University *of Ljubljana*
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SEMINAR

Statistical mechanics of money, wealth and income

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Abstract

This seminar is concerned with the application of physics to the statistical models for money, wealth and income distributions, as an example of the methods of econophysics (the history of which is briefly reviewed). By analogy with the Boltzmann-Gibbs distribution of energy in the canonical ensemble, the distribution of money is shown to be exponential for certain classes of models. Alternatives are also reviewed. With respect to income and wealth distributions, empirical data reveal a two-class distribution with the majority of the population exhibiting an exponential distribution and a small fraction in the upper class being governed by a power-law tail. A model is presented which explains these basic characteristics for income distributions.

Contents

1 Introduction

Econophysics is a relatively recent interdisciplinary research field, which seeks to apply the methods of statistical mechanics in particular, and those of theoretical physics in general, to the study and modeling of (complex) economic phenomena and financial systems.

The application of physics to different fields of science is not at all uncommon - astrophysics, geophysics and biophysics, to name just a few, fit this designation. It should be emphasized however, that, by contrast with (say) astrophysics; econophysics does not literally apply the laws of physics, such as e.g. those of quantum mechanics, to humans; rather it uses mathematical methods developed in statistical physics to study complex systems consisting of a large number of interacting economic agents (individuals, corporations, . . .). In this sense, econophysics might be considered an applied branch of probability theory. "However, statistical physics is distinctly different from mathematical statistics in its focus, methods and results," [1, page 1][1]. On the other hand (from the economists' perspective) econophysics is closer to econometrics, rather than the verbose style of classical political economy, emphasizing quantitative analysis of large amounts of economic and financial data.

Historically, statistical mechanics was developed in the second half of the nineteenth century, primarily by James Clerk Maxwell, Ludwig Boltzmann and Josiah Williard Gibbs. Its origins were not without connection to social statistics - indeed [1, page 2] cites Ludwig Boltzmann as saying:

> "The molecules are like individuals, . . . and the properties of gases only remain unaltered, because the number of these molecules, which on the average have a given state, is constant,"

as well as, with regard to the systematic development of statistical mechanics by Gibbs:

> "This opens a broad perspective, if we do not only think of mechanical objects. Let's consider to apply this method to the statistics of living beings, society, sociology and so forth."

Many now-famous economists originated from physics and engineering. For example, Vilfred Pareto earned a degree in mathematical sciences and a doctorate in engineering. He was the first to discover that income follows a power law probability distribution, [4].

With time, however, both physics and economics became more specialized, detached from one another, and the social origin of statistical physics was largely forgotten - only to await rediscovery with the advent of econophysics. The beginnings of the latter may be found largely during the twentieth century in a series of books and articles attempting to (once again) bring together the two disciplines, but it was not until the late 1990s that a robust movement of econophysics was finally brought to life. Indeed the term "econophysics" first appeared in print in 1996 in an article by Stanley *et al.*, [5], wherein a manifesto of the new field was presented, arguing that:

> "behavior of large numbers of humans (as measured, e.g., by economic indices) might conform to analogs of the scaling laws that have proved useful in describing systems composed of large numbers of inanimate objects."

Shortly thereafter the first econophysics conferences were organized and the book *An introduction to Econophysics* by Mantegna and Stanley was published.

[1]We should note here that the seminar is based to a great extent on [1] and [2]. Thus, whenever claims are made but not shown, they are generally so given with gratitude and reference to these two sources (unless otherwise stated).

In what follows we shall consider the application of the methods of statistical physics to the study of the distribution of money, wealth and income[2] in a given society of interacting economic agents - displaying the type of results which are obtained with a sensible application of the methods of physics to the analysis of socioeconomic phenomena. Unlike many papers in economic literature, which use a stochastic process to derive wealth and income distributions, considering the latter's fluctuations independently and separately for each economic agent (what could be called a one-body approach); econophysics introduces an alternative two-body approach, in which agents perform pairwise economic transactions, whereby transferring money holdings from one to another (not unlike molecules in a gas, which transfer kinetic energy during an elastic collision).

2 The Boltzmann-Gibbs distribution of energy

In this section we offer a brief recap of the central result of equilibrium statistical mechanics, namely the *Boltzmann-Gibbs* distribution of energy and the *canonical ensemble*. The latter represents a probability measure on the statistical ensemble of a large number of mental copies of a system, considered all at once, each of which represents a possible (microscopic) state that the real (macroscopic) system might be in (the so-called *phase space*). We introduce ρ so that $\rho \mathrm{d}\Gamma$ represents the probability that the system is located in the $\mathrm{d}\Gamma$ vicinity of a point in the phase space at a certain point in time. Naturally $\int \rho \mathrm{d}\Gamma = 1$. Under the assumption that we are dealing with a large number of members constituting the system under consideration (statistical character) and that the average energy of the system over time is constant (with only minor variations as a result of the interaction of the system with its neighborhood) the *canonical distribution* may be derived, giving ρ as a function of the energy ϵ of the system only (the *canonical ensemble*):

$$\rho(\epsilon) = c e^{-\epsilon/k_B T} \tag{2.1}$$

where k_B is Boltzmann's constant and T is the temperature of the system under consideration, c is a normalizing constant (cf. e.g. [10, page 117]). The expected value of any physical variable x is then obtained *via*:

$$\langle x \rangle = \frac{\sum_k x_k e^{-\epsilon_k/k_B T}}{\sum_k e^{-\epsilon_k/k_B T}}, \tag{2.2}$$

where the sum (possibly integral) is taken over all the possible states of the system. Whenever energy E is given simply by the sum of the respective energies of the constituent particles and the $\mathrm{d}\Gamma$-element of the phase space may be suitably factored (as is the case, e.g. for the classical ideal gas), it makes sense to consider the probability that a given (randomly chosen) particle is in a state i having energy ϵ - the latter then clearly being proportional to:

$$P(\epsilon) \propto e^{-\frac{\epsilon}{k_B T}}, \tag{2.3}$$

the *Boltzmann-Gibbs distribution of energy* (here $\langle E \rangle \sim k_B T$ up to a numerical coefficient of order 1). (2.3) may be derived in different ways with all derivations including two main ingredients: the statistical character of the system and conservation of energy. One of the shortest derivations, albeit a naïve one, considers the system divided into two parts. The total energy is the sum of the parts $\epsilon = \epsilon_1 + \epsilon_2$, whereas

[2] One should recognize that these three categories are distinctly different, even if interconnected. Wealth has money holdings of an individual as a (proper) part and includes all material possessions, whereas income is the gross "inflow" of money per unit of time.

the probability is the product of probabilities $P(\epsilon) = P(\epsilon_1)P(\epsilon_2)$. Thus $P(\epsilon_1+\epsilon_2) = P(\epsilon_1)P(\epsilon_2)$, whence $\ln P(\epsilon_1+\epsilon_2) = \ln P(\epsilon_1) + \ln P(\epsilon_2)$ and upon differentiation with respect to ϵ_1 and ϵ_2, $\frac{P'(\epsilon_1)}{P(\epsilon_1)} = \frac{P'(\epsilon_2)}{P(\epsilon_2)} = \frac{P'(\epsilon_1+\epsilon_2)}{P(\epsilon_1+\epsilon_2)}$, which is possible only if all expressions are equal to some constant, say $-\beta$, whence $P(\epsilon) \propto e^{-\beta\epsilon}$.

