University of Ljubljana
FACULTY OF MATHEMATICS AND PHYSICS
Department of physics

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The Free Will Theorem

Timon MEDE
mentor: prof. Rudolf Podgornik

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Abstract

I begin with description of quantum entanglement and its relevance to the EPR-type (thought and real) experiments which played a decisive role in rejecting determinism and hidden-variable theories.

Then I proceed with deriving the main object of my paper, the recently published Conway and Kochen's Free Will Theorem (by first proving the Kochen-Specker paradox, on which it is founded). There it is being shown on the basis of three physical axioms, following from the special theory of relativity and quantum mechanics, that if the choice for a particular type of spin 1 experiment is not a function of the information accessible to the experimenters, then its outcome is equally not a function of the information accessible to the particles. Therefor the response of the particles can be seen as free.

In conclusion I review some significant remarks that followed in responses on this article and mention the relevant points from the subsequent discussion and the philosophical aspects and implications of this theme.

Contents

Chapter 1

Introduction

I would like to begin by presenting some of the reasons that motivated me for choosing this topic and point out some of the goals of this paper.

First of all, there was this rather intriguing title, that sounded very promising in a way that allowed me to combine my interest in physical theory (quantum mechanics) and philosophy and to confirm my firm belief that at the forefront of every science they go hand-in-hand with eachother and become inseparable, perhaps even indistinguishable.

Besides that I was given an excellent opportunity to learn more about the foundations of the quantum mechanical theory and some of its most controversial conceptual yet unsolved problems that are usually the main reason why this theory strikes most of the people, even physicists, as odd. Even Richard Feynman once remarked: "I think I can safely say that nobody understands quantum mechanics."

The main problem is, that quantum mechanics contrary to our intuition shows that the world is objectivelly random, not determenistic. This is closely linked with the infamous measurement problem i.e. the problem of the wave function collapse, better known as the paradox of Schrodinger's cat. Heisenberg's uncertainty principle (complementarity) seems to lie in the heart of this matter, and not just being a consequence of our imperfect experimental methods. And finally, how to understand objective randomness of the single event in connection with entangled states in e.g. EPR-setup, where the result of measurement of a random event on one end is determined by the result of preceding measurement of another random event on other's end of the experimental setup, Einstein's relativity of time making it even stranger, because the direction of the influence is completely (reference) frame dependent.

Next I wish to mention two very influential gedanken-experiments that greatly changed our understanding of physics: Bell inequality and the Kochen-Specker paradox showed a way how to experimentally resolve some age-old philosophical dilemmas.

And last but not least, I also wanted to present an excerpt from contemporary debate about the foundations of quantum theory, to show the course it's taking and to emphasize that this quest is far from over.

This paper is mainly based upon the recent work by Conway and Kochen in their 2006 article "The Free Will Theorem" and the subsequent replies by Bassi, Ghirardi, Tumulka, Adler,...

Chapter 2

Entangled states

2.1 Joint state space for two subsystems[3]

The representation of physical states of two independent (noninteracting) quantum systems, labelled A and B, can be considered in two independent Hilbert spaces by choosing their state vectors:

$$|\Psi_A\rangle \in H_A$$

and

$$|\Psi_B\rangle \in H_B$$

When we bring these two systems together and let them interact, the joint state space (the Hilbert space of the composite system) for systems A and B corresponds to the **tensor product** of H_A and H_B, denoted

$$H_{AB} = H_A \otimes H_B$$

Let N_A be the dimension of H_A, and N_B the dimension of H_B. If $\{|1\rangle_A, |2\rangle_A, |3\rangle_A, \cdots\}$ is a complete orthonormal basis for H_A and $\{|1\rangle_B, |2\rangle_B, |3\rangle_B, \cdots\}$ is a complete orthonormal basis for H_B, then $H_A \otimes H_B$ is the Hilbert space of dimension $N_{AB} = N_A N_B$, spanned by the vectors of the form $|i\rangle_A \otimes |j\rangle_B$. Hence arbitrary states in H_{AB} have the form

$$|\Psi_{AB}\rangle = \sum_{i=1}^{N_A} \sum_{j=1}^{N_B} c_{ij} |i\rangle_A \otimes |j\rangle_B$$

As long as we fix an ordering for the new basis states $|i\rangle_A \otimes |j\rangle_B$, the set of $N_A N_B$ complex coefficients c_{ij} can be used as a vector representation for kets in H_{AB}.

2.2 Entanglement[4]

As a consequence of this mathematical rule for representation of joint states, there exist beside separable i.e. product states of composite system also 'nonfactorizable' states $|\Psi_{AB}\rangle \in H_{AB}$ that cannot be expressed as the tensor product of a state $|\Psi_A\rangle \in H_A$ with a state $|\Psi_B\rangle \in H_B$. Such states are said to be entangled. For example, let's consider two three-dimensional (e.g. spin 1) systems. Say we have chosen orthonormal bases

$$\{|-1\rangle_A, |0\rangle_A, |1\rangle_A\} \text{ for } H_A \qquad \text{and}$$

$$\{|-1\rangle_B, |0\rangle_B, |1\rangle_B\} \text{ for } H_B.$$

Then H_{AB} is spanned by the nine states:
$\{|-1\rangle_A \otimes |-1\rangle_B,\ |-1\rangle_A \otimes |0\rangle_B,\ |-1\rangle_A \otimes |1\rangle_B,\ |0\rangle_A \otimes |-1\rangle_B,\ |0\rangle_A \otimes |0\rangle_B,$
$|0\rangle_A \otimes |1\rangle_B,\ |1\rangle_A \otimes |-1\rangle_B,\ |1\rangle_A \otimes |0\rangle_B,\ |1\rangle_A \otimes |1\rangle_B\}$

Factorizable (non-entangled) states in H_{AB} are all of the form

$$\begin{aligned}
|\Psi_A^{fac}\rangle &= (c_{-1}^A|-1\rangle_A + c_0^A|0\rangle_A + c_1^A|1\rangle_A) \otimes (c_{-1}^B|-1\rangle_B + c_0^B|0\rangle_B + c_1^B|1\rangle_B) = \\
&= c_{-1}^A c_{-1}^B|-1\rangle_A \otimes |-1\rangle_B + c_{-1}^A c_0^B|-1\rangle_A \otimes |0\rangle_B + c_{-1}^A c_1^B|-1\rangle_A \otimes |1\rangle_B + \\
&+ c_0^A c_{-1}^B|0\rangle_A \otimes |-1\rangle_B + c_0^A c_0^B|0\rangle_A \otimes |0\rangle_B + c_0^A c_1^B|0\rangle_A \otimes |1\rangle_B + \\
&+ c_1^A c_{-1}^B|1\rangle_A \otimes |-1\rangle_B + c_1^A c_0^B|1\rangle_A \otimes |0\rangle_B + c_1^A c_1^B|1\rangle_A \otimes |1\rangle_B
\end{aligned}$$

