Models of the Small World

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Abstract

It is believed that almost any pair of people in the world can be connected to one another by a short chain of intermediate acquaintances, of typical length about six. This phenomenon, colloquially referred to as the "six degrees of separation", has been the subject of considerable interest within the physics community in recent years.

1 Introduction

The world of human society has become quite large in recent times, but people routinely claim that it's still a small world we live in - and in a certain sense they could be right. Despite the enormous number of people on the planet, the topology and structure of social networks is such that we are all closely connected to one another.

1.1 Milgram's experiment

One of the first quantitative studies of the structure of social networks was performed in the late 1960's by Stanley Milgram of Harvard University [1]. He took a number of letters addressed to a stockbroker in Boston and distributed them to a random selection of people in Nebraska and Kansas. His instructions were that the letters were to be sent to the stockbroker by passing them from person to person and, in addition, could be sent only to someone whom the current holder knew on a first-name basis. Since it was not likely that the initial recipients of the letters were on a first-name basis with a Boston stockbroker, their best strategy was to pass their letter to someone they presumed was more likely to know the person to whom the letter was ultimately addressed.
By requiring each intermediary to report their receipt of the letter, Milgram kept track of the letters and the demographic characteristics of their handlers.

A reasonable number of letters eventually reached their destination, with a median chain length of about six. Milgram concluded that six was therefore the average number of acquaintances separating any two people in the entire world. This situation has been labeled "six degrees of separation", a phrase which has passed into popular folklore.

There were certainly enough possible sources of error in Milgram's experiment, from his sample selection to the fact that it was confined to the United States, to suspect that the figure six is probably not a very accurate one. However, the general result that two randomly chosen human beings can be connected by only a short chain of intermediate acquaintances has been subsequently verified and is now widely accepted. This result is referred to as the small-world effect.

1.2 Importance of social networks

But why should a serious scientist care about the structure of social networks? Because such networks are crucially important for communications. Most human communication takes place directly between individuals. The spread of news, rumours, fashions all take place by contact between individuals. Even more importantly, the spread of disease also occurs by person-to-person contact and the structure of networks of such contacts has a huge impact on the nature of epidemics. In a highly connected network, diseases like HIV virus or this year's superflu, SARS, can spread faster than in a network where the paths between individuals are relatively long.

2 Networks and graphs

Networks are omnipresent. The brain is a network of neurons, organizations are networks of people, the global economy is a network of national economies and markets.

Any kind of network can be represented by a graph, composed of nodes or vertices and a set of lines joining the nodes. The nodes can represent members of a population and the edges stand for their interpersonal ties. Traditionally, networks are modeled as either completely ordered or completely random.

2.1 Random graphs

The simplest explanation for the small-world effect uses the idea of a random graph [1]. Let N be the number of people in the world, who on average have z acquaintances. This means that there are $\frac{1}{2}Nz$ connections between people in the entire population. The number z is called the *coordination number* of the network.

The probability p that an edge is present between two vertices in such a network is $p = z/(N\text{-}1)$, which is usually approximated by z/N for large N. The number of edges connected to any particular vertex is called the degree k of that vertex, and has a probability distribution P_k given by

$$P_k = \binom{N}{k} p^k (1-p)^{N-k} \approx \frac{z^k e^{-z}}{k!} \ ,$$

where the second equality becomes exact in the limit of large *N*. We recognize this distribution as the Poisson distribution – the ordinary random graph has a Poisson distribution of vertex degrees.

A very simple model of a network represented by a random graph can be made by taking *N* dots and drawing ½*Nz* lines between randomly chosen pairs to represent connections (fig. 1).

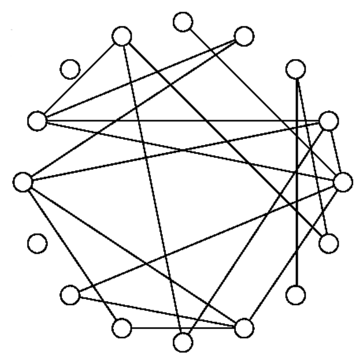

Figure 1. A schematic representation of a random graph, the circles representing vertices and lines edges.