A perhaps more convincing account is that which uses the concept of entropy. Consider N particles with a total energy E. Let N_k denote the number of particles with energy between ϵ_k and $\epsilon_k + \Delta\epsilon$ (so that the ratio $\frac{N_k}{N} = P_k$ gives the probability of a particle having energy in the interval $[\epsilon_k, \epsilon_k + \Delta\epsilon]$). The number of permutations of the particles, which keep the energy distribution constant, among the "energy bins" of size $\Delta\epsilon$ is

$$W = \frac{N!}{\Pi_k N_k!}.$$

The natural logarithm of the multiplicity W is called the entropy (up to a multiplicative constant)

$$S = \ln W. \tag{2.4}$$

As $N \to \infty$ we may use Stirling's approximation $\ln N! \approx N(\ln N - 1)$, and we have

$$\frac{S}{N} \approx \frac{N(\ln N - 1)}{N} - \sum_k \frac{N_k(\ln N_k - 1)}{N} = \sum_k \frac{N_k}{N}\ln N - \sum_k \frac{N_k}{N}\ln N_k = -\sum_k \frac{N_k}{N}\ln\frac{N_k}{N} = -\sum_k P_k \ln P_k,$$

where we have taken into account that $\sum_k N_k = N$. We seek the distribution of energies for which entropy is maximal. Applying the method of Lagrange's multipliers (with constraints $\sum_k N_k\epsilon_k = E$ and $\sum_k N_k = N$), we have the Lagrangian $L = -\sum_k P_k \ln P_k + \lambda(\sum_k P_k - 1) + \mu(\sum_k P_k\epsilon_k - \frac{E}{N})$ and the partial derivative with respect to P_k yields $-\ln P_k - 1 + \lambda + \mu\epsilon_k = 0$ and hence $P_k \propto e^{\mu\epsilon_k}$, which is just the Boltzmann-Gibbs distribution.

3 Statistical mechanics of money distribution

The derivations put forth under Section 2 in connection to the Boltzmann-Gibbs distribution, are very general and use as arguments only the statistical character of the system and conservation of energy. As such one may legitimately expect the conclusions to hold true (by analogy) of any other statistical system with a conserved quantity. Certainly the economy is a big statistical system with millions of participating agents ($1 \ll N$, for our purposes N shall be constant) and so presents itself as a possible and promising target for the application of statistical mechanics. All that remains is to find a sensible conserved quantity!

3.1 Conservation of money

Money m seems to be a viable option in this respect. Indeed ordinary economic agents, excluding central banks, can only receive money from and give money to other agents. Unlike central banks, they are not allowed to produce money "out of thin air", by e.g. printing euro bills. Thus, in any pairwise transaction of two economic agents we have a *local conservation of money*. More precisely, if agents i and j perform a transaction in which the first pays the second an amount of money Δm, then the money balances change as follows:

$$\begin{aligned} m_i \to m_i' &= m_i - \Delta m \\ m_j \to m_j' &= m_j + \Delta m \end{aligned} \tag{3.1}$$

and as regards local conservation of money,

$$m_i + m_j = m_i' + m_j'. \tag{3.2}$$

The rule for the transfer of money is analogous to the transfer of energy from one molecule to another in molecular collisions in a gas and (3.2) is analogous to conservation of energy in said collision. Enforcement of (3.2) is key to a successful functioning of a *fiat* money regime (wherein the value of the money is not tied to some material entity, e.g. gold (viz. the gold standard), but declared to be legal tender by the central bank[3]). If economic agents were, as a rule, allowed to "manufacture" money, they would be printing it by the dozens and purchasing goods and acquiring services therewith - seeing as how the process of printing (or indeed merely changing the balance on a computerized bank account) requires little to no effort, whereas in the process of purchasing a good or service, actual wealth (utility for the consumer) is gained.

It should be emphasized that in the model described above, money flows represent payments for goods and services - and yet the model only keeps track of the first and not of the second. One important reason why this is so, is the fact that many goods disappear after consumption, and in addition services (e.g. haircuts) can hardly be considered tangible assets, which would allow for a sensible quantitative treatment. Because they are not conserved and because they are measured in different physical units, they are not very practical aggregates, the distribution of which would then be analyzed.

Another very important remark is that the addition of material wealth to an economy does not automatically translate into an increase in the money supply - ordinary economic agents can (are allowed to) "grow apples on trees, but cannot grow money on trees," as Yakovenko and Rosser eloquently put it in [1, page 5]. Indeed only the CB of some monetary area (ECB for the Eurozone, the Fed for the USA, etc.) is allowed to increase the monetary base M_b (also base money, money base, high-powered money; comprising coins, paper money and commercial banks' reserves with the CB) - debt and credit issues are deferred until Subsection 3.3.

With regard to *global conservation of money* the following two points are thus pertinent. First, unlike ordinary economic agents, the central bank has the exclusive right and privilege of injecting new money into the system, whereby changing the total amount of money in an economy. This process is analogous to an influx of energy into a system from external sources. In dealing with such a situation, physicists will tend to ignore the influx on a temporary basis, treating the system as being closed and in thermal equilibrium and only then generalizing to an open system. Thus, as long as the *influx of money from CB sources is slow enough* compared with relaxation processes in the economy and does not cause hyperinflation, the system is in a quasi-stationary statistical equilibrium with slowly changing parameters (temperature T in a gas, which is being slowly heated by an external heat flow: or the average money per individual in an economy). Second, money flows may also occur between different monetary areas. This process involves issues of exchange rates between different currencies and is set aside here - whereby an idealization of a *closed economy with a single currency* is made (not at all uncommon in economic literature).

Indeed the concept of equilibrium is very common to economic discussions. In what follows the usual supply-demand "mechanical" equilibrium is extended to a statistical equilibrium, characterized by a stationary probability distribution of money $P(m)$ (here $P(m)\mathrm{d}m$ is the probability of a randomly chosen economic agent having a money balance between m and $m + \mathrm{d}m$).

[3]Abbreviated sometimes to CB.