That is, a certain relationship exists between the coefficients of the nine basis states in H_{AB}. An example of an entangled state, whose coefficients do not exhibit the above relationship, is

$|\Psi_{AB}\rangle = \frac{1}{\sqrt{3}}|1\rangle_A \otimes |-1\rangle_B + \frac{1}{\sqrt{3}}|-1\rangle_A \otimes |1\rangle_B - \frac{1}{\sqrt{3}}|0\rangle_A \otimes |0\rangle_B \neq |\Psi_A\rangle \otimes |\Psi_B\rangle;$

$|\Psi_A\rangle \in H_A,\ |\Psi_B\rangle \in H_B$

which can easily be seen if we look at the equations:

$$\begin{aligned}
c_1^A c_{-1}^B &= \frac{1}{\sqrt{3}} \neq 0 \\
c_{-1}^A c_1^B &= \frac{1}{\sqrt{3}} \neq 0 \\
c_0^A c_0^B &= -\frac{1}{\sqrt{3}} \neq 0
\end{aligned}$$

that show that none of the coefficients $c_{-1}^A, c_0^A, c_1^A, c_{-1}^B, c_0^B, c_1^B$ is equal to zero, so if $|\Psi_{AB}\rangle$ was factorizable, it should also contain the terms $|-1\rangle_A \otimes |-1\rangle_B, |-1\rangle_A \otimes |0\rangle_B, |0\rangle_A \otimes |-1\rangle_B, |0\rangle_A \otimes |1\rangle_B, |1\rangle_A \otimes |0\rangle_B, |1\rangle_A \otimes |1\rangle_B$ which it doesn't. When the joint state of two subsystems is entangled, there is no way to assign a definite pure quantum state to either subsystem alone - the entangled states of both subsystems are inseparable. Instead, they are superposed with one another. This interconnection leads to correlations between observable physical properties (e.g. spin or charge) of remote systems, where the spatial separation between the two individual objects that represent those two subsystems (e.g. two particles) is being irrelevant. That bothered Albert Einstein so much that he denoted entanglement as "spukhafte Fernwirkung" or "spooky action at a distance", but every subsequent experiment confirmed in every detail the idea of quantum entanglement. The current state of belief is that although two entangled systems appear to interact across large distances instantaneously, no useful information can be transmitted in this way, meaning that causality cannot be violated through entanglement. This is the statement of the no-communication theorem[2] - quantum entanglement does not enable the transmission of classical information faster than the speed of light because a classical information channel is required to complete the process.

Pairs of entangled particles are generated by the decay of other particles, naturally or through induced collision and this quantum entanglement has applications in the emerging technologies of quantum computing and quantum cryptography (superdense coding), and has been used to realize quantum teleportation experimentally. At the same time, it prompts some of the more philosophically oriented discussions concerning quantum theory.

2.3 Singlet state of two spin 1 particles

An example of a spin 1 system is an atom of orthohelium[8] - the form of the helium atom in which the spins of the two electrons are parallel ($S = 1$, triplet state) which implies lower energies than for the form with antiparallel spins ($S = 0$, singlet state) called parahelium.

We use Clebsch-Gordan's coefficients to write down the singlet state of a twinned pair of such interacting spin 1 particles i.e. the state with total spin 0 corresponding therefor to quantum numbers $S = 0$ and $M = 0$ and we get exactly the state already mentioned above as an example of an entangled state:

$$|S^{ab} = 0, M_w^{ab} = 0\rangle = \frac{1}{\sqrt{3}}[|M_w^a = 1\rangle|M_w^b = -1\rangle + |M_w^a = -1\rangle|M_w^b = 1\rangle - |M_w^a = 0\rangle|M_w^b = 0\rangle]$$

This state is independent of the direction w, because $S_w^a (= S_w \otimes I)$ and $S_{w'}^b (= I \otimes S_{w'})$ act on different Hilbert spaces and are therefor commuting operators for any directions w and w'. This means, that singlet state looks the same for every basis - we can for example write it down using eigenstates of spin along x- or z-axis and the form stays the same.

Components of spin for each entangled particle are indeterminate until some physical intervention is made to measure them. Then the wavefunction collapse causes the system to leap into one of superposed basic states and acquire the coresponding values of the measured property. When the two members of a singlet pair are measured, they will always be found in opposite states. The distance between the two particles is irrelevant.

Singlet states have been experimentally achieved for two spin 1/2 particles separated by more than 10km. Presumably a similar singlet state for distantly separated spin 1 particles will be attained with sufficient technology.

Chapter 3

EPR paradox and the hidden variable theories

3.1 EPR paradox[9]

In quantum mechanics, the EPR paradox is a thought experiment introduced in 1935 by Einstein, Podolsky, and Rosen[5] to argue that quantum mechanics is not a complete physical theory.

Quantum theory and quantum mechanics do not account for single measurement outcomes in a deterministic way. According to an accepted interpretation of quantum mechanics known as the Copenhagen interpretation, a measurement causes an instantaneous collapse of the wave function describing the quantum system, and the system after the collapse appears in a random state. Einstein did not believe in the idea of genuine randomness in nature. In his view, quantum mechanics was incomplete and suggested that there had to be 'hidden' variables responsible for random measurement results. In their paper, mentioned above, they turned this philosophical discussion into a physical argument. They claimed that given a specific experiment (like the one described below), in which the outcome of a measurement could be known before the measurement actually takes place, there must exist something in the real world, an "element of reality", which determines the measurement outcome. They postulated that these elements of reality are local, in the sense that they belong to a certain point in spacetime. This element may only be influenced by events which are located in the backward light cone of this point in spacetime. Even though these claims sound reasonable and convincing, they are founded on assumptions about nature which constitute what is now known as local realism and has been later experimentally proven wrong.

Locality seems to be a consequence of special relativity, which states that information can never be transmitted faster than the speed of light without violating causality. It is generally believed that any theory which violates causality would also be internally inconsistent, and thus deeply unsatisfactory. It turns out that quantum mechanics only violates the principle of locality without violating causality.

The EPR paradox draws on quantum entanglement, to show that measurements performed on spatially separated parts of a quantum system can apparently have an instantaneous influence on one another. This effect is now known as ”nonlocal behavior” (or ”spooky action at a distance”). In order to illustrate this, let us consider a simplified version of the EPR thought experiment put forth by David Bohm.

