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TURBULENCE AND THE NAVIER-STOKES EQUATIONS

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Abstract

The scope of this seminar is fully developed homogeneous and isotropic turbulence in incompressible viscous fluids. We shortly describe semi-empirical classical theory of turbulence which correctly describes many phenomena, but it is incomplete since it does not explain the so-called intermittency. The intermittency is an universal feature of the turbulent flows and should be included in any reasonable theory of turbulence. The attempts of direct simulations are shortly discussed the effective dimension of the phase space is estimated. At the end we describe the multyscale analyses of the Navier-Stokes equations developed by S. Grossman et al. and discuss their results. The ambitious goal to give a fair review of present work and a solid answer to the question: 'How much do we understand about turbulence?' is not fullfilled for two reasons. The first is the enormous amount of literature about the subject and the second incoherence between different literature. We may conclude that the answer to the above question is not clear yet.

Contents

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1 Introduction

Most flows in nature and in engeneering are turbulent. The motion of the air in the atmosphere and the oceanic currents are turbulent. The boundary layer on an aircraft wing will in most cases be turbulent aswell. A notable exceptions are biological flows, which are not turbulent. When we stir cofee in a cup, we create turbulence. This increases mixing (by orders of magnitude in comparisson with molecular mixing) and the sugar gets dissolved quicker. Life on Earth could not be possible without the turbulent mixing, mass and heat transfer in the atmosphere. On the other hand turbulence can be harmful, too. When gas or oil flow through the pipes or a vessel moves in a fluid, the wall friction is large owing to turbulence and the energy losses are gigantic. Understanding the nature of the turbulent flow including the details is thus of practical importance, just as much as of theoretical interest. So far there is no complete theory of turbulence. It is a completely classical problem, no quantum mechanics, no relativity, but we still know very little about it. Turbulent flows seem irregular, random and unpredictable. They are highly nonlinear, which means that small perturbations grow exponentially. The perturbations, however, frequently reach an equilibrium, but this new equilibrium can become unstable to new more complicated perturbations and so on and on, finally reaching a chaotic state. Turbulent flow is made up of eddies of different sizes. There is a vast number of scales (sizes of eddies) present in a turbulent flow. The largest eddies are of the size of the flow region. They contain most of the energy. The energy is passed from the large to the small eddies by nonlinear interactions until it is dissipated by viscosity in the smallest eddies, which are usually about a milimeter large. Turbulent flow requires a continuous supply of the energy to make up for the viscous losses.

1.1 History

The first systematic work on turbulence was done by O. Reynolds in 1883. He showed that the pipe flow becomes turbulent when the nondimensional ratio called the Reynolds number becomes large enough. Reynolds argued that flows of the same kind with the same Reynolds numbers are similair.

The idea of correlation functions was introduced by G.I. Taylor in 1921. Later he introduced the concepts of homogeneous and isotropic turbulence and the turbulence spectrum.

L. Richardson in 1922 wrote the first book on the numerical weather prediction. He proposed the idea of spectral energy cascade, which is the heart of our present understanding of the turbulent flow.

Kolmogorov followed Richardson's idea of the energy transport from large to small eddies. He supposed that the statistics of the small scales are isotropic and only depend on the viscosity and the rate of dissipation. On dimensional grounds he derived the order of the smallest eddy size λ_0, called later the Kolmogorov microscale. The scales smaller than λ_0 form the dissipation range. The energy is dissipated due to viscosity there. The largest scales which contain almost all energy are the energy range. In between them there is the inertial range, where the energy dissipation ϵ is the only important parameter. Kolmogorov and Obukhov independently derived the spectrum in the inertial range, which is proportional to $\epsilon^{2/3}K^{-5/3}$, where K is the wavenumber. This five-third law has later been confirmed by the observations.

1.2 The Equations of Motion

A fluid flow is described by the **velocity field** $\vec{u}(\vec{r},t)$. The equations of motion are the Navier-Stokes equations

$$\frac{\partial \vec{u}}{\partial t} = -(\vec{u}\cdot\nabla)\vec{u} - \frac{1}{\rho}\nabla p + \nu\nabla^2\vec{u} + \vec{f} \tag{1}$$