3.2 Boltzmann-Gibbs distribution of money

In this subsection the simplest model of money distribution is reviewed, one in which no debt or credit is allowed (and the economy is closed, with the influx of new money from the CB allowing for an equilibrium and not causing hyperinflation[4]). Under such a regime an individual's money holdings are not allowed to drop below zero, $m_i \geq 0$, and transaction (3.1) is only allowed when the agent paying has enough money to do so without "going into debt". The boundary $m = 0$ is equivalent to the ground state energy in physics (or, say, the positive kinetic energy of the molecules). If an agent spends all his money, his balance drops to zero and he can no longer buy goods or services, though he may still receive money in payment of selling/providing them. In the real world, this situation is not uncommon for people living from paycheck to paycheck.

Under the assumptions stipulated and with the results of Section 2 and the discussions of Subsection 3.1 firmly in place, it is not unreasonable to expect that the distribution of money m is exponential, i.e. given by

$$P(m) = ce^{-\frac{m}{T_m}}, \tag{3.3}$$

whenever $m \geq 0$ and $P(m) = 0$ otherwise (c is a normalizing constant). If we approximate sums by integrals, we have $\int_0^\infty ce^{-\frac{m}{T_m}} \mathrm{d}m = 1$ and hence $c = \frac{1}{T_m}$; and for the average money holdings of an individual we obtain (*via per partes* - $\frac{\mathrm{d}}{\mathrm{d}m}(me^{-m/T_m}) = e^{-m/T_m} - \frac{m}{T_m}e^{-m/T_m}$) $\frac{M}{N} = \int_0^\infty \frac{m}{T_m}e^{-m/T_m}\mathrm{d}m = T_m$ - where M is now the monetary base and N the number of agents - T_m being equal to the average amount of money per agent.

To verify this conjecture of sorts Drăgulescu and Yakovenko, [2, pages 2-3], performed agent-based computer simulations. All the agents are initially given the same amount of money, say $M/N = 1000$ dollars and then a pair of agents (i, j) is randomly selected and the amount Δm is transferred from the first to the second (if condition $m_i \geq 0$ is fulfilled, otherwise the transaction does not take place). The process is then repeated. Several different rules for Δm may be used. In the simplest Δm is held fixed at say 1 dollar (economically all agents sell and buy products for the same price) and while the example is instructive it is not very realistic. In another model Δm in each transaction is taken to be a random fraction ν of the average amount of money per agent, i.e. $\Delta m = (M/N)\nu$, where ν is uniformly (continuously) distributed on the interval $[0, 1]$ (reflecting different prices for different goods and services, as well as different quantities bought and sold). In yet another variation, Δm equaled a random fraction of the average amount of money of the two agents $\nu(m_i + m_j)/2$. The final distribution was universal despite these different rules and the result is shown in Figure 1.

One might (correctly) argue that the pairwise exchange of money corresponds more to a medieval market rather than the modern big-company driven economy. To such an end a model may be considered in which one agent at a time is appointed to become a firm. The firm borrows capital $K \geq 0$ from another agent and returns it with interest hK, $h \geq 0$, hires $L \geq 0$ agents and pays them wages $\omega \geq 0$, manufactures $Q \geq 0$ items of a product and sells them at a price $p \geq 0$ for a total profit $F = pQ - \omega L - hK$ (the net result of the company is a "many-body" transfer of money, which still satisfies the conservation law). The aggregate demand-supply curve for the product may be taken in the form $p(Q) = \frac{v}{Q^\eta}$ - here Q is the quantity consumers are willing to buy at price p, $\eta > 0$ and $v > 0$ are some parameters. The production function of the firm may be taken to have the traditional Cobb-Douglas form $Q(L, K) = L^\chi K^{1-\chi}$, where $0 \leq \chi \leq 1$ is a parameter. The profit of the firm is then maximized with respect to K and L. Computer

[4] Assumptions made throughout.

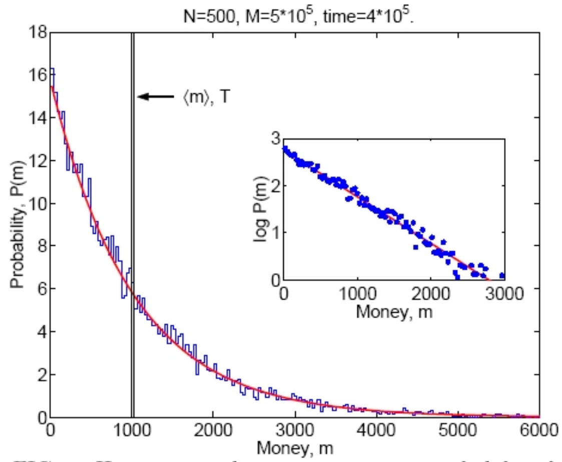

Figure 1: Stationary probability distribution of money $P(m)$ obtained *via* computer simulations (blue histogram) - the distribution is the same irrespective of the three different rules for Δm. Solid curves are fits to the Boltzmann-Gibbs law (3.3), the vertical lines indicate the initial distribution of money, $\delta(m - \frac{M}{N})$. The time $4 \cdot 10^5$ is simply a measure of the number of transactions that were taken to arrive at the equilibrium distribution shown on the graph. In the analysis of non-equilibrium phenomena attention should also be given to the realistic time scales in which transactions actually occur in the real world. The focus here is exclusively on equilibrium distributions. Image reproduced from [2].

simulations, [2], again produce the same exponential distribution independently of the parameters.

3.3 Models with debt

We now introduce debt into our considerations. From the perspective of an individual economic agent the latter may be viewed (justly so) as *negative money*. When an agent borrows money from a bank[5], his cash balance increases, but at the same time the agent acquires a corresponding negative debt obligation - whence his total balance (net worth of sorts) remains constant. The act of borrowing therefore does not violate the global conservation law, which is now taken to mean the conservation of *total* money, i.e. the sum of positive (cash M) and negative (debt D) contributions: $M - D = M_b$. The boundary $m_i \geq 0$ is no longer applicable. In this sense, allowing debt, is not to violate conservation (now of net worth), but rather to modify the boundary conditions. When an agent wishes to perform a transaction for which he does not have sufficient positive cash holdings, he may go into debt - and upon transaction the agent then has negative net worth. If the simulation is allowed to continue without any restriction on the maximum amount of debt, the distribution spreads indefinitely towards $m = +\infty$ and $-\infty$ - with some agents becoming ever richer at the expense of other agents going further into debt!