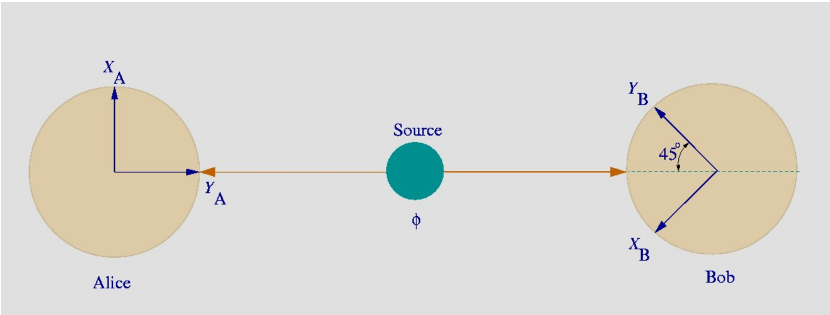

Figure 1. The EPR experimental setup

We have a source that emits pairs of spin 1/2 particles (e.g. electrons), with one particle sent to destination A, where there is an observer named Alice, and another sent to destination B, where there is an observer named Bob. Our source is such that each emitted electron pair is in an entangled quantum state called a spin singlet:

$$\begin{aligned}|S^{ab}=0,M_w^{ab}=0\rangle &= \frac{1}{\sqrt{2}}[|M_w^a=\frac{1}{2}\rangle|M_w^b=-\frac{1}{2}\rangle-|M_w^a=-\frac{1}{2}\rangle|M_w^b=\frac{1}{2}\rangle]= \\ &= \frac{1}{\sqrt{2}}[|\uparrow\downarrow\rangle-|\downarrow\uparrow\rangle]\end{aligned}$$

which can be viewed as a quantum superposition of two states, one with electron A having spin up along the z-axis ($+z$) and electron B having spin down along the z-axis ($-z$) and another with opposite orientations. Therefore, it is impossible to associate either electron in the spin singlet with a state of definite spin until one of them is being measured and the quantum state of the system collapsed into one of the two superposed states.

Since the quantum state determines the probable outcomes of any measurement performed on the system, if Alice measured spin along the z-axis and obtained the outcome $+z$, Bob would subsequently obtain $-z$ measuring in the same direction with 100 % probability and similarly if Alice obtained $-z$, Bob would get $+z$. And because the singlet state is symetrical with respect to rotation, the same is true for any choice of direction of spin measurement.[1]

[1]modest suggestion for proper expressions by Conway and Kochen to avoid the conceptual problems

The problem now comes down to this: because the spin along x-axis and spin along z-axis are "incompatible observables", subjected to Heisenberg uncertainty principle, a quantum state cannot possess a definite value for both variables. So how does Bob's electron's spin, measured in x-direction, instantaneously know, which way to point, if it is not allowed to know (locality) whether Alice decided to measure spin along x-axis (Bob's x-spin measurement will with certainty produce opposite result) or z-axis (Bob's x-spin measurement will have a 50 % probability of producing $+x$ and a 50 % probability of $-x$)? Using the usual Copenhagen interpretation rules that say the wave function "collapses" at the time of measurement, there must be action at a distance or the electron must know more than it is supposed to.

According to its authors the EPR experiment yields a dichotomy. Either

1. The result of a measurement performed on one part A of a quantum system has a non-local effect on the physical reality of another distant part B, in the sense that quantum mechanics can predict outcomes of some measurements carried out at B; or...

2. Quantum mechanics is incomplete in the sense that some element of physical reality corresponding to B cannot be accounted for by quantum mechanics (that is, some extra variable is needed to account for it.)

Incidentally, although we have used spin as an example, many types of physical quantities — what quantum mechanics refers to as "observables" — can be used to produce quantum entanglement. The original EPR paper used momentum for the observable. Experimental realizations of the EPR scenario often use photon polarization, because polarized photons are easy to prepare and measure.

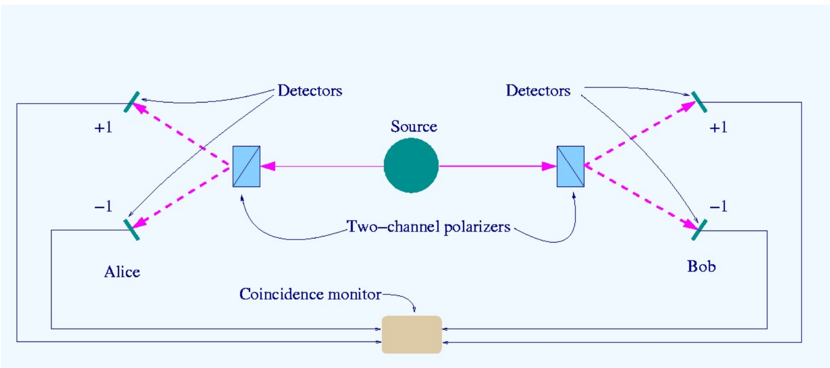

Figure 2. Measurements on a pair of entangled photons

The EPR paradox is a paradox in the following sense: if one takes quantum mechanics and adds some seemingly reasonable (but actually wrong, or questionable as a whole) conditions (referred to as locality, realism, counter factual definiteness[2][10] and completeness), then one obtains a contradiction. However, quantum mechanics by itself does not

[2]counter factual definiteness is the ability to speak meaningfully about the definiteness of the results of measurements, even if they were not performed

appear to be internally inconsistent, nor — as it turns out — does it contradict relativity. As a result of further theoretical and experimental developments since the original EPR paper, most physicists today regard the EPR paradox as an illustration of how quantum mechanics violates classical intuitions.

3.2 Hidden variable theories and no-go theorems

There are several ways to resolve the EPR paradox. The one suggested by EPR is that the statistical (probabilistic) nature of quantum mechanics indicates, despite its enormous success in experimental predictions, that quantum mechanics is an ”incomplete” description of reality and there should exist some yet undiscovered more fundamental theory of nature, that can due to its dependence on some hidden factors that govern the behaviour of quantum world, always predict the outcome of each measurement with certainty. Such theories are called hidden variable theories.[11] Although determinism (Albert Einstein insisted that, ”God does not play dice”) was initially a major motivation for physicists looking for hidden variable theories, nondeterministic theories trying to explain what the supposed reality underlying the quantum mechanics formalism looks like, later also emerged.

Bell, Kochen and Specker, and others then produced some powerful no-go theorems[12] that dispose of the most plausible hidden variable theories by showing that even though they may look attractive are in fact not possible. First in 1964, John Bell showed through his famous theorem[6] that if local hidden variables exist, certain experiments could be performed where the result would have to satisfy a Bell inequality[7]. If, on the other hand, quantum entanglement is correct the Bell inequality would be violated. Experiments have later been performed that have found violations of these inequalities up to 242 standard deviations and thus with great certainty ruled out local hidden variable theories. Another no-go theorem concerning hidden variable theories is the Kochen-Specker theorem, described in detail below.