Here $\vec{u}(\vec{r},t)$ is the velocity field, $p(\vec{r},t)$ the pressure, ρ the density of the fluid and $\nu = \eta/\rho$ the kinematic viscosity, the only coefficient to determine the viscosity of an incompressible fluid. Typically, $\nu = 15mm^2/s$ for air and $\nu = 1mm^2/s$ for water at room temperatures. $\vec{f}$ is the volume external forcing. The incompressibility of the fluid is described by the condition $\nabla\cdot\vec{u} = 0$. The equations (1) are completed by the boundary conditions $\vec{u}(\vec{r}\in\delta\Omega,t) = \vec{U}_L(t)$, $\delta\Omega$ is the boundary of the flow region Ω of volume V.
We can introduce the **stress tensor** σ_{ij},

$$\sigma_{ij} = -p\delta_{ij} + 2\rho\nu e_{ij}\ ,$$

where $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ is the symmetric **viscous stress tensor**. The Navier-Stokes equations written in the compact form are

$$\frac{D}{Dt}u_i = \frac{1}{\rho}\partial_j\sigma_{ij} + f_i\,. \tag{2}$$

Here $\frac{D}{Dt}$ is the substantial derivative, $\frac{Du_i}{Dt} = \frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}$. If we multiply (2) by u_i and sum over i we get the equation for the kinetic energy balance

$$\frac{D}{Dt}\left(\tfrac{1}{2}u_i^2\right) = \frac{\partial}{\partial x_j}\left(-\frac{pu_i}{\rho} + 2\nu u_i e_{ij}\right) - 2\nu e_{ij}e_{ij}\,. \tag{3}$$

The left side represents the rate of change of the kinetic energy and the right side the mechanisms that bring about this change. The first two terms in the brackets are in the 'flux divergence form'. Integrating the equation (3) over all space would give the rate of change of the total kinetic energy. The divergence terms can be transformed into a surface integral by Gauss's theorem. If the flow is confined to a limited region with $\vec{u} = 0$ at sufficient distance, this terms do not contribute to the change of the total kinetic energy. It follows that this two terms can only redistribute or transport the energy from one region to another and can not generate or dissipate it. The third term on the left represents the loss

Figure 1: The Karman vorteces at two different Reynolds numbers (140 on the upper and 2300 on the lower picture). Water is flowing past a cylinder on the left). Behind the cylinder the flow structure is visualised by laser induced fluorescence.

of the energy due to viscosity, the **viscous dissipation**. We have denoted this quantity by ϵ earlier, so

$$\epsilon = 2\nu e_{ij} e_{ij} \, . \tag{4}$$

The Navier-Stokes equations are most revealing in the dimensionless form. Let us introduce new dimensionless variables :

$$X = \frac{r}{l}\,, \qquad U = \frac{u}{u_0}\,, \qquad P = \frac{p}{\rho u_0^2}\,, \qquad \tau = \frac{\nu t}{l^2}\,.$$

The time scale $\frac{l^2}{\nu}$ is a typical viscous time scale. With this definitions the equations (1) are

$$\frac{\partial U_i}{\partial \tau} = -\frac{u_0 l}{\nu}\left(U_j \frac{\partial U_i}{\partial X_j} + \frac{\partial P}{\partial X_i}\right) + \Delta U_i \, . \tag{5}$$

We see that (given the geometry and the boundary conditions) the solutions will depend only on the dimensionless parameter in front of the first two terms on the left of (5). This parameter is the basic parameter of the Navier-Stokes equations and is called the **Reynolds number**. We shall return to it shortly.

1.3 Correlations and Spectra

The velocity in a turbulent flow fluctuates chaotically with time about some mean value at each point in the flow. We call this regime **fully developed turbulence**. We first introduce the **mean velocity**, obtained by averaging the actual velocity at each point over long intervals of time

$$\vec{u}(\vec{r}, t) = \vec{u}_0(\vec{r}, t) + \vec{v}(\vec{r}, t)\, .$$

The mean velocity $\vec{u}_0$ varies smoothly from point to point. The difference $\vec{v} = \vec{u} - \vec{u}_0$ varries irregularly and we call it the **fluctuating part** of the velocity or simply the **fluctuation**. The mean value of the fluctuation is zero.
The autocorrelation function of the velocity is

$$R(\tau) = \overline{u(\vec{x}, t) u(\vec{x}, t + \tau)} \tag{6}$$

and its normalized version

$$r(\tau) = \frac{\overline{u(\vec{x}, t) u(\vec{x}, t + \tau)}}{\overline{u^2}}\, . \tag{7}$$

It goes to zero as $\tau \to \infty$. A measure of the width of the correlation function is

$$T = \int_0^\infty r(\tau) d\tau \,.$$

The Fourier transform of the autocorrelation function (6) is

$$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega t} R(\tau) d\tau \,. \tag{8}$$

Inversley it is true that

$$R(\tau) = \int_{-\infty}^{\infty} e^{i\omega t} S(\omega) d\omega \,. \tag{9}$$

If we put $\tau = 0$ we get

$$\overline{u^2} = \int_{-\infty}^{\infty} S(\omega) d\omega \,, \tag{10}$$

which shows that $S(\omega)d\omega$ is proportional to the energy in a frequency interval $d\omega$ around ω. We call the function $S(\omega)$ the **energy spectrum** and we have shown that it is simply the Fourier transform of the autocorelation function.
We can also define the autocorrelation as a function of the space coordinates,

$$R(\vec{x}) = \overline{u(\vec{x}_0, t) u(\vec{x}_0 + \vec{x}, t)} \,. \tag{11}$$

The energy spectrum is now the function of the wavenumber $\vec{K}$,

$$S\left(\vec{K}\right) = \frac{1}{(2\pi)^{\frac{1}{3}}} \int_{-\infty}^{+\infty} e^{-i\vec{K}\cdot\vec{x}} R(\vec{x}) d\vec{x} \,. \tag{12}$$

To measure the energy spectrum we need instantaneous measurements of the velocity at different positions. We approach this by lansing a fast enough probe and so we measure the 'frozen' velocity field.