Indeed common sense indicates that an economic system cannot be stable if unlimited amounts of debt are allowed. Consider only the effects of the current financial and economic crisis, which in the opinion of this author, has as its root-cause over-leveraged financial institutions and debt-wise over-stretched consumers, especially in the USA. Or as Yakovenko and Rosser put it in [1, page 7] and with which the

[5]Which is considered to be a big reservoir of money, outside the system of agents; the debt of the agents is an asset for the bank and bank deposits are liabilities of the bank.

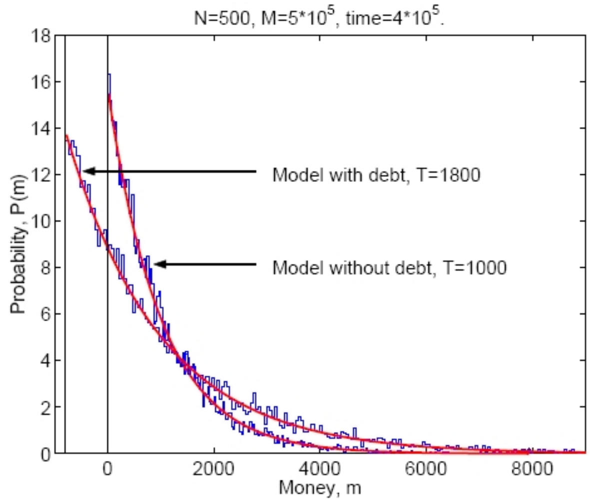

Figure 2: Histograms: stationary distributions of money with ($m_d = 800$) and without debt obtained *via* computer simulations. Solid curves are fits to the Boltzmann-Gibbs law (3.3) with money temperatures $T_m = 1800$ and $T_m = 1000$, respectively. Image reproduced from [2].

author could not agree more:

> "Arguably, the current financial crisis was caused by the enormous debt accumulation in the system, triggered by sub-prime mortgages and financial derivatives based on them. A widely expressed opinion is that the current crisis is not the problem of liquidity, i.e., a temporary difficulty in cash flow, but the problem of insolvency, i.e., the inherent inability of many participants /to/ pay back their debts."

Going back to our model, some form of modified boundary conditions need be imposed in order to ensure overall stability of the system and prevent unlimited growth of debt. A simple enough way to achieve this, is to demand that all agents' debts are limited by a certain amount m_d (i.e. $m_i \geq -m_d$). Interest rates are set to zero for the sake of simplicity. One would again expect (3.3) to hold, just that now for all $m \geq -m_d$ (and with $P(m) = 0$ for $m < -m_d$). The normalizing condition becomes $\int_{-m_d}^{\infty} ce^{-m/T_m}\mathrm{d}m = 1$, whence $c = \frac{1}{T_m}e^{-m_d/T_m}$ and the average amount of total money per agent is now $\frac{M_b}{N} = \int_{-m_d}^{\infty} \frac{m}{T_m} e^{-(m_d+m)/T}\mathrm{d}m = T_m - m_d$, i.e. $T_m = m_d + \frac{M_b}{N}$. Thus the distribution is essentially the same as in the model without debt, merely "translated" and scaled by a higher "money temperature" T_m: the distribution is thus broader - i.e. debt increases economic inequality. See Figure 2 for the results of computer simulations.

Models with a more realistic boundary conditions, say on the total debt of all agents, go beyond the scope of this seminar (the reader shall find a review in [1] and certainly in the extensive references quoted therein). Let us mention briefly only the reason as to why the limit on total debt is more sensible. Banks are required by law to set aside a fraction R of their deposits, whereas the remaining money may be loaned further. If the initial money in the system (base money) is M_b and one assumes that all debts are converted into bank deposits, one obtains for the total amount of positive money $M = M_b + M_b(1-R) + M_b(1-R)^2 + \cdots = M_b\frac{1}{1-(1+R)} = \frac{M_b}{R}$ - a process called money multiplication, and the factor $1/R$ is the money multiplier. Maximal total debt is then $S = M_b/R - M_b$. We mention it only in passing that another direction in which models with debt may be investigated is by introducing interest rates.

In the rest of this section only models without debt are considered!

3.4 Another model without debt, the Boltzmann kinetic equation

In the models of money transfer discussed in Subsection 3.2 the transferred amount Δm was typically independent of the money balances of the agents involved in the transaction, or was dependant on the money balance symmetrically. A qualitatively different model, called the *multiplicative asset exchange model*, sets Δm equal to a fixed fraction of the payers money,

$$\Delta m = \gamma m_i \tag{3.4}$$

The stationary distribution in this model is similar, but not exactly equal, to the Gamma distribution:

$$P(m) = cm^{\beta} e^{-m/T}, \tag{3.5}$$

for $m \geq 0$ and 0 otherwise. The distribution differs from that under (3.3) by the power-law prefactor m^{β}. An example of a stationary distribution is shown in Figure 3. A formula connecting γ introduced in (3.4)

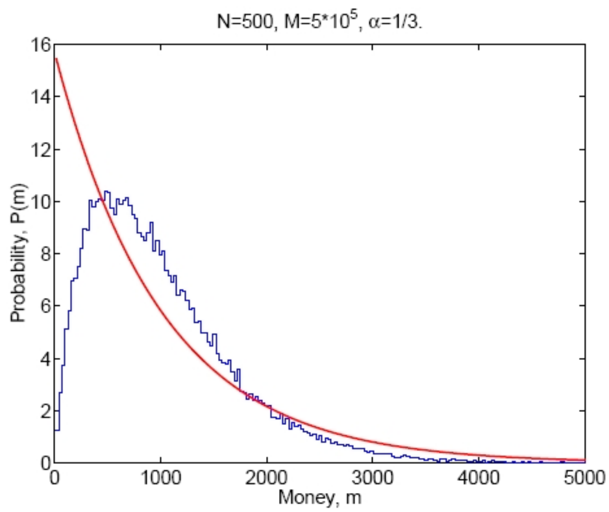

Figure 3: Histogram: stationary distribution of money for the multiplicative asset exchange model (3.4) with $\gamma = \frac{1}{3}$ obtained *via* computer simulations. The solid curve is the exponential Boltzmann-Gibbs law (3.3). Image reproduced from [2].

and β from (3.5) may be derived, [1, page 9]: $\beta = -1 - \ln 2 / \ln(1 - \gamma)$. One can show without much effort, that $0 < \beta$, precisely when $\gamma < 1/2$, and in such an instance $P(0) = 0$, i.e. when payers spend a relatively "modest" amount of their money holdings, the population with low money balances is reduced. (3.4) may be connected with the surplus theory of social stratification, which argues that inequality in human society develops when people can produce more than is required for minimal subsistence. This additional wealth can be transferred from original producers to other people, generating inequality. Different models may be obtained by taking γ constant, or randomly distributed, or even unequal for different groups of people (modeling the effects of education etc.). All of these models generate a Gamma-like distribution, [1, page 9]. Naturally one can come up with any number of other (more complicated) models, but their study (indeed even mention) would take us to far off course.