No-go theorems together with experiments have shown a vast class of hidden variable theories to be incompatible with observations. However, there are a couple of ways around that. Theoretically, though highly improbable, there could be experimental deficiencies called loopholes, that affect the validity of the experimental findings. Another way of blocking no-go theorems that hidden variable theories have proposed is “contextuality”– that the outcome of an experiment depends upon hidden variables in the apparatus.

A hidden-variable theory which is consistent with quantum mechanics would have to be non-local, maintaining the existence of instantaneous or faster than light nonlocal relations (correlations) between physically separated entities (information travel is still restricted to the speed of light). The first hidden-variable theory was the pilot wave theory of Louis de Broglie, dating from the late 1920s. The currently best-known hidden-variable theory is David Bohm’s Causal Interpretation of quantum mechanics, created in

1952, that gives the same answers as quantum mechanics, thus invalidating the famous theorem by von Neumann that no hidden variable theory reproducing the statistical predictions of QM is possible. Bohm's (nonlocal) hidden variable is called the quantum potential.

3.3 Kochen-Specker paradox

In quantum mechanics, the Kochen-Specker theorem[14, 13] is a certain "no go" theorem proved by Simon Kochen and Ernst Specker in 1967.[30] It places certain constraints on the permissible types of hidden variable theories which try to explain the apparent randomness of quantum mechanics as a deterministic theory featuring hidden states, defining definite values of observables at all times. The theorem is a complement to Bell's inequality.

The theorem excludes noncontextual hidden variable theories intending to reproduce the results of quantum mechanics and requiring values of a quantum mechanical observables to be **noncontextual** (i.e. independent of the measurement arrangement).(However, note, that it remains possible, that the value attribution may be context-dependent, i.e. observables corresponding to equal vectors in different measurement arrangements need not have equal values.)

The Kochen-Specker theorem was an important step forward in the debate on the (in)completeness of quantum mechanics, since it demonstrated the impossibility of Einstein's assumption, made in the famous Einstein-Podolsky-Rosen paper from 1935 (creating the so-called EPR paradox), that quantum mechanical observables represent 'elements of physical reality'. Bohr[1] tried to overcome that problem by introducing contextuality which in exchange implied nonlocality (or spooky action at a distance, as Einstein loved to call it). In the 1950's and 60's two lines of development were open for those being fond of metaphysics, both lines improving on a "no go" theorem presented by von Neumann, trying to prove the impossibility of hidden variable theories yielding the same results as does quantum mechanics. Bell assumed that quantum reality is nonlocal, and that probably only local hidden variable theories are in disagreement with quantum mechanics. More importantly did Bell manage to lift the problem from the metaphysical level to the physical one by deriving an inequality, the Bell inequality, that can be experimentally tested.

A second line is the Kochen-Specker one. The essential difference with Bell's approach is that there is no implication of nonlocality because the proof refers to observables belonging to one single object, to be measured in one and the same region of space. And while Bell inequality gives only statistical restrictions on the results of measurements (their statistical distributions differ in both types of theory), the Kochen-Specker paradox states that certain sets of QM observables cannot be assaigned values at all. Contextuality is related here with incompatibility of quantum mechanical observables, incompatibility being associated with mutual exclusiveness of measurement arrangement. By using the so-called yes-no observables, having only values 0 and 1, corresponding to projection operators on the eigenvectors of a certain orthogonal basis in a three-dimensional Hilbert space, Kochen and Specker were able to find a set of 117 such projection operators, not allowing to assign to each of them in an unambiguous way either value 0 or 1. But here we reproduce one of the similar though much simpler proofs given much later by Asher Peres.

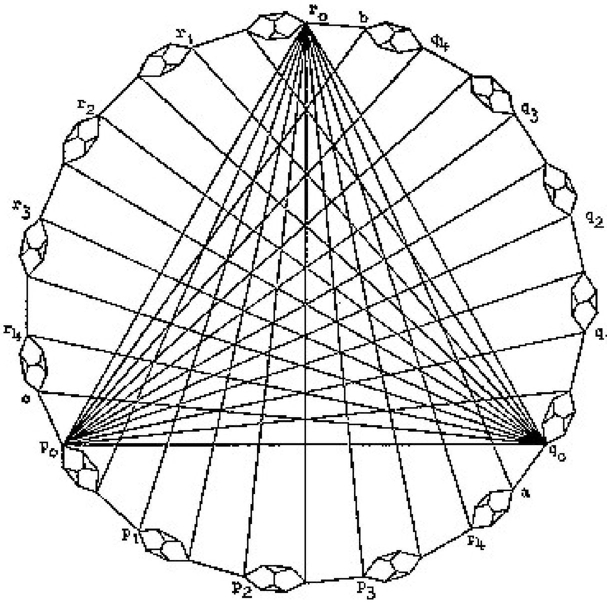

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Figure 3. The original Kochen-Specker's 117 directions

3.3.1 Kochen-Specker paradox for Peres' 33-direction configuration

Kochen-Specker paradox for Peres' 33-direction configuration:
"There is no 101-function for the ± 33 directions of Figure 4."

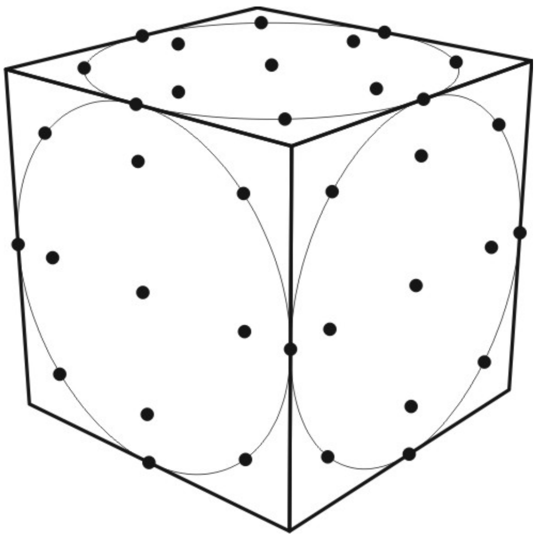

Figure 4. The ± 33 directions are defined by the lines joining the center of the cube to the ± 6 mid–points of the edges and the ± 3 sets of 9 points of the 3×3 square arrays shown inscribed in the incircles of its faces.

For the triple experiment[3], noncontextuality doesn't allow the particle's spin in the z-direction (say) to depend upon the measurement frame (x, y, z). This means that we can assign definite values of squares of components of spin to every direction from Peres' set. For a spin 1 particle such a symmetrical function of direction is called "101-function". One of its properties is that for three orthogonal directions we always get some permutation of a "canonical outcome" $(1, 0, 1)$. That entails that we can not get the outcome $(0, 0)$ for two orthogonal directions.