To characterize the structure of the fluid the velocity field is not the only usefull information. Often its change by distance v_r

$$v_r = |\vec{u}(\vec{x} + \vec{r}, t) - \vec{u}(\vec{x}, t)| \tag{13}$$

is studied. It increases linearly with r for small r and less on larger distances (see figure (5). One usually calculates the moments

$$\langle |v_r|^m \rangle$$

which are called m-th order **structure function**.

2 The Classical Theory of Turbulence

There are no rigorous existence and uniqueness theorems for the Navier-Stokes equations. Also, there seem to be too many degrees of freedom to allow for any rigorous analitycal approach. We have to use intuition and qualitative, many times dimensional arguments to learn about turbulent flows. The classical theory of turbulence in its heart is intuitive and qualitative, but it gives some good results. The major work was certainly done by Kolmogorov. In these section we shall shortly summarize the classical theory of turbulence.

2.1 The Structure of the Flow

Let us think of a flow of an incompresssible fluid past a solid body. If the shape of the body is given, its geometrical properties are determined by one linear dimension, l. Any flow is specified by three parameters : l, ν and u_0, the mainstream velocity. By comparing the units of these quantities,

$$[\nu] = m^2 s^{-1}\,, \quad [l] = m\,, \quad [u_0] = ms^{-1}\,,$$

it is easy to construct the only dimensionless parameter,

$$Re = \frac{ul}{\nu}\,, \tag{14}$$

called the **Reynolds number**. If we are interested in other type of flow but flow past a solid body, we can define the Reynolds number in a similair way. The parameter l is always a typical linear dimension of the flow and u_0 is a typical velocity int the flow. The velocity field which is the solution of (1) will be a function of the form

$$\vec{u} = u_0 \vec{f}(\vec{r}/l, Re)\,.$$

In two different flows with the same geometry and boundary conditions the velocities $\vec{u}/u_0$ are thus the same functions of the ratio $\vec{r}/l$, provided the two Reynolds numbers are the same. The flows of the same type (flows past bodies of the same shape or flows in pipes of the same shape, for example) with the same Reynolds numbers are **similair**, even if the two fluids are different. This is called the law of similarity and was first stated by O. Reynolds in 1883.

It is well known that the fluid flow becomes turbulent if the Reynolds number is sufficiently large. The fluctuating part of the velocity may be qualitatively regarded as the superposition of **turbulent eddies** of different sizes. The size of an eddy is the order of magnitude of the distance over which the velocity varies appreciably. In the fully developed turbulence there is a hierarchy of eddies, from the largest to the smalest. The size of the largest eddies is of the order of the dimension of the region in which the flow takes place. We denote this with l. The velocity in this largest eddies is comparable to the variation of the mean velocity over the distance l, Δu. The frequency corresponding to an eddy is of the order of u/l, because it corresponds to the period with which the flow pattern is repeated, being observed from some fixed frame of reference.

Figure 2: The size of the largest eddy is of the order of the dimension of the flow region, in this case (flow in a boundary layer) of the order of the boundary thickness.

We can introduce different Reynolds numbers for different eddies. If λ is the size of the eddy and v_λ its velocity (orders of magnitude, of course), the corresponding Reynolds number is

$$Re_\lambda = \frac{\lambda v_\lambda}{\nu}\,. \tag{15}$$

This number decreases with the size of the eddy. For the largest eddies the Re_λ is large, which corresponds to small viscosity, so we can conclude that viscosity is unimportant for the largest eddies. The viscosity becomes important for the small eddies whose Re_λ is comparable to unity. We shall denote the size of these eddies by λ_0. The picture of the energy transport in the turbulent flow we have arrived at is as follows. The energy is transported from large eddies to the smaller ones practically without dissipation, there is a continuous flow of energy from large to small eddies. This energy is dissipated in the smallest eddies (the kinetic energy is transformed into heat). Of course, there have to be external sources present to supply the energy to the largest eddies. The range of scales $\lambda \approx l$ (the largest eddies) is called the **energy range**, almost all kinetic energy of the fluid is concentrated there. The scales where viscosity is important ($\lambda \leq \lambda_0$) form the **viscous range**. For large Reynolds numbers R these two ranges are far appart and between them is the **inertial range**. If the Reynolds number is large enough the inertial range is quite broad.
In the inertial range the viscosity is unimportant. The scales are still much smaller than l, so the flow might well be treated as isotropic and homogeneous. The only parameter relevant