Rather we seek to motivate the reason as to why distributions found in the multiplicative asset exchange model differ qualitatively from those of Subsection 3.2. To such an end we consider the *Boltzmann kinetic equation*, which describes the time evolution of the distribution function $P(m)$ due to pairwise interactions:

$$\frac{\mathrm{d}P(m)}{\mathrm{d}t} = \int\int\{-f_{[m,m']\to[m-\Delta,m'+\Delta]}P(m)P(m') + f_{[m-\Delta,m'+\Delta]\to[m,m']}P(m-\Delta)P(m'+\Delta)\}\mathrm{d}m'\mathrm{d}\Delta, \tag{3.6}$$

where $f_{[m,m']\to[m-\Delta,m'+\Delta]}$ is the probability per unit time, of transferring money Δ from an agent with money m to an agent with money m' (i.e. $P(m)\delta mP(m')\delta m' f_{[m,m']\to[m-\Delta,m'+\Delta]}\delta t\delta\Delta$ is the probability that a randomly chosen individual shall have money in the δm interval around m and another randomly chosen individual shall have money in the $\delta m'$ interval around m' and that they shall perform a transaction in the δt-time interval around t, in which the first shall pay the second (!) a sum from the $\delta\Delta$-interval around Δ) - a similar definition pertains to the second "f" in the above equation. Consider now an interval of time around t of length δt and individuals having money in the δm-interval around m - there are $P(m)\delta mN$ of them (if N is the total number of individuals). There are two ways in which a change in their number can occur in the time interval considered. First of all they may be involved in a transaction in which they pay a certain sum in the $\delta\Delta$-interval around Δ to some other individual having money from the $\delta m'$-interval around m', and do so with probability $P(m')\delta m' f_{[m,m']\to[m-\Delta,m'+\Delta]}\delta t\delta\Delta$, by definition, for a total loss of $NP(m)\delta mP(m')\delta m' f_{[m,m']\to[m-\Delta,m'+\Delta]}\delta t\delta\Delta$ individuals. Secondly those having money in the δm interval around $m-\Delta$ (there are $NP(m-\Delta)\delta m$ of them) may be involved in a transaction with those with money from the $\delta m'$-interval around $m'+\Delta$, in which a sum from the $\delta\Delta$-interval around Δ is transferred from the second to the first (!); and do so with probability $P(m'+\Delta)\delta m' f_{[m-\Delta,m'+\Delta]\to[m,m']}\delta t\delta\Delta$ by definition, for a total gain of $P(m-\Delta)\delta mNP(m'+\Delta)\delta m' f_{[m-\Delta,m'+\Delta]\to[m,m']}\delta t\delta\Delta$ individuals. It follows that the change in the number of individuals having money in the δm-interval around m in a time interval δt around t is an appropriate integral over all possible m' and Δ (net gain)

$$\begin{aligned} X &= \int\int P(m-\Delta)\delta mNP(m'+\Delta)\mathrm{d}m' f_{[m-\Delta,m'+\Delta]\to[m,m']}\delta t\mathrm{d}\Delta - \\ &\quad \int\int NP(m)\delta mP(m')\mathrm{d}m' f_{[m,m']\to[m-\Delta,m'+\Delta]}\delta t\mathrm{d}\Delta. \end{aligned}$$

Now $X = \delta(P(m)\delta mN)$ (where the first δ pertains to the change in time), whence

$$\frac{\delta P(m)}{\delta t} = \int\int\{-f_{[m,m']\to[m-\Delta,m'+\Delta]}P(m)P(m') + f_{[m-\Delta,m'+\Delta]\to[m,m']}P(m-\Delta)P(m'+\Delta)\}\mathrm{d}m'\mathrm{d}\Delta$$

and in the limit as $\delta t \to 0$, (3.6) obtains.[6] It will be noticed that when the process is such that it has the "time-reversal" property, namely:

$$f_{[m,m']\to[m-\Delta,m'+\Delta]} = f_{[m-\Delta,m'+\Delta]\to[m,m']}, \tag{3.7}$$

then the condition $P(m-\Delta)P(m'+\Delta) = P(m)P(m')$ is sufficient for $P(m)$ to be stationary (an equilibrium). The exponential distribution satisfies this condition, which motivates its robustness for the models

[6] Clearly the above is a physicists' (and not a mathematicians') argument.

of Subsection 3.2 (these all satisfy (3.7)[7]). In the multiplicative asset exchange model presented above, Δm depends asymmetrically on m_i and m_j, time-reversal is not satisfied and thus it is not surprising that the exponential distribution is not stationary in this particular instance. The proportional rule (multiplicative models) typically violates time-reversal, and so the stationary distribution in these models tends not to be the exponential one. "Making the transfer dependent on the money balance of the payer effectively introduces Maxwell's demon into the model," [1, page 11].

The above remarks show that the Boltzmann-Gibbs distribution is universal in a limited way, in the sense that it holds true of a broad variety of models in which time-reversal symmetry given by (3.7) is not broken. In the absence of a detailed knowledge of the actual "microscopic" dynamics of money transfer it thus seems to be a natural starting point and as good a first approximation as any! To once again quote [1, page 11]: " /.../ one can argue that parsimony is the virtue of a good mathematical model, not the abundance of additional assumptions and parameters, whose correspondence to reality is hard to verify."

Finally let us note that in the different models that we have considered in this section, (some) randomness was always assumed in the exchange of money. The results thus obtained "would apply the best to the probability distribution of money in a closed community of gamblers," [2, page 728]. Indeed in accordance with traditional economics, money is exchanged not randomly, but rather following deterministic strategies obtained e.g. from the maximization of utility functions. It is not completely unreasonable to expect however, that when a large group of heterogeneous agents deterministically interact with one another and spend various amounts of money, the exchange is (overall) *effectively* random. The case is analogous to the one in which atoms (classically speaking) follow fully deterministic equations of motions, and yet the overall result of the energy exchange between them is *as if* said exchange were random.[8]

4 Wealth distribution

Having considered the statistical mechanics of money, we now turn to the issue of wealth. The terms "money" and "wealth" need not and should not be used interchangeably - for, in a sense, the first is a subset of the second - money holdings being some (usually proper) part of the entire wealth of an individual, which may consist also of durable material goods, such as houses, cars but also financial instruments, such as stocks, bonds and their derivatives (e.g. options). This other part constituting wealth w is not immediately liquid and needs typically be sold (converted into money) in order for it to be used for other purchases. In the simplest model we would consider just one type of property, distinct from money, say stocks s. In such an instance the wealth of an agent i would be given by

$$w_i = m_i + ps_i, \tag{4.1}$$

where p is the price of the stock (common for all individuals and determined by some market process) and the rest of the notation is self-explanatory.