Proof. Assume that a 101–function θ is defined on these ± 33 directions. If $\theta(W) = i$, we write $W \to i$. Altogether there are 40 different orthogonal triples - 16 inside the Peres configuration together with the 24 that are obtained by completing its 24 remaining orthogonal pairs. The orthogonalities of the triples and pairs used below in the proof of a contradiction are easily seen geometrically. For instance, in Figure 2, B and C subtend the same angle at the center O of the cube as do U and V, and so are orthogonal. Thus A,B,C form an orthogonal triple. Again, since rotating the cube through a right angle (90°) about OZ takes D and G to E and C, the plane orthogonal to D passes through Z,C,E, so that C,D is an orthogonal pair and Z,D,E is an orthogonal triple. As usual, we write "wlog" to mean "without loss of generality".

[3]defined in SPIN

The orthogonality of X, Y, Z implies $X \to 0$, $Y \to 1$, $Z \to 1$ wlog (symmetry)

The orthogonality of X, A and X, A' implies $A \to 1$ and $A' \to 1$

The orthogonality of A, B, C and A', B', C' implies $B \to 1$, $C \to 0$ wlog (symmetry with respect to rotation through an angle 180° about OX) and $B' \to 1$, $C' \to 0$ wlog (symmetry with respect to rotation through an angle 90° about OX)

The orthogonality of C, D and C', D' implies $D \to 1$ and $D' \to 1$

The orthogonality of Z, D, E and Z, D', E' implies $E \to 0$ and $E' \to 0$

The orthogonality of E, F and E, G and similarly E', F' and E', G' implies $F \to 1$, $G \to 1$ and $F' \to 1$, $G' \to 1$

The orthogonality of F, F', U implies $U \to 0$

The orthogonality of G, G', V implies $V \to 0$

and since U is orthogonal to V , this is a contradiction that proves the above assertion (Lemma).

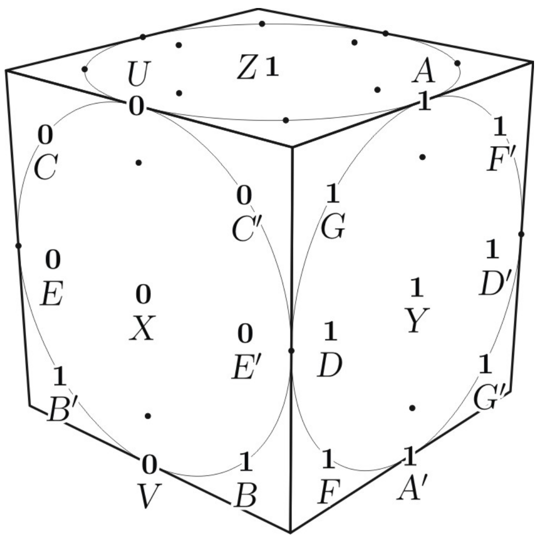

Figure 5. Spin assignments for Peres' 33 directions

Chapter 4

The Free Will Theorem

4.1 Introduction to The Free Will Theorem[15]

Do we really have free will, or is it all just an illusion? And what exactly is free will if it exists and how is it possible to reconcile it with determinism? What if everything is predetermined but we still can't have the knowledge of the future even in theory? Fate, destiny?

Since the dawn of time the question of our free will has been one of the most controversial ones and it still remains that way in present day physics which hasn't yet found the proper place for it inside the theory.

The Free Will Theorem of John Conway and Simon Kochen[23] doesn't answer to this ancient riddle but merely states that, if we have a certain amount of "free will", then so do some elementary particles. Thus from explicitly assumed small amount of human free will, which is only that we can freely choose to make any one of a small number of observations, it deduces free will of the particles all over the universe i.e. the particles' response to a certain type of experiment is not determined by the entire previous history of that part of the universe accessible to them, so we can describe that response as free. If the questions aren't determined ahead of time, so are not the outcomes of measurement. But that does not mean that free will exists at all. A fully deterministic view of the universe could imply that both our questions about the particles and the answers to those questions are pre-ordained.

What follows is, that if their physical axioms are even approximately true, together with the Free Will Assumption, they imply, that no theory, (whether it extends quantum mechanics or not), can correctly predict the results of future spin experiments, since these results involve free decisions that the universe has not yet made. Therefor this failure to predict them should no longer be regarded as a defect of theories extending quantum mechanics, but rather as their advantage.

The strength of this no-go theorem lies in that it doesn't presuppose any physical theory but refers actually only to the predicted macroscopic results of certain possible experiments. (The theorem refers to the world itself, rather than to some theory of the world.)

4.2 Philosophical thoughts on the question of Free Will

Hume, compatibilism, determinism, Dolev

The definition of "free will" used in the proof of the Free Will Theorem is simply that an outcome is "not determined" by prior conditions, and may therefore be equivalent to the possibility that the outcome is simply random, whatever that means. Thus, the definition of "free will" may not coincide with other definitions or intuitions about free will. (Indeed, some philosophers strongly dispute the equivalence of "not determined" with the existence of free will.)

We can make our assumption explicit:

Free Will Assumption:
The experimenter can freely choose to make any one of a small number of observations. By freely we mean, that his choice is not a function of the information accessible to him.

4.3 Stating the theorem

The experimental setup in this case is similar than the one in EPR thought experiment, only this time we are considering a pair of particles, each of total "spin 1" that are "twinned," meaning that they are in "the singlet state," i.e. with total spin 0, travelling in opposite directions and performing upon them an operation called "measuring the square of the component of spin in a direction w" which always yields one of the answers 0 or 1. We will indicate the result of this operation by writing $w \to i$; ($i = 0$ or 1). We call such measurements for three mutually orthogonal directions x, y, z a triple experiment for the frame (x, y, z).

Despite the use of quantum mechanical terminology in the above statement, we don't refer to any particular theoretical concepts, but merely to the locations of the spots on a screen that are produced by suitable beams in the above kinds of experiment. No certain theory has to be preassumed and the axioms used thus only refer to the predicted macroscopic results of certain possible experiments.

The SPIN axiom:
A triple experiment for the frame (x, y, z) always yields the outcomes $1, 0, 1$ in some order. We can write this as: $x \to j$, $y \to k$, $z \to l$, where j, k, l are 0 or 1 and $j + k + l = 2$.

Note that, even though sentences involving definite values for $\hat{\mathbf{S}}_\mathbf{x}$, $\hat{\mathbf{S}}_\mathbf{y}$, $\hat{\mathbf{S}}_\mathbf{z}$ are meaningless, since these operators do not commute, for a spin 1 particle their squares do commute[1] and hence, correspond to compatible observables, not subjected to Heisenberg uncertainty principle. Since the same eigenstates belong to all of them, they can be simultaneously measured[2] and attributed measured values 0 or 1.

[1] for proof of that statement look in the **Appendix**

[2] e.g. by an electrical version of the Stern–Gerlach experiment, by interferometry that involves co-

The twinned particles are in the state with total spin 0, meaning if we were to measure the component of spin in some arbitrary direction w for both distantly separated particles we would obtain opposite outcomes, but since we measure their squares, they give the same answers to corresponding questions.

