navier-Stokes equation (1), the heat and continuity equation.

$$\begin{aligned}
\frac{\partial \tilde{u}_i}{\partial t} + \tilde{u}_j \frac{\partial \tilde{u}_i}{\partial x_j} &= -\frac{1}{\varrho_0}\frac{\partial \tilde{p}}{\partial x_i} - g\left[1 - \alpha\left(\tilde{T} - T_0\right)\right]\delta_{i3} + \nu\nabla^2\tilde{u}_i, \qquad (4)\\
\frac{\partial \tilde{T}}{\partial t} + \tilde{u}_j \frac{\partial \tilde{T}}{\partial x_j} &= \kappa\nabla^2\tilde{T},\\
\frac{\partial \tilde{u}_i}{\partial x_i} &= 0,\\
\tilde{\varrho} &= \varrho_0\left[1 - \alpha\left(\tilde{T} - T_0\right)\right] \quad , \quad \varrho_0 \text{ density at } T_0.
\end{aligned}$$

The total flow variables (background plus perturbation) are represented by a tilde (˜). We decompose the motion into a background state of no motion plus perturbations:

$$\begin{aligned}
\tilde{u}_i &= 0 + u_i(x,t),\\
\tilde{T} &= \bar{T}(z) + T'(x,t),\\
\tilde{p} &= P(z) + p(x,t).
\end{aligned}$$

4.2 Solution of the Problem

We introduce the method of normal modes on variables of temperature T' and velocity w, where w is vertical component of velocity. To obtain the solution, we put this in equations (4) and linearize them. For detailed steps see [3], pages 352 to 360.

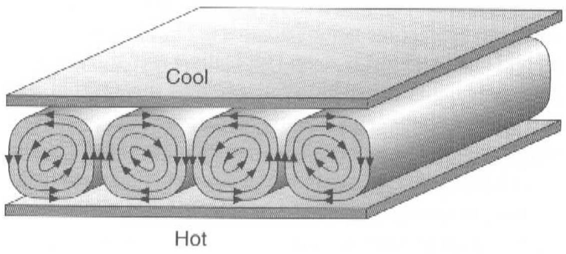

Figure 5: Convection roll cells are roughly square in cross-section, and adjacent cells rotate in opposite directions. Picture from [1].

$$
\begin{aligned}
w &= w(z)e^{ikx+ily+\sigma t} \\
T' &= T(z)e^{ikx+ily+\sigma t} \\
K &= \sqrt{k^2+l^2}
\end{aligned}
$$

The solution of eigenvalue problem gives a relation between Ra and the corresponding eigenvalue K.

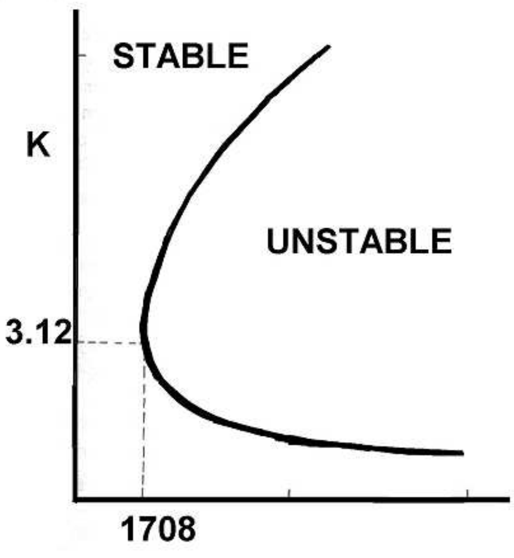

Figure 6: Stable and unstable regions for two rigid plates. [3]

1. We first give the solution (Fig. 6) for the case that is easiest to realise in a laboratory experiment, namely a layer of fluid confined between two rigid plates where no-slip conditions are satisfied. The lowest value of Ra is found to be

$$Ra_{cr} = 1708 \quad , \quad K_{cr} = 3.12.$$

   Since all values of K are allowed by the system, the flow first becomes unstable when the $Ra = Ra_{cr}$. The wavelength at the onset of instability is

$$\lambda = \frac{2\pi d}{K_{cr}} \approx 2d.$$

2. We now give the solution for a layer of fluid with stress-free surfaces. This case can be approximately realized in a laboratory experiment if a layer of liquid is floating on top of a somewhat heavier liquid. The solution is

$$Ra_{cr} = \frac{27}{4}\pi^4 = 657 \quad , \quad K_{cr} = \frac{\pi\sqrt{2}}{2}.$$

3. For a layer with free upper surface (where the stress is zero) and a rigid bottom wall, the solution gives

$$Ra_{cr} = 1101 \quad , \quad K_{cr} = 2.68.$$

   This case was observed by Bénard.