in the inertial range is the energy dissipation ϵ. Let us use some simple arguments to derive the order of magnitude of the energy dissipation. The energy dissipation ϵ is the amount of energy dissipated in a unit of time and in a unit of mass of the fluid. Its unit is $[\epsilon] = m^2 s^{-3}$. The dissipation takes place in the smallest eddies, but the energy comes from the largest, so ϵ can only depend on the quantities characterizing the largest eddies. These are the fluid density ρ, the velocity Δu and the size l. From the dimensions we see that

$$\epsilon \approx \frac{(\Delta u)^3}{l} .$$

2.2 The Inertial Range

Let us examine the properties of the turbulence in the inertial range. The natural assumption is that such local turbulence is homogeneous and isotropic. The local properties thus do not depend on the direction of the mean velocity. Of course this holds for the relative motion of the fluid in the small eddies and not the absolute motion of the eddies themselves which is due to the larger eddies. If the eddies are much smaller than the dimension of the flow region ($\lambda \ll l$) but still much larger than the scale at which dissipation is appreciable, the parameters which define the local properties of the fluid are ϵ and ρ. The energy flux ϵ continually passes from larger to smaller eddies and thus it determines the properties of the eddies larger than λ_0. The velocity variations over distances λ, v_λ, can only depend on ϵ and λ. Again, by counting units we find the relation

$$v_\lambda \propto (\epsilon\lambda)^{\frac{1}{3}} . \tag{16}$$

The relation was first obtained by Kolmogorov and Obukhov in 1941. The velocity variation over small distances ($\lambda_0 \ll \lambda \ll l$) is proportional to the cube root of the distance. The quantity v_λ may also be regarded as the velocity of the turbulent eddies of the size λ. The variation of the mean velocity over small distances is small compared to variation of the fluctuating part and may be neglected.
Next we can find the range λ_0 at which viscosity becomes important. The order of magnitude for λ_0 is that for which Re_{λ_0} is about one. From the relation (16) and the definitions of the Re_λ, (15) we find

$$\lambda_0 = \left(\frac{\nu^3}{\epsilon}\right)^{\frac{1}{4}} . \tag{17}$$

Expressing the relation with the Reynolds number for the flow as a whole (14) gives

$$\lambda_0 \approx \frac{l}{Re^{\frac{3}{4}}} . \tag{18}$$

In the viscous range the energy is dissipated due to viscosity of the fluid. The velocity v_λ depends linearly on the distance λ. We have seen that in the inertial range the velocity

Figure 3: The velocity spectrum measured by Gagne (reference 27 in [2]). Here η is the Kolmogorov scale (λ_0). In the inertial range the measured spectrum is very close to the Kolmogorov's $K^{-5/3}$ law. In the energy range the spectrum is a little steeper.

increases with the cube root of the distance. The crossover between this two ranges is expected to be around the Kolmogorov scale, λ_0.

In the inertial range, according to the theory of Kolmogorov, (16) should hold. We can rewrite (16) as

$$\langle |v_r|^m \rangle \propto r^{\frac{m}{3}} \,, \tag{19}$$

the m-th order structure function is a $\frac{m}{3}$ power of the distance r.

2.3 The Inertial Range Spectrum

We defined the wavenumber spectrum $S(\vec{K})$ in (12). If the flow is isotropic, the spectrum depends only on the magnitude of the wavenumber K. In this case we can write

$$\overline{u^2} = \int_0^\infty S(K) dK \,.$$

We can associate the wavenumber K with an eddy of size K^{-1}. Small eddies are represented by large wavenumbers. The spectrum in the inertial range does not depend on how much energy is presented at the large scales, nor can it depend on the viscosity. It can only depend on the parameters which characterize the flow,

$$S = S(K, \epsilon) \ , \qquad\qquad l^{-1} \ll K \ll \lambda_0^{-1} \, .$$

The unit of S is $\frac{m^3}{s^2}$, the unit of ϵ is $\frac{m^2}{s^3}$ and the unit of K is m^{-1}. Counting the units leads to

$$S = A\epsilon^{\frac{2}{3}} K^{-\frac{5}{3}} \, . \tag{20}$$

The dimensionless constant A has been found universal for all turbulent flows ($A \simeq 1.7$ to 2.4, the exact value is still discussed in the literature). The relation (20) is usually called **Kolmogorov's $K^{-5/3}$ law**. It has been verified by the experiments and by further theoretical modeling.