If the same triple x, y, z were measured for each particle, possibly in different orders or even simultaneously, then the two particles' responses to the experiments in individual directions would be the same. For instance, if measurements in the order x, y, z for one particle produced $x \to 1$, $y \to 0$, $z \to 1$, then measurements in the order y, z, x for the second particle would produce $y \to 0$, $z \to 1$, $x \to 1$.

The TWIN axiom:
For twinned spin 1 particles, if the first experimenter A performs a triple experiment for the frame (x, y, z), producing the result $x \to j$, $y \to k$, $z \to l$ while the second experimenter B measures in a single direction w, then if w is one of x, y, z, its result is that $w \to j$, k or l, respectively.

The FIN Axiom:
There is a finite upper bound to the speed with which information can be effectively transmitted. (effective transmission of information applies to any realistic physical transmission)

This is, of course, a well–known consequence of relativity theory, the finite bound being the speed of light, with all the usual relativistic terminology of past and future light–cones. FIN is not experimentally verifiable, even in principle (unlike SPIN and TWIN). Its real justification is that it follows from theory of relativity and what we call "effective causality,"[3] that effects cannot precede their causes irrespective of Lorentz reference frame observing them.

The Free Will Theorem (assuming SPIN, TWIN, and FIN):
If the choice of directions in which to perform spin 1 experiments is not a function of the information accessible to the experimenters, then the responses of the particles are equally not functions of the information accessible to them.

We explicitly assumed that experimenters have sufficient free will to choose the settings of their apparatus (directions of measurements) in a way that is not completely determined by past history (accessible information) and expressed that in a Free Will Assumption. The Free Will Theorem then deduces that the spin 1 particles' responses are also not determined by past history. Thus they possess exactly the same property, which for experimenters is an instance of what is usually called "free will," so we use the same term also for particles.

herent recombination of the beams for $S_x = +1$ and $S_x = -1$, or finally by the "spin-Hamiltonian" type of experiment, that measures an expression of the form $a\mathbf{\hat{S}_x^2} + b\mathbf{\hat{S}_y^2} + c\mathbf{\hat{S}_z^2}$

[3]explanation of effective notions

However, we can also produce a modified version of the theorem without the use of the Free Will Assumption. It invalidates certain types of theories, called the “free state theories”, which describe the evolution of a state from an initial arbitrary or “free” state according to laws that are themselves independent of space and time.

The Free State Theorem (assuming SPIN, TWIN and FIN):
No free state theory can exactly predict the results of twinned spin 1 experiments for arbitrary triples x, y, z and vectors w. In fact it cannot even predict the outcomes for the finitely many cases used in the proof.

4.4 The proof of the Free Will Theorem

We consider experimenters A and B performing the pair of experiments described in the TWIN axiom on separated twinned particles a and b, and assert that responses of a and b cannot be functions of all the information available to them.

Let’s first show that the value $\theta(w)$ of “the squared spin in direction w” doesn’t (already) exist prior to its measurement, for if it did, the function $\theta(w)$ would be defined for each direction w, and would have according to SPIN the following interesting properties:

1. Its values on each orthogonal triple would be 1, 0, 1 in some order.
   This easily entails two further properties:
2. We cannot have $\theta(x) = \theta(y) = 0$ for any two perpendicular directions x and y.
3. Because $\theta(w)$ is “the value of the SQUARED spin in direction w”, we have $\theta(w) = \theta(-w)$ for any pair of opposite directions w and $-w$.
   Consequently, $\theta(w)$ is really defined on “$\pm$ directions.”

We call a function on a set of directions that has all three of these properties a “**101-function**.” However, we already know from the Kochen-Specker paradox (proven above for Peres’ 33-direction configuration) that this certain geometric combinatorial puzzle has no solution. This disposes of the above naive supposition about existence of $\theta(w)$ prior to its measurement.

We are now left only with the possibility that the particles’ responses are not predetermined (ahead of time), but we will try to prove that they also can not be a function of the information available to them.

So let us suppose, that particle a’s response is a function $\theta_a(\alpha)$ of the information α available to it and show that this assumption leads to contradiction. Information α is determined by the choice of the certain triple x, y, z and all the information α' that was accessible just before and independent of that choice, i.e. the information in the past light cone of a (FIN).[4] So we can express it as a function

$$\theta_a(x, y, z; \alpha') = \{x \to j, y \to k, z \to l\}$$

or if we refine this notation to indicate the result of measurement in any particular one of the three directions by adjoining a question–mark to it, thus

$$\begin{aligned}\theta_a(x?, y, z; \alpha') &= j \\ \theta_a(x, y?, z; \alpha') &= k \\ \theta_a(x, y, z?; \alpha') &= l\end{aligned}$$

Similarly we express b's responses as a function of the direction w and the information β' available to b before w was chosen

$$\theta_b(w; \beta') = \{w \to m\}$$

or alternatively

$$\theta_b(w?; \beta') = m$$

The TWIN axiom then implies that

$$\theta_b(w?; \beta') = \begin{cases} \theta_a(x?, y, z; \alpha') & \text{if } w = x, \\ \theta_a(x, y?, z; \alpha') & \text{if } w = y, \\ \theta_a(x, y, z?; \alpha') & \text{if } w = z, \end{cases}$$

According to the Free Will Assumption A and B can freely choose any direction w and triple of orthogonal directions x, y, z from Peres' set of ± 33 directions, but whenever the directions coincide, the corresponding measurements must yield the same answers regardless of the information α' and β'. Now, the important thing is, that not only is the information α' by definition independent of the choice of x, y, z, but it is also independent of the choice of w, since in some coordinate frames B's experiment happens later than A's and in order to preserve causality can not influence A. For the same reason, β' is independent of x, y, z as well as w.

Now, to conclude the proof, let us first notice that responses of paricles a and b to measurement of the squares of components of spin are of the form

$$\theta_a = \theta_a(x, y, z; \alpha') \neq f(w; \beta')$$

$$\theta_b = \theta_b(w; \beta') \neq f(x, y, z; \alpha')$$

but at the same time have to fulfil TWIN, therefor it is evident that they can only be fuction of direction of measurement, independently of the measurement arrangement, i.e.