4.3 Which pattern is chosen?

Experiments show, that many patterns are possible in Bénard convection. A pattern formation is like a phase transition and breaks a symmetry. In our case we see transition from an isotropic state to a roll-like or a hexagon-like pattern, where some directions are special. In the same way during the freezing of water the crystal structure of ice picks out certain directions. In both cases we cannot predict this directions. A linear stability analysis of convection does not help us. So the question of whether rolls or hexagons will be preferred has no universal answer.

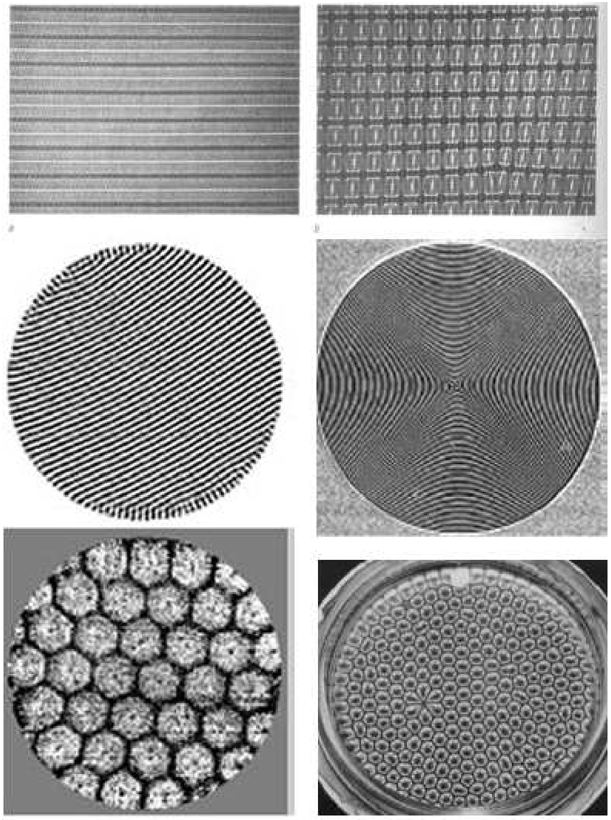

Figure 7: Different patterns in Bénard convection. Pictures in the middle and at bottom left are from [5]. Pictures on top are from [1] and picture at bottom right is from [5].

5 Other Instabilities

Now we will have a quick look on some other type of instabilities.

5.1 Taylor Instability

We shall consider a flow between concentric rotating cylinders, a problem first solved by G. I. Taylor[6] in 1923. In this flow the source of instability is the adverse gradient of angular momentum. We are interested, when the centrifugal force overcomes the viscous force. The critical number (called Taylor number) and corresponding wavenumber are (from [3])