3 Beyond the Classical Theory

3.1 Intermittency

Let us have a look at data from an experiment. The turbulence was created in a flow which passed trnsversely installed grid with characteristic scale l. The results are ploted on figure (4). There are variations of nine different quantities shown in the figure. On the top there is the velocity component u_i as a function of time. If we plotted the probability distribution of the velocity using this time signal, it would be a Gaussian distribution with the root mean sqare $\sqrt{\overline{u_i^2}}$. The next time signal is of the vorticity component $\omega_i = ((\mathrm{curl}(\vec{u}))_i$. It is not as Gaussian as the first one, but shows sharper variations. There are more quantities plotted further which correspond to higher powers of the velocity derivatives. We see sharper variations the higher the order. We notice the sharp and high peaks separated by smooth periods with relatively low amplitudes. Also the time signal for ϵ, the energy dissipation is clearly non-Gaussian and shows intense variations from the mean. This type of behaviour of the dynamical quantities is called **intermittent** and the phenomenom **intermittency**. Intermittency is common for nonlinear dynamical systems and is a typical quality of the dynamical chaos. Quantitatively intermittence is particularly shown in the fact that the moments of the respective quantities are much larger than their corresponding Gaussian values. Even though the turbulence is characterized by a large number of degrees of freedom, the presence of the intermittency tells us that there is similarity between behaviour of the turbulent flows and the dynamical chaos in simple nonlinear systems.

The velocity field in a turbulent flow is dominated by large scale fluctuations, while the velocity derivatives are dominated mostly by the small scale variations (small eddies). From experimental data (figure 4) thus follows that large scale turbulent eddies are weakly correlated while small eddies are strongly coupled and in consequence non-Gaussian. On figure

Figure 4: Experimental time signal for different quantities in grid generated turbulence (from Dracus, [3]).

(8) there is a time signal of the velocity component $u_i^{(l)}$ on different scales l obtained from numerical simulation. The details will be explained later. For now let us only notice that intermittency becomes stronger on the smaller scales. According to Grossman [2] intermittency results from the competition between the actions of the nonlinearity and the viscous damping. Viscosity dampes the amplitudes and pulls out energy, which is restored by energy transport downscale. The viscous time scale is $(\nu p^2)^{-1}$ and the interaction time scale $(pu(p))^{-1}$. The intermittency becomes important when the interaction time scale becomes larger than the viscous time scale. It happens for large p or at small eddies. This intermittency is therefore called **small scale intermittency**.

The classical theory of turbulence does not give intermittency and is thus incomplete. It still correctly describes the large-scale fluctuations and so the low-order statistics and is still widely used and of great value for engineering problems, but since intermittency is present in every turbulent flow the real and complete model of turbulence should, by all means, correctly predict it.

We have not tried to describe anisotropic turbulence here, so let us just mention why is intermittency important in engineering. The turbulent motion near the walls (boundaries) is always highly anisotropic. The eddies are streched in the direction parallel to the walls and compressed in the perpendicular direction. Such structures can be interpreted as large-scale long lived fluctuations from the mean flow [3]. The intermittent peaks in time signals are associated with the instability of theese structures which suddenly decompose into smaller eddies in violent bursts. During these bursts the slowly moving fluid near the walls is ejected into the regions further away from the walls which are moving faster. As a consequence, the faster moving parts of fluid are slowed down which results in the turbulent drag. The intermittency thus enlarges the drag force and the energy losses. Since the energy losses are usually very large, the reduction for some percents would already be of great practical importance. A lot of effort is (for obvious reasons) put into controling the turbulence. By keeping the intermittency as low as possible one can save a lot. There are different suggestions how one could control the intermittency. One of them is installing the so-called micromechanical devices or hydrodynamic chips (a silicon chip capable of mechanical movement when triggered electrically). By use of these one can cause the fluctuations at the desired place and time. Of course, one first has to understand much more about turbulence and intermittency to efficently control such a complicated system.

3.2 Stability Analyses

Figure (5) shows the second-order structure function. The crossover from viscous to inertial range is clearly noticable. According to the theory of Kolmogorov we would expect the crossover between the two ranges to happen at the Kolmogorov scale $r \approx \lambda_0$. From the figure we see that the crossover happens at about $10\lambda_0$,

$$r_c \approx 10\lambda_0 \,.$$

Figure 5: The second order structure function versus distance. There are different experimental values together with the theoretical prediction of Kolmogorov (full line). The crossover from the viscous to the inertial range is clearly seen and it is at about $10\lambda_0$.