[4] even if some additional piece of information is created after the choice of the triple, it must clearly be of form $i(x, y, z; \alpha')$, meaning that we can still describe response of a as some function of x, y, z and α': $\theta_a(x, y, z; \alpha', i(x, y, z; \alpha')) = \theta'_a(x, y, z; \alpha')$

the response is noncontextual. To see that, let us consider for example experimenter A measuring the triple x, y, z and obtaining in x direction the result

$$\theta_a(x?, y, z; \alpha') = j$$

From TWIN it follows, that if experimenter B also measured in the same direction, he would obtain the same answer

$$\theta_b(x?; \beta') = j$$

and because that response is not a function of x, y, z, he would obtain it regardlessly of that, what kind of measurement (x, y, z) if any at all performed A. But on the other hand this entails that also A's measurement of any other triple that includes x would produce the outcome j in x direction

$$\theta_a(x?, \widetilde{y}, \widetilde{z}; \alpha') = j$$

because it must correspond to possible, but at the same time independent B's measuremnt. We have thus proven that in every single case (for any fixed α', β') the outcome $\theta(w)$ is completely determined for any direction.w[5] Because of the SPIN it must be described by the "101-function" of direction, which doesn't exist for Peres' configuration. So we arrived at contradiction. The crucial moment in this derivation was to realize, that the outcomes on both sides are independent of each other but at the same time because of quantum entanglement yield same values in the same directions, so our inital assumption that responses of particles are a function of our choice of a triple turned wrong.

We have thus first refuted the possibility that the outcomes of measurement are predetermined, and now also the case in which they are determined by its history and our free choice (triple x, y, z and the state of the universe before the choice of that triple). This leaves only the case in which some of the information used (say, by a) is spontaneous, i.e. is itself not determined by any earlier information whatsoever. This spontaneous information arises in the moment when the universe takes a free decision and contibutes an additional input to the result of the measurement.

This completes the proof of the Free Will Theorem, except for a brief remark on that in the statements of axioms used above and throughout the proof we have made some tacit idealizations that might worry someone. For example, we have assumed that the spin experiments can be performed instantaneously, and in exact directions. Both assumptions and subsequent proof can be replaced by more realistic ones that account for both, the approximate nature of actual experiments and their finite duration. The idea is to take into account that the twinned pair might only be nominally in the singlet state and that orthogonality of frame (x, y, z) and parallelism between w and one of the triple x, y, z might be only approximate. Redefined SPIN and TWIN axioms take into consideration that expected ideal results are achieved only in a portion of experiments. By estimating inaccuracy in angle it is possible to show, that with sufficient precision, the probabilty of discrepancy of outcomes from those obtained in the ideal conditions can be dropped below 1/40. Namely any function of direction must fail to have the 101–property

[5] $\theta_a(w)$ is completely determined by possible measurement $\theta_b(w; \beta')$

for at least one of 40 particular orthogonal triples (the 16 orthogonal triples of the Peres' configuration and the triples completed from its remaining 24 orthogonal pairs).

So, what we have proven is, that if there are experimenters with a modicum of free will, then response of elementary particles is also free. Close inspection reveals that Free Will Assumption was only needed to force the functions θ_a and θ_b to be defined for all of the triples x, y, z and vectors w from a certain finite collection and some fixed values α' and β' of other information about the world. Now we can take these as the given arbitrary initial conditions that enter into some putative free state theory that explains the results of the measurement and then the same procedure proves also the Free State Theorem.

Chapter 5

Conclusion

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Chapter 6

Appendix

The Proof of commutation of squares of components of spin in three orthogonal directions for S=1 particles

In case of S=1/2 or S=1: $[\hat{\mathbf{S}}_{\mathbf{i}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{j}}^{\mathbf{2}}] = 0$; $\quad \forall\, i.j \in \{x, y, z\}$,

but in general this doesn't hold true.

To show that, we take into consideration the following rules for commutators:

$$\begin{aligned}
[\hat{\mathbf{S}}_{\mathbf{i}}, \hat{\mathbf{S}}_{\mathbf{j}}] &= i\hbar\varepsilon_{ijk}\hat{\mathbf{S}}_{\mathbf{k}} \\
\left[\hat{\mathbf{S}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{i}}\right] &= 0 \\
\left[\hat{\mathbf{S}}_{\mathbf{z}}, \hat{\mathbf{S}}_{\pm}\right] &= \pm\hbar\hat{\mathbf{S}}_{\pm} \\
\left[\hat{\mathbf{S}}_{\pm}, \hat{\mathbf{S}}_{\mp}\right] &= \pm 2\hbar\hat{\mathbf{S}}_{\mathbf{z}} \\
\left[\hat{\mathbf{S}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\pm}\right] &= 0
\end{aligned}$$

...and those for operators:

$$\begin{aligned}
\hat{\mathbf{S}}_{\mathbf{z}}|s, m\rangle &= \hbar m|s, m\rangle \\
\hat{\mathbf{S}}^{\mathbf{2}}|s, m\rangle &= \hbar^2 s(s+1)|s, m\rangle \\
\hat{\mathbf{S}}_{\pm}|s, m\rangle &= \hbar\sqrt{s(s+1) - m(m\pm 1)}|s, m\pm 1\rangle \ ; \quad \hat{\mathbf{S}}_{\pm} = \hat{\mathbf{S}}_{\mathbf{x}} \pm i\hat{\mathbf{S}}_{\mathbf{y}}
\end{aligned}$$

$[AB, C] = A[B, C] + [A, C]B \Longrightarrow [A, BC] = B[A, C] + [A, B]C$

$S = 1 \Longrightarrow (2S + 1 = 3 : \quad M = -1, 0, 1)$

$\hat{\mathbf{S}}_{\mathbf{z}}^{\mathbf{2}}|1, M\rangle = \hbar^2 M^2|1, M\rangle \in \{0,\ \hbar^2|1, M\rangle\}$

$\hat{\mathbf{S}}^{\mathbf{2}}|1,M\rangle = (\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}} + \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}} + \hat{\mathbf{S}}_{\mathbf{z}}^{\mathbf{2}})|1,M\rangle = \hbar^2 S(S+1)|1,M\rangle = 2\hbar^2|1,M\rangle$

So, we see that when measuring the squares of the components of spin in three orthogonal directions, they yield the outcomes 1,0,1 in some order.

$$\begin{aligned}
\hat{\mathbf{S}}_+|1,-1\rangle &= \sqrt{2}\hbar|1,0\rangle \\
\hat{\mathbf{S}}_+|1,0\rangle &= \sqrt{2}\hbar|1,1\rangle \\
\hat{\mathbf{S}}_+|1,1\rangle &= 0 \\
\hat{\mathbf{S}}_-|1,-1\rangle &= 0 \\
\hat{\mathbf{S}}_-|1,0\rangle &= \sqrt{2}\hbar|1,-1\rangle \\
\hat{\mathbf{S}}_-|1,1\rangle &= \sqrt{2}\hbar|1,0\rangle \\
\hat{\mathbf{S}}_{\mathbf{z}}|1,-1\rangle &= -\hbar|1,-1\rangle \\
\hat{\mathbf{S}}_{\mathbf{z}}|1,0\rangle &= 0 \\
\hat{\mathbf{S}}_{\mathbf{z}}|1,1\rangle &= \hbar|1,1\rangle
\end{aligned}$$