[6]Geofrey Ingram Taylor, March 7, 1886 – June 27, 1975

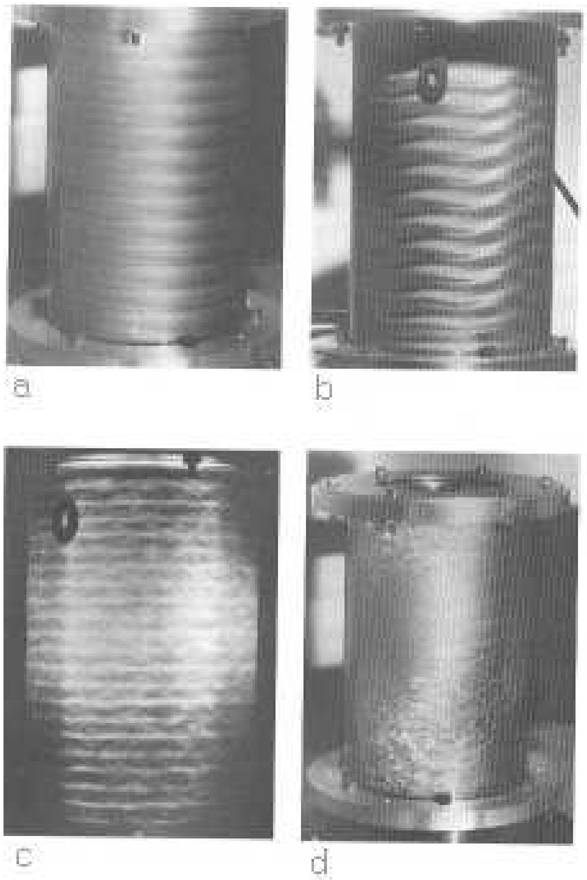

Figure 8: A flow between two concentric rotating cylinders. The Reynolds number increases from (a) to (d). [1]

$$\begin{aligned} Ta_{cr} &= \frac{1708}{\frac{1}{2}\left(1+\frac{\Omega_1}{\Omega_2}\right)}, \\ Ka_{cr} &= 3.12 \quad , \quad \lambda_{cr} \approx 2d, \end{aligned}$$

where Ω_1 is angular velocity of the inner cylinder and Ω_2 of the outer cylinder.

Let us first keep the outer cylinder fixed, and just rotate the inner one. At low speeds the fluid simply tracks the rotation. But when we turn up the speed, something happens. Not only is the fluid carried arround in circles, but it is simultaneously forced outwards. When the centrifugal force can no longer be resisted, patterns start to appear. These are rool-like vortices in which fluid circulates in alternate directions. We call them the Taylor

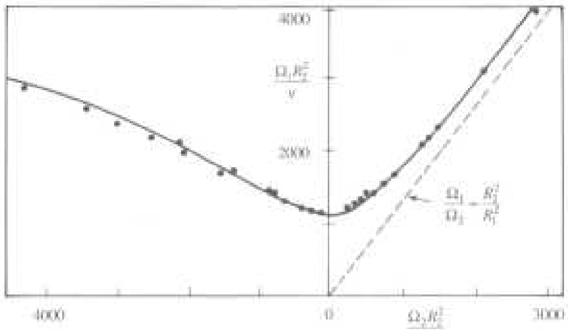

Figure 9: G. I. Taylor's observation and narrow-gap calculation of marginal stability in rotating flow of water. The ratio of radii is $R_2/R_1 = 1.14$. The region above the curve is unstable. The dashed line represents Rayleigh's inviscid criterion. [3]

vortices (Fig. 8). Increasing the Reynolds number further brings a series of instabilities. First they go wavy, undulating up and down arround the cylinder. Then everything gets more complex and more or less turbulent.

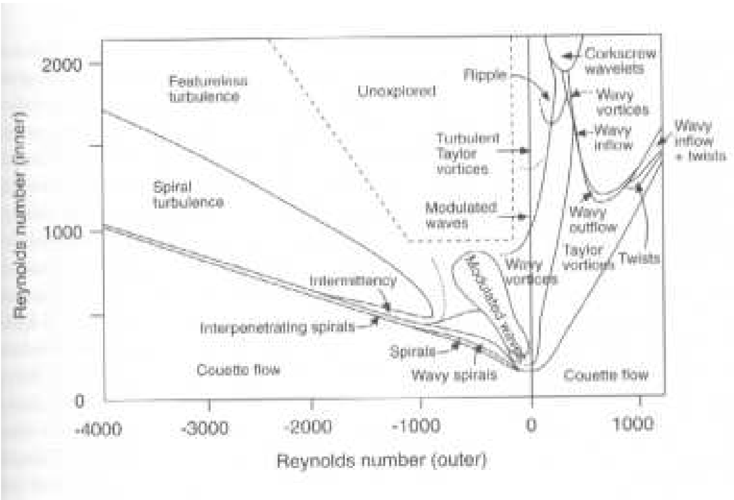

Figure 10: Complexity of flow between two rotating cylinders. [1]