We would like to explain this by analysing the stability of the solutions of the Navier-Stokes equations. We consider small disturbances $\vec{v}$ of a velocity field $\vec{u_1}$ which is a solution of the Navier-Stokes equations (1),

$$\vec{u} = \vec{u_1} + \vec{v}\,.$$

We get the linear deviation equation when we put the new velocity in the Navier-Stokes equations and keep only the first-order terms. We are only interested in the integral $\overline{u^2}(t) = \int dV u^2(\vec{r}, t)$ which is a measure of the fluctuations, $\overline{u^2} - \overline{u_1^2}$ is some kind of standard deviation from the mean distribution $\overline{u_1^2}$. We thus are interested only in the Hermitean part of the linear deviation equation, which is derived in the Appendix A. The result is

$$\frac{\partial v_i}{\partial t} = \nu \Delta v_i - \tfrac{1}{2} v_j \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) = \nu \Delta v_i - v_j e_{ij} \tag{21}$$

The first term on the right without the second one leads to viscous damping, all eigenvalues are negative (the Laplacean operator is negative definitive). The second term on the right is the local shear. It can increase disturbances and shift the spectrum to positive λ's. The mean effect of destabilization by the second term is $\sqrt{\frac{\epsilon}{2\nu}}$, as is evident from (4). One expects the onset of turbulence when the mean shear is strong enough to shift the least damped eigenvalue to the positive λ-range. The least damped eigenvalue of the operator $\nu\Delta$ is $\nu\left(\frac{2\pi}{l}\right)^2$, where l is the linear extension of the flow volume. So, we expect the onset of turbulence if

$$\sqrt{\frac{\epsilon}{2\nu}} \geq \nu\left(\frac{2\pi}{l}\right)^2 . \tag{22}$$

Let us express the above condition with the viscous scale λ_0. The order of magnitude of the velocity v_{λ_0} is given by (16), $v_{\lambda_0} = (\epsilon\lambda_0)^{\frac{1}{3}}$. Inserting (16) into the condition (22) and having in mind the definition of λ_0 ($Re_{\lambda_0} = 1$) we get

$$\lambda_0 \leq 2^{-4}\frac{l}{2\pi} \approx \frac{l}{8} \tag{23}$$

as the condition for instability. We can confine on a subvolume of the flow and see that we can expect the onset of instability at scales $r \geq 8\lambda_0$. The crossover from the stable viscous range to the turbulent inertial range is thus expected to happen at the scale of about $8\lambda_0$. This result in a way confirms the Kolmogorov's prediction (λ_0) based on scaling arguments and also gives a numerical prefactor which is roughly in agreement with the numerical and experimental data.

3.3 Direct Numerical Simulations

We want to analyse the Navier-Stokes equation in more details to get a deeper insight into the dynamics of the turbulent flow. The first question we should ask ourselves is how many degrees of freedom are neccessary to correctly describe a turbulent flow. In principle a fluid flow has continuously many degrees of freedom. What can we do? The first major reduction of the degrees of freedom is a consequence of the nature of the flow. In the phase space of the turbulent flow there is a strange attractor. The trajectories are bound to some lower dimensional hypersurface in the phase space and do not fill the entire phase space volume in time. An estimate of the dimension of the attractor can be obtained from the linear deviation equation (21). Each eigenvalue corresponds to a Lyapunov exponent. A positive Ljapunov exponent leadds to exponential divergence of the nearby trajectories, the negative Ljapunov exponent means exponential convergence. The phase space so shrinks in the direction of the negative Ljapunov exponents. The remaining directions form a subspace which eventually gets filled by trajectories. We order the Lyapunov exponents in the decreasing order. The first ones are positive and the last ones negative. The sum of the first s Lyapunov exponents is still positive for $s = m$ but already negative for $s = m+1$. The dimension of the attractor [1], [2] is approximately the number m plus a noninteger part m_1 determined as follows:

$$\sum_{s=1}^{s=m} \lambda_s + m_1\lambda_{m+1} = 0 .$$

$$d_{Lyap} = m + m_1 .$$

The result for the Navier-Stokes equation (using the linear deviation equation (21)) is

$$\frac{d_{Lyap}}{V} = 0.173 {\lambda_0}^{-3} . \tag{24}$$

The calculation is described in [2] and in the references there. The result for the ONERA windtunel is $d \approx 10^{12}$ and $\frac{d}{V} \approx 5 \cdot 10^5 m^{-3}$ for the atmosphere. This is a big reduction of the degrees of freedom, but is it possible to simply solve the Navier-Stokes equations numerically now? Let us estimate further. We can use the equation (18) and rewrite (24):

$$d_{Lyap} \propto Re^{9/4} . \tag{25}$$

The attractor dimension increases rapidly with the Reynolds number. The number of the operations in the computerwork is proportional to the number of the degrees of freedom and to the time of the calculation. A calculation for one eddie turnover time and for the Reynolds number $Re = 5 \cdot 10^6$ tells us that a machine with 400 Megaflops (operations per second) would take 200000 years CPU time for such a calculation! The Los Alamos Connection Machine has 128 Gigaflops, this reduces the time to about 10000 years CPU. If we want to reduce the CPU time to, say, 1 day, we would need a machine with $32 \cdot 10^{15}$ Flops, 32 Million Gigaflops! This finally convinces us that the dream is still utopic and we should approach the problem by other means. In the next section we will describe an approximate method which considers a tractable number of the degrees of freedom and resolves all the neccessary scales.