We use the matrix representation of operators

$\boxed{\hat{\mathbf{A}}|\Psi\rangle = \sum_n |n\rangle(\sum_m \langle n|\hat{\mathbf{A}}|m\rangle\langle m|\Psi\rangle) = \sum_n |n\rangle(\sum_m A_{nm}c_m) = \sum_n |n\rangle d_n}$

to obtain:

$$\hat{\mathbf{S}}_+ = \hbar\sqrt{2}\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad \hat{\mathbf{S}}_- = \hbar\sqrt{2}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\hat{\mathbf{S}}_{\mathbf{x}} = \frac{1}{2}(\hat{\mathbf{S}}_+ + \hat{\mathbf{S}}_-) = \frac{\hbar\sqrt{2}}{2}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \hat{\mathbf{S}}_{\mathbf{y}} = -\frac{i}{2}(\hat{\mathbf{S}}_+ - \hat{\mathbf{S}}_-) = \frac{\hbar\sqrt{2}}{2}\begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix}$$

$$\hat{\mathbf{S}}_{\mathbf{z}} = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

$$\begin{aligned}
[\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}] &= \hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{y}}[\hat{\mathbf{S}}_{\mathbf{x}}, \hat{\mathbf{S}}_{\mathbf{y}}] + \hat{\mathbf{S}}_{\mathbf{x}}[\hat{\mathbf{S}}_{\mathbf{x}}, \hat{\mathbf{S}}_{\mathbf{y}}]\hat{\mathbf{S}}_{\mathbf{y}} + \hat{\mathbf{S}}_{\mathbf{y}}[\hat{\mathbf{S}}_{\mathbf{x}}, \hat{\mathbf{S}}_{\mathbf{y}}]\hat{\mathbf{S}}_{\mathbf{x}} + [\hat{\mathbf{S}}_{\mathbf{x}}, \hat{\mathbf{S}}_{\mathbf{y}}]\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{x}} \\
&= i\hbar(\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{z}} + \hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{z}}\hat{\mathbf{S}}_{\mathbf{y}} + \hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{z}}\hat{\mathbf{S}}_{\mathbf{x}} + \hat{\mathbf{S}}_{\mathbf{z}}\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{x}}) \\
&= i\hbar(\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{z}} + \hat{\mathbf{S}}_{\mathbf{x}}(\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{z}} - i\hbar\hat{\mathbf{S}}_{\mathbf{x}}) + \hat{\mathbf{S}}_{\mathbf{y}}(\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{z}} + i\hbar\hat{\mathbf{S}}_{\mathbf{y}}) + (\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{z}} + [\hat{\mathbf{S}}_{\mathbf{z}}, \hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{x}}])) \\
&= i\hbar(\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{z}} + \hat{\mathbf{S}}_{\mathbf{x}}(\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{z}} - i\hbar\hat{\mathbf{S}}_{\mathbf{x}}) + \hat{\mathbf{S}}_{\mathbf{y}}(\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{z}} + i\hbar\hat{\mathbf{S}}_{\mathbf{y}}) + (\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{z}} + i\hbar\hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}} - i\hbar\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}})) \\
&= 2i\hbar(\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{z}} + \hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{z}} - i\hbar\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}} + i\hbar\hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}})
\end{aligned}$$

For S=1 this expression becomes:

$$
\begin{aligned}
[\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}] &= 2i\hbar \cdot (\frac{\hbar^3}{2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \\
&+ \quad \frac{\hbar^3}{2} \begin{bmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} + \\
&+ \quad -\frac{i\hbar^3}{2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \frac{i\hbar^3}{2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}) \\
&= 2i\hbar \cdot \frac{i\hbar^3}{2} (\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 1 \end{bmatrix}) \\
&= 0
\end{aligned}
$$

$$
[\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}] = 0 \implies \begin{array}{l} [\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{z}}^{\mathbf{2}}] = [\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}}, \hat{\mathbf{S}}^{\mathbf{2}} - \hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}} - \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}] = -[\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}] = 0\hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{z}}^{\mathbf{2}} > \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{z}}^{\mathbf{2}} \\ = [\hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}, \hat{\mathbf{S}}^{\mathbf{2}} - \hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}} - \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}] = +[\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}] = 0 \end{array}
$$

Therefor in case of S=1/2 or S=1: $[\hat{\mathbf{S}}_{\mathbf{i}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{j}}^{\mathbf{2}}] = 0$; $\quad \forall\, i.j \in \{x, y, z\}$
$\hat{\mathbf{S}}_{\mathbf{i}}^{\mathbf{2}}|1, M\rangle = \hbar^2 M^2 |1, M\rangle \in \{0,\ \hbar^2|1, M\rangle\}$; i=x,y,z
$\hat{\mathbf{S}}^{\mathbf{2}}|1, M\rangle = (\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}} + \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}} + \hat{\mathbf{S}}_{\mathbf{z}}^{\mathbf{2}})|1, M\rangle = \hbar^2 S(S+1)|1, M\rangle = 2\hbar^2|1, M\rangle$
because $\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{z}}^{\mathbf{2}}$ are commuting operators, the same eigenstate corresponds to all of them.
Let's cover also the case S=1/2 by showing $[\hat{\mathbf{S}}_{\mathbf{i}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{j}}^{\mathbf{2}}] = 0$:

$$
\hat{\mathbf{S}}_{\mathbf{x}} = \frac{\hbar}{2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad \hat{\mathbf{S}}_{\mathbf{y}} = \frac{\hbar}{2} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \qquad \hat{\mathbf{S}}_{\mathbf{z}} = \frac{\hbar}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}
$$

$$
\begin{aligned}
[\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}}, \hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}] &= 2i\hbar(\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{z}} + \hat{\mathbf{S}}_{\mathbf{y}}\hat{\mathbf{S}}_{\mathbf{x}}\hat{\mathbf{S}}_{\mathbf{z}} - i\hbar\hat{\mathbf{S}}_{\mathbf{x}}^{\mathbf{2}} + i\hbar\hat{\mathbf{S}}_{\mathbf{y}}^{\mathbf{2}}) \\
&= 2i\hbar \cdot \frac{i\hbar^3}{8} (\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}) \\
&= 0
\end{aligned}
$$

$\hat{\mathbf{S}}_{\mathbf{i}}^{\mathbf{2}}|\frac{1}{2}, m\rangle = m^2\hbar^2|\frac{1}{2}, m\rangle = \frac{\hbar^2}{4}|\frac{1}{2}, m\rangle$; m=$\pm\frac{1}{2}$
$\hat{\mathbf{S}}^{\mathbf{2}}|\frac{1}{2}, m\rangle = s(s+1)\hbar^2|\frac{1}{2}, m\rangle = \frac{3\hbar^2}{4}|\frac{1}{2}, m\rangle$