5.2 Kelvin-Helmholtz Instability

Instability at the interface between two horizontal parallel streams of different velocities and densities, with the heavier fluid at the bottom is called the

Figure 11: Kelvin-Helmholtz instability generated by tilting a horizontal channel containing two liquids of different densities. [6]

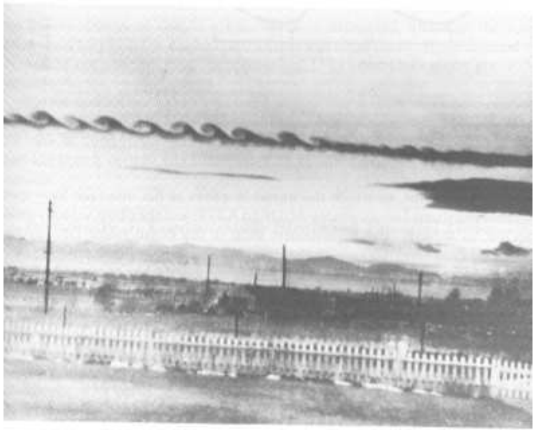

Figure 12: Kelvin-Helmholtz instability: billow cloud near Denver, Colorado. [3]

Kelvin-Helmholtz[7] instability (Fig. 11).

Instability is caused by the destabilizing effect of shear, which overcomes the stabilizing effect of stratification. This kind of instability is easy to generate in the laboratory by filling a horizontal glass tube containing two liquids of slightly different densities and gently tilting it. This starts a current in the lower layer down the plane and a current in the upper layer up the plane.

[7]William Thomson Kelvin, June 26, 1824 – December 17, 1907
Herman Ludwig Ferdinand von Helmholtz, August 31, 1821 – September 8, 1894

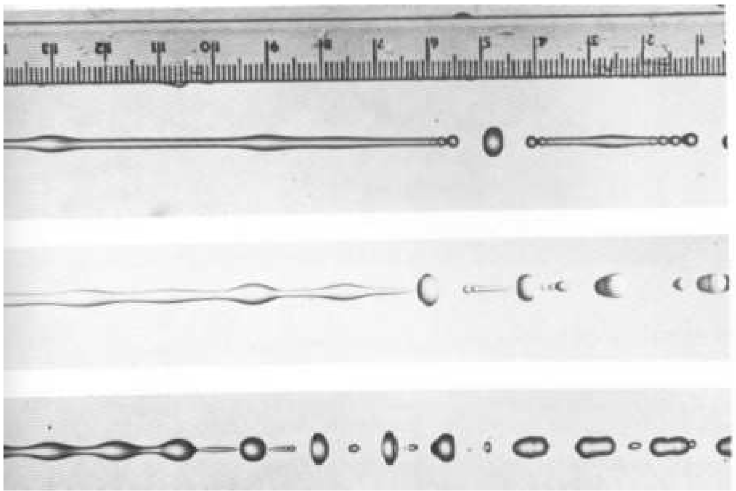

Figure 13: Water forced from a $4mm$ tube is perturbed at various frequencies by a loudspeaker. The wavelength is 42, 12.5 and 4.6 diameters, the last being nearly Rayleigh's value for maximum growth of disturbances. [4]

5.3 Rayleigh instability

Suppose a thin, cylindrical column of liquid falling down. It is unstable in the sense of tiny disturbances. If the column developes a slightly wavy unevenness, surface tension acts to accentuate the spherical droplet (Fig. 13). There is a certain wavelength, that is the most stable and this determines the size of droplets.

6 Conclusion

We have seen many patterns and their complexity in different flows. More attention was put to experiments than to the mathematical analysis. This was done for practical reasons because the theory is far too complex even in its simplest form. However, we described the priciples of the normal modes method. The experimental results for the Bénard and the Taylor problem agree well with theory predictions. Nevertheless, the method can describe only the early stage of instability. What is beyond and how the turbulence (most flows in practice are turbulent) is developed, we can see only through experiment and the theory of Kolmogorov from 1941. In this field of physics many open questions wait to be answered.

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[5] http://tweedledee.ucsb.edu/ guenter/picturepage1.html

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[7] http://www-history.mcs.st-andrews.ac.uk/history/index.html