3.4 Multiscale Analysis of the Navier-Stokes Equations

The idea of the multiscale analysis is to reduce the number of degrees of freedom in such a way that still all relevant scales sre present. The approximation is called the **reduced wave vector set approximation**. It was developed by S. Grossman and his group [2].

We decompose the velocity into its Fourier components:

$$\vec{u}(\vec{r}, t) = \sum_{\vec{p}} \vec{u}(\vec{p}, t) e^{i\vec{p}\cdot\vec{r}} .$$

The scale is now measured in p^{-1}. We next write down the Navier-Stokes equations in the Fourier space (more detailes can be found in [2] and in the references there) and restrict the number of allowed wave vectors $\vec{p}$ in such a way that all scales $\frac{1}{l} < p < \frac{1}{\lambda_0}$ will be present. We first define the wave vector set

$$K^{(0)} = \{\vec{p}_n^{(0)} ; n = 1, 2, ..., n_{max}\} ,$$

which properly represents the largest scales ($p_n^{-1} \approx l$). The choice of the wave vectors in this set is rather arbitary. To take care of the smaller scales we construct wave vector sets with larger p as geometrically scaled $K^{(0)}$:

Figure 6: A set of allowed wave vectors (2D projection). There are five subsets $K^{(l)}$ in this figure.

Figure 7: The time signal of a velocity component $u_1(t)$ obtained by the multiscale analyses of the Navier-Stokes equations. The time scale is blown up twice by a factor of 20.

$$K^{(l)} = \{\vec{p}_n^{(l)} = 2^l \vec{p}_n^{(0)}\,; n = 1, 2, ..., n_{max}\}\,.$$

For sufficently large l_{max} the scale $p_{l_{max}}^{-1}$ is sufficently smaller than the Kolmogorov scale λ_0. All the wave vectors

$$\vec{p} \in K = \cup_{l=0}^{l_{max}} K^{(l)}$$

are now admitted in the Fourier-transformed Navier-Stokes equations. In this way all the neccessary scales are resolved and there is still only a tractable number of degrees of freedom. An example of a wave vector set is presented in figure (6).

In the calculations published in [2] they took for $Re = 2 \cdot 10^6$ $n_{max} = 86$ and $l_{max} = 12$. The number of degrees of freedom was $d = 2n_{max}(l_{max} + 1) \approx 2200$. The results are presented in figures (7) and (8).
In the figure (8) there are time signals of the velocity components at different scales. We can clearly see that the signal becomes intermittent on smaller scales.
The intermittency affects the probability distribution function. We recall that the Kolmogorov classical theory of turbulence predicts for the m-th order structure function the power law behaviour, $\langle |v_r|^m \rangle \propto r^{\frac{m}{3}}$. This changes because of intermittency. The power law remains, but the exponent is different,

$$\langle |v_r|^m \rangle \propto r^{\xi_m}\,. \tag{26}$$

The exponent is

$$\xi_m = \frac{m}{3} + \delta\xi_m\,,$$

Figure 8: Scale resolved velocity component $u_1^{(l)}$ versus time. The ordinate shows the sum of the amplitudes $u_1^{(l)}$ of a certain scale $\vec{p} \in K^{(l)}$ divided by the root mean square of $u_1^{(l)}$.

where $\delta\xi_m$ is the **intermittency correction**. The results for the intermittency correction obtained by multiscale analyses [2] are plotted in figure (9). The results are suprising. The intermittency correction is almost zero in the inertial range, which means that there is no intermittency there.
There is intermittency in the viscous range (large p) and there is some intermittency in the energy range. The result was carefully checked many times. Some possible explanations for the missing or unexpectably small intermittency in the inertial range are:

1^{st} possibility: **there is indeed no intermittency in the inertial range. The experimentally observed intermittency is probably small scale intermittency (viscous range), because the Reynolds numbers in the experiments are usually too small to allow for a well extended inertial range.**

2^{nd} possibility: **the number of allowed modes is too small. If the number of modes is increased drastically (towards d_{Lyap}), the intermittency in the inertial range could increase very slowly. It would mean that we have no hope to correctly describe intermittency by a tractable number of degrees of freedom.**

3^{rd} possibility: **the Navier-Stokes equation is free of inertial range intermittency and the experimentally observed intermittency is due to plumes and swirls from**

Figure 9: The intermittency correction $\delta\xi_m(p)$ versus the scale On. phe energy range is on the left side of the figure and the viscous range is on the right side. We see that there is some intermittency in the energy range and more in the viscous range, while in the inertial range the corrections are almost zero.

the physical boundary layers of real turbulence. We do not simulate real turbulence with this approach, because there is no boundary layer considered.

Further work is neccessary to find out, which of these or possibly others possible conclusions is the right one.