It is reasonable to start with a model in which both total money $m = \sum_i m_i$ and total stock $s = \sum_i s_i$ are conserved. The agents pay money to buy stock and sell stock to obtain money. Importantly, while

[7]Whenever it is the case, that of the two agents involved in a transaction, Δm is independent of m_i and m_j, in such an instance (3.7) is certainly satisfied, and if Δm depends only on say the sum of m_i and m_j, the claim is clearly still valid.

[8]Nevertheless the assumption of effective randomness may fail to be representative of actual individuals. In this respect an analogue of a molecular dynamics simulation, in which agents would follow deterministic strategies given by their utility functions, would be helpful (even if extremely complex to produce). The final arbiter of course is the actual distribution of money in the real economy - due to privacy issues such statistics (unfortunately) is either not collected or else is not (does not seem to be) publicly available, [1, page 14].

common stock and money are conserved, the total wealth $W = m + ps = \sum_i w_i$ need not be, due to possible price fluctuations; indeed the wealth of an individual does not change in a transaction but rather during a period of time in which the individual holds stock, but the price changes. Thus redistribution of wealth in this model is directly dependant on changes in the price of the stock p. In certain cases (depending on the nature of the process in which p is determined, [1, page 12]) the stationary distribution $P(w)$ in this model, is given by the Gamma distribution. Also, because m and s are conserved, one expects that W shall remain constant on average.

Brief remarks on some other models may again be found in [1, page 12], together with references to other papers in relation to the study of wealth distribution - we mention only that models with stochastic growth of wealth tend to produce a power-law tail for $P(w)$. We would digress to far off course (given the confines of this seminar), were we to go into any further detail here. Instead we turn to empirical data on wealth distribution.

Unlike income, wealth is not routinely reported by the majority of individuals to the government and consequently statistics regarding it is difficult to come by. Nevertheless, in some countries, when a person dies, all assets must be reported for the purposes of inheritance tax. In principle then, one can find good statistics on the wealth distribution among dead people. Even though the latter may not in itself be representative of the entire (living) population, estimates may be made via adjustment procedures taking into account age, gender, etc. (see [12] for further details). Figure 4 shows cumulative

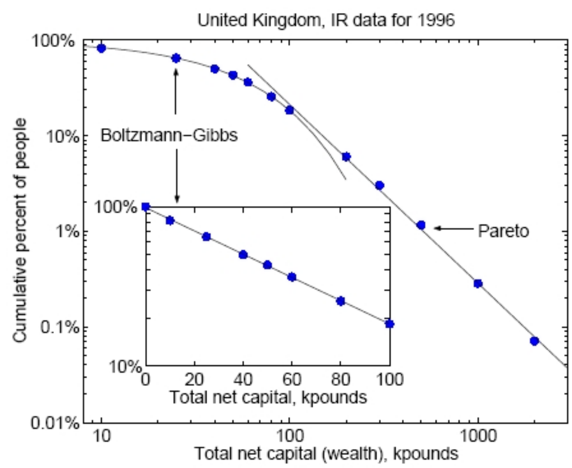

Figure 4: Cumulative probability distribution C as a function of total net wealth w for the UK, for the year 1996, on log-log and log-linear (inset) scales (points represent the data from Inland Revenue). Solid lines are fits to exponential (Boltzmann-Gibbs; inset and upper-left part) and power (Pareto; straight line) laws. On the log-linear scale a straight line represents an exponential dependence, and on a log-log scale a power-law dependence. Image reproduced from [13].

probability $C(w) = \int_w^\infty P(w')\mathrm{d}w'$ as a function of a person's net personal wealth w composed of assets (cash, stocks, property, household goods etc.) and liabilities (mortgages and other debts) for the UK in 1996. It demonstrates that the distribution of wealth follows a power-law $C(w) \propto 1/w^\alpha$ with $\alpha = 1.9$ for wealth greater than about 100k£. Below this value, the data is fitted well by an exponential distribution $C(w) \propto e^{-w/T_w}$ with the effective "wealth temperature" $T_w = 60$k£. To sum up - the distribution of wealth is characterized by the Pareto power law in the upper tail of the distribution and the exponential

Boltzmann-Gibbs law in the lower part of the distribution for the great majority (roughly 90%) of the population.

One might speculate that wealth in the lower part of the distribution is dominated by the distribution of money (because the less well-off people do not have significant other assets) - and hence the results of Section 3 are pertinent. On the other hand in the upper part of the distribution, wealth is dominated by investments (power-law tails for models with stochastic growth of wealth).

5 Data and models for income distributions

5.1 Empirical data on income distribution

In contrast to money (privacy issues) and wealth distributions, a lot more data is available for the distribution of (annual) income r from tax agencies, as well as population surveys. Just as was the case for wealth distribution, the lower part of the income distribution $P(r)$ can be approximated well by an exponential, whereas the upper part admits for a power-law fit - this can be seen e.g. in Figure 5. Indeed the observation that the bulk of the income (or for that matter wealth) distribution is described by a Boltzmann-Gibbs (possibly log-normal or Gamma) law, crossing over at the very high income range to a power law, seems to be a universal feature: "from ancient Egyptian society through nineteenth century Europe to modern Japan. The same is true across the globe today: from the advanced capitalist economy of USA to the developing economy of India," [3].

We introduce $C(r) = \int_r^\infty P(r')\mathrm{d}r'$ - the cumulative probability distribution of income. Parameters T_r ("income temperature") and α are introduced so that in the lower part $C(r) \propto e^{-r/T_r}$ and in the upper part $C(r) \propto 1/r^\alpha$ (whence it follows upon differentiation, that $P(r) \propto e^{-r/T_r}$ in the lower part and $P(r) \propto 1/r^{\alpha+1}$ in the upper part). The fact that income distribution is governed by two distinct

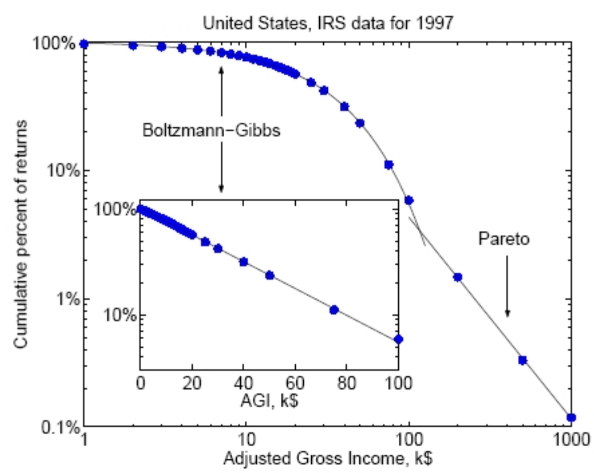

Figure 5: $C(r)$ as a function of gross income r for the USA in 1997 shown on log-log (main panel) and log-linear (inset) scales. Points represent IRS data and solid lines are fits to exponential (Boltzmann-Gibbs; inset and upper-left part) and power (Pareto; straight line) laws. Image reproduced from [1].

laws (exponential and power law) indicates a two-class structure of the American society. The boundary between the two can be defined by the intersection of the Boltzmann-Gibbs and Pareto fits. For the case in Figure 5 the annual income separating both groups was about 120 thousand US dollars, with roughly 97% of the population belonging to the lower part and only about 3% of the population to the upper.