4 Conclusions

The topic of the seminar was in some respect unfortunate. We could not present a nice and complete theory with firm theoretical results and experimental confirmations. The complete theory of turbulence is still missing. We tried to give a phenomenological review and to show what is going on in the field of research. The problem has been and still is discussed a lot in the literature in the last more than hundred years. It is one of the few completely classical problems which we do not understand. Besides the theoretical interest there is a strong need to understand the nature of turbulent flows in the engineering. Once understanding it one could effectively reduce the energy losses. Despite the lack of a complete theory we see some improvement in the recent years. The question which often stormed the brains of scientists years ago, how can a flow be unpredictable if it is described by deterministic laws, was answered by the outburst of the chaos theory in the seventies and eighties. The intermittency is not a surprising phenomenom once we are familiar with the behaviour of the deterministic nonlinear systems. There are some indications that the small scale turbulence and the intermittency could be of crucial importance in developing the theory of turbulence if one is to emerge. Anyway, we are still far from the final success and many new bright ideas and hard work will be neccessary to acomplish the mission.

5 Appendix A: The Deviation Equation

We start from the Navier-Stokes equations and disturb the velocitz $\vec{u}(\vec{r},t)$ a little:

$$\vec{u}(\vec{r},t) \to \vec{u}(\vec{r},t) + \delta\vec{u}(\vec{r},t)\,.$$

The so-obtained deviation equation is, considering onlz the linear terms,

$$\frac{\partial \delta u_i}{\partial t} = -\left[u_j \frac{\partial \delta u_i}{\partial x_j} + \delta u_j \frac{\partial u_i}{\partial x_j}\right] + \nu \Delta \delta u_i\,. \tag{27}$$

We want to find an operator $\widehat{L}$ which satisfies the relation

$$\frac{\partial \delta\vec{u}}{\partial t} = \widehat{L}\delta\vec{u}\ .$$

The eigenvalues of the operator $\widehat{L}4$ are the spectrum of disturbances $\delta\vec{u}$. In fact, we are only interested in how the magnitude of disturbances changes with time and therefore have to study the distribution

$$\langle \delta\vec{u}|\frac{\partial \delta\vec{u}}{\partial t}\rangle = \tfrac{1}{2}\frac{\partial}{\partial t}\langle \delta\vec{u}|\delta\vec{u}\rangle = \tfrac{1}{2}\frac{\partial}{\partial t}|\delta\vec{u}|^2\,,$$

In order to analyze how the magnitude changes with time, it is enough to consider the Hermitean part of the operator $\widehat{L}$, $\widehat{A} = \frac{1}{2}(\widehat{L} + \widehat{L}^\dagger)$. It is true that

$$\langle \delta\vec{u}|\widehat{L}|\delta\vec{u}\rangle = \langle \delta\vec{u}|\widehat{A}|\delta\vec{u}\rangle\,.$$

We first multiply the linear deviation equation (27) by δu_i (and sum over i) and integrate over the flow region. We obtain

$$\int dV\,\delta u_i \frac{\partial \delta u_i}{\partial t} = -\int dV\,\delta u_i\left[u_j \frac{\partial \delta u_i}{\partial x_j} + \delta u_j \frac{\partial u_i}{\partial x_j} - \nu\Delta\delta u_i\right]\,.$$

The first term on the right is zero, which can quickly be checked by per partes integration. We decompose the second term into the Hermitean and anti-Hermitean part,

$$\frac{\partial \delta u_i}{\partial x_j} = \tfrac{1}{2}\left[\tfrac{\partial \delta u_i}{\partial x_j} + \tfrac{\partial \delta u_j}{\partial x_i}\right] + \tfrac{1}{2}\left(\tfrac{\partial \delta u_i}{\partial x_j} - \tfrac{\partial \delta u_j}{\partial x_i}\right)\,.$$

The antisymmetric part is zero after the integration and we are thus left with

$$\int dV \delta u_i \frac{\partial \delta u_i}{\partial t} = \int dV \delta u_i \left[\nu \Delta \delta u_i - \tfrac{1}{2} \delta u_j \left(\tfrac{\partial \delta u_i}{\partial x_j} + \tfrac{\partial \delta u_j}{\partial x_i} \right) \right] .$$

The Hermitean operator $\widehat{A}$ is

$$A_{ij} = \nu \delta_{ij} \Delta - \tfrac{1}{2} \left(\tfrac{\partial \delta u_i}{\partial x_j} + \tfrac{\partial \delta u_j}{\partial x_i} \right)$$

and the linear deviation equation (27) transforms into

$$\frac{\partial \delta u_i}{\partial t} = \nu \Delta u_i - \tfrac{1}{2} \delta u_j \left(\tfrac{\partial \delta u_i}{\partial x_j} + \tfrac{\partial \delta u_j}{\partial x_i} \right) .$$

In order to analyze stability of the solutions of the Navier-Stokes equations we have to study the eigenvalues of this equation.

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