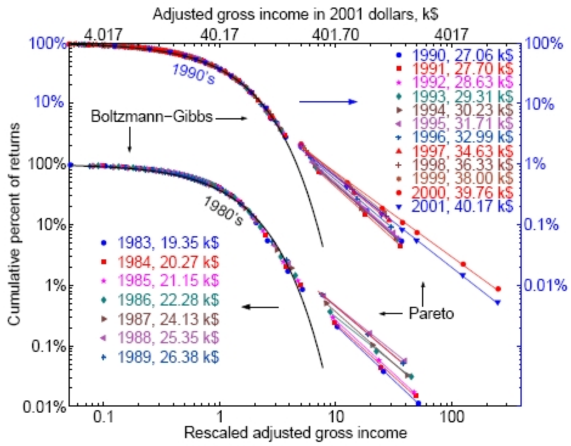

Figure 6: $C(r)$ as a function of gross income r/T_r for the USA in several years shown on log-log scales. The columns of numbers give values for T_r in the different years (the IRS data points are for 1983-2001). Plots for the 1980s and 1990s are shifted vertically for greater clarity. The structure is qualitatively the same for all years. More importantly the lower part data points collapse on the same exponential curve for all years, demonstrating the stability of this part of the distribution, which suggests "thermal equilibrium". By contrast the upper part does not rescale and changes significantly over time with the exponent α decreasing from 1.8 to 1.4 from 1983 to 2000, indicating the "fattening" of the upper tail, i.e. greater income inequality. By contrast in 2001 after the stock market crash α increased. The upper tail is thus highly dynamical and not stationary, responding to e.g. speculative bubbles. Image reproduced from [1].

It is worthwhile considering the temporal evolution of $C(r)$. Because of inflation, income r is rescaled in units of T_r obtained by a fit for every year. See Figure 6, which shows *inter alia* that the relative income inequality within the lower class remains stable, while overall income inequality in the USA has increased significantly due to the tremendous growth of income of the upper class.

Measures of income inequality include total income of the upper class as a fraction f of total income in the system, *Lorenz plots* and the *Gini coefficient*. The Lorenz plot is a parametric one, with r as parameter and $x(r) = \int_0^r P(r')\mathrm{d}r'$, the fraction of the population with income below r, and $y(r) = \frac{\int_0^r r'P(r')\mathrm{d}r'}{\int_0^\infty r'P(r')\mathrm{d}r'}$, the fraction of the income which this population accounts for. As r changes from 0 to ∞, x and y grow monotonically from 0 to 1. In a society in which every individual has the same income, clearly the plot thus obtained is $y(r) = x(r)$, i.e. a straight line. When $P(r) = \frac{1}{T_r}e^{-r/T_r}$ (for $0 \leq r$; $P(r) = 0$ for $r < 0$, since there can be no negative gross income) we have $x(r) = 1 - e^{-r/T_r}$ and $y(r) = \frac{T_r(1-e^{-r/T_r}) - re^{-r/T_r}}{T_r}$, whence

$$y = x + (1-x)\ln(1-x). \tag{5.1}$$

When f is significant and so long as the fraction of the population in the upper class is small enough (5.1) may be modified *cum grano salis* to (H is the Heaviside function equal to 0, for $x < 0$ and 1 otherwise)

$$y = (1-f)[x + (1-x)\ln(1-x)] + f\mathrm{H}(x-1). \tag{5.2}$$

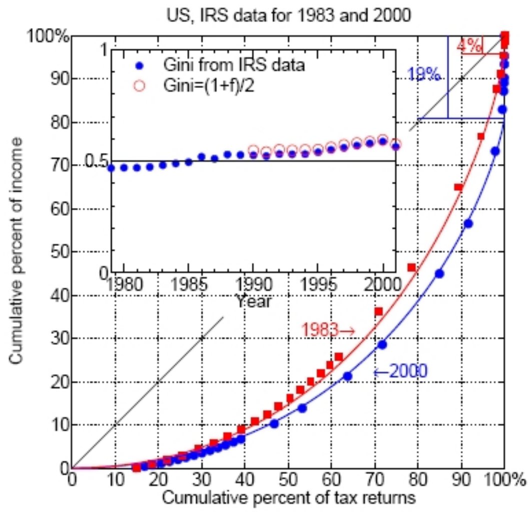

Figure 7: Main panel shows Lorenz plots for the years 1983 and 2000 in the USA, data points are from IRS, the 1983 curve is fitted well by (5.1), whereas the 2000 curve requires (5.2). Inset shows the Gini coefficient: full circles for empirical data, open circles for the theoretical formula (5.3). Income inequality has been increasing for the last 20 years but decreased after the stock market crash in 2001. Image reproduced from [1].

The Gini coefficient is defined to be the area between the Lorenz curve and the diagonal line (deviation from the equal income scenario) divided by the area of the triangle beneath the diagonal line (1/2), i.e. $G = 1 - 2\int_0^1 y\mathrm{d}x$, so that $0 \leq G \leq 1$.[9] The greater the Gini coefficient the greater income inequality (with $G \to 1$ indicating perfect economic inequality wherein income is concentrated at the smallest fraction of the population and everyone else receiving little to no income). Via (5.1) and *per partes* it is easily verified that for the purely exponential distribution $G = \frac{1}{2}$. Consequently with (5.2) one obtains $G = 1 - 2(\frac{1}{2}(1 - \frac{1}{2})(1 - f))$, i.e.

$$G = \frac{1}{2}(1 + f). \tag{5.3}$$

See Figure 7 for an example of a Lorenz plot, the Gini coefficient and application of the fits hereabove considered.

5.2 Theoretical models of income distribution

In the previous subsection much attention was devoted to empirical data regarding income distribution. Let us now briefly examine a theoretical model, which seeks to explain this distribution. Income r_i is the influx of money per unit time to agent i (hence *monthly* income, *annual* income etc.). Thus, if m_i is analogous to energy, r_i is analogous to power, which is the flux of energy per unit time. It is clear then that a conceptual difference exists between the two concepts. Moreover, while money is regularly

[9] Indeed $y(r) \leq x(r)$ as we shall demonstrate, whence $0 \leq G \leq 1$ follows at once. Denote $p = \int_0^r P(r')\mathrm{d}r'$, $W_{\leq} = \int_0^r r'P(r')\mathrm{d}r'$ and $W_{\geq} = \int_r^\infty r'P(r')\mathrm{d}r'$. Then $W_{\geq} \geq r(1-p)$ and $W_{\leq} \leq rP$. It follows when $W_{\leq} \neq 0$ that $\frac{W_{\leq}}{W_{\geq}+W_{\leq}} = \frac{1}{\frac{W_{\geq}}{W_{\leq}}+1} \leq \frac{1}{\frac{r(1-p)}{rp}+1} = p$, i.e. $W_{\leq} \leq Wp$, where $W = \int_0^\infty r'P(r')\mathrm{d}r'$. The latter certainly holds when $W_{\leq} = 0$ as well. But then $y(r) = \frac{W_{\leq}}{W} \leq p = x(r)$, as required.

transferred from one agent to another in pairwise transactions, it is not typical for individuals to trade portions of their income. Nevertheless indirect transfers between individuals do exist when for example one employee is promoted and another is demoted, or when one keeps his job when another loses it etc. Having said that, the treatment of an individuals' income as a stochastic process still seems reasonable. By studying a suitable continuous Markov process in which in a small time interval Δt, r changes by a small amount Δr (details go beyond the scope of this seminar, what shall be mentioned is only to motivate the end result - further information is to be found in [1, page 18]) one obtains the following equation:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial r}[AP + \frac{\partial(BP)}{\partial r}], \tag{5.4}$$

where $A = -\frac{\langle \Delta r \rangle}{\Delta t}$ and $B = \frac{\langle (\Delta r)^2 \rangle}{2\Delta t}$, determined, respectively, by the first and second moments of income changes per unit time. The stationary distribution[10] is given by the condition $\partial_t P = 0$ and hence by $\frac{\partial(BP)}{\partial r} = -AP$. The last equation may be rewritten in the form $\frac{(BP)'}{BP} = -\frac{A}{B}$ and then integrated to obtain the general solution:

$$P(r) = \frac{c}{B(r)} \exp(-\int^r \frac{A(r')}{B(r')})\mathrm{d}r'), \tag{5.5}$$

where c is some constant. For the lower part of the distribution it is reasonable to expect that changes in income are independent of income (wages, salaries; the *additive* process) and hence $A = A_0$ and $B = B_0$ are constants, which immediately yields the exponential distribution $P(r) \propto e^{-r/T_r}$, where $T_r = \frac{B_0}{A_0}$. On the other hand, for the upper tail of income distribution (bonuses, investments, capital gains) it is sensible to assume $\Delta r \propto r$ (the *multiplicative* process). In this case $A = ar$ and $B = br^2$ and hence (5.5) yields $P(r) \propto 1/r^{\alpha+1}$ with $\alpha = 1 + a/b$. The additive and multiplicative process may coexist. One way to take this into account is to put $A = A_0 + ar$ and $B = B_0 + br^2 = b(r_0^2 + r^2)$. In this case (5.5) yields (upon simple enough integration)

$$P(r) = c\frac{e^{-(r_0/T_r)\arctan(r/r_0)}}{[1 + (\frac{r}{r_0})^2]^{1+a/2b}}, \tag{5.6}$$

where $T_r = \frac{B_0}{A_0}$ and c is a normalizing constant. Finally, to conclude this section, we offer Figure 8 - it is quite simply spectacular how well (5.6) fits the data for income distributions.

6 Conclusions

The use of the methods of (theoretical) physics in matters pertaining to economics and finance is perhaps not wholly expected. After all economics is a science which deals with human individuals and their interactions, and in this sense resembles more psychology than it does, say, chemistry. And yet it is also a quantitative science, studying different aggregates (GDP growth, unemployment rates, inflation, money aggregates etc., etc.) of a large number of individuals - as such it is amenable to the methods of physics and statistical mechanics in particular. Several examples of how this is so, were given in the above and we shall not belabour the reader by repeating them. Suffice it to say that the subject of money, income and wealth distributions has always been important and will likely be relevant for a long time to come in relation to the study of *social inequality*. Although social classes are an old concept, the realization that

[10]Here only such are considered. One may nevertheless have that the parameters (here A and B) change "slowly" over time - slowly enough that is, to allow equilibrium to arise at individual "points" in time. Compare the same situation and the comments made in relation to the stationary distribution of money in Subsection 3.1.

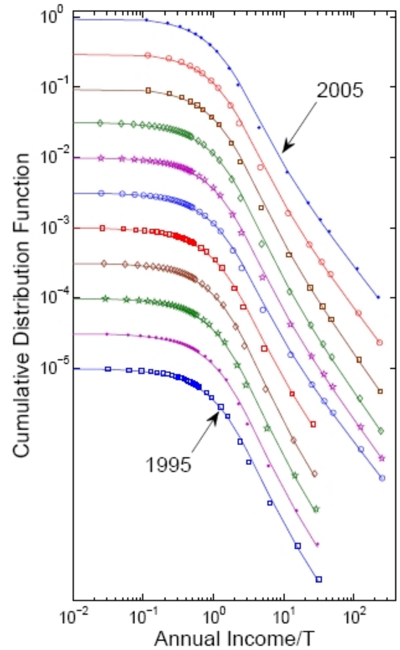

Figure 8: Fits of the IRS data for income distribution using (5.6). Plots for different years are shifted vertically for clarity. Image reproduced from [1].

they are described by simple mathematical laws is quite new and the demonstration of the ubiquitous nature of the exponential distribution is one of the new contributions produced by econophysics.

From the practical perspective the study of money, wealth and income distributions is relevant in at least two obvious instances: *deposit insurers* (e.g. FDIC in the USA, which insures bank deposits up to a certain limit) could/can estimate their potential exposure from data on the distribution of bank account balances; and from the apposite distributions of wealth and income, *governments* can properly estimate tax revenue for particular systems of (wealth, inheritance or income) tax rates.

What is finally worth noting, is that in economics the verbose style of the political economist and the more precise and at times technical approach of the physicist are not rivals but friends - neither can function well without the other - and both risk detaching themselves to far from reality if they ignore their "friend". We can only hope then, that a symbiosis of sorts shall (continue or come) to exist between the two disciplines - either one can probably only stand to gain from such a relationship!

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