It is easy to see that a random graph shows the small-world effect. If a person A, represented by a node, has z neighbours on such a graph, and each of A's neighbours also has z neighbours, then A has about z^2 second neighbours, z^3 third neighbours and so on. Most people have between a hundred and a thousand acquaintances, so z^3 is already between about 10^8 and 10^{12}, which is comparable to the population of the world. The diameter of the graph, D, is given by $z^D = N$, which implies that

$$D = \frac{\log N}{\log z} \ .$$

This logarithmic increase in the diameter of the graph and distance between the nodes is typical of the small-world effect. Since log *N* increases only slowly with *N*, it allows the distances to be quite small even in very large systems.

However, random graphs are not a good model of social networks. People's circles of acquaintances tend to overlap to a great extent. Since people meet most new friends through existing friends, the networks are locally ordered. The outcome of local ordering in such a network is that one individual's friends are more likely than not to know one another – a characteristic that is called *clustering*. Clustering means that if A is connected to B and B is connected to C, then A is more likely to be connected to C than to some other random node. Therefore in a real social network it is not true to say that A has z^2

second neighbours, since many of those second neighbours are already first neighbours of A.

We define a clustering coefficient C as the average fraction of pairs of neighbours of a node which are also neighbours of each other [1]. In a fully connected network, in which everyone knows everyone else, $C = 1$.
A random graph does not show clustering. In a random graph the probability that two of A's friends will be friends of one another is no greater than the probability that two randomly chosen people will be - C is equal to p, $C = z/N$, which is very small for a large network.

In real-world networks it has been found that, while C is significantly less than 1, it is much greater than the random graph value z/N.

2.2 Ordered lattice

In order to model the real-world networks, graphs must have both clustering and small-world properties. Random graphs show the small-world effect – average vertex-to-vertex distances increase only logarithmically with N – but they do not show clustering.

The opposite of a random graph is a completely ordered lattice, the simplest example of which is a one-dimensional lattice (fig. 2a). If we connect each vertex to z vertices closest to it, it is easy to see that most of immediate neighbours of any site are also neighbours of one another – it shows clustering properties. Just for convenience, we apply periodic boundary conditions to the lattice (fig. 2b).

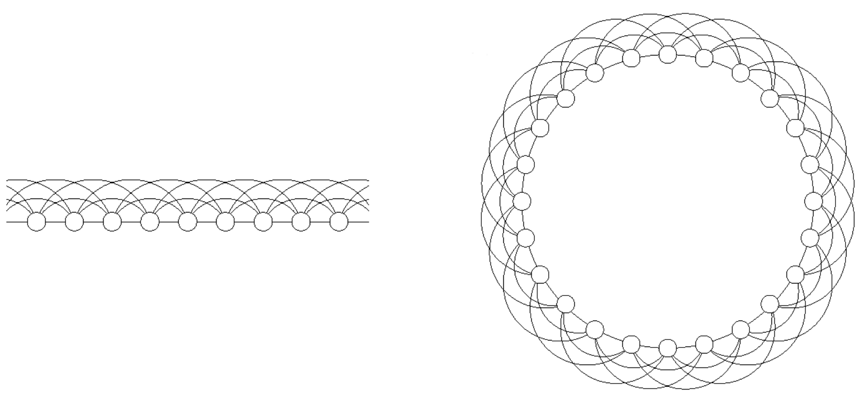

Figure 2. (a) A one-dimensional lattice with each site connected to its z nearest neighbors, where in this case z = 6. (b) the same lattice with periodic boundary conditions.

For such a lattice the clustering coefficient C can be calculated exactly [1],

$$C = \frac{3(z-2)}{4(z-1)} \ ,$$

which tends to ¾ in the limit of large *z*. We can also build networks out of higher-dimensional lattices, such as square or cubic lattices, and these also show the clustering property. The value of clustering coefficient in general dimension *d* is

$$C = \frac{3(z-2d)}{4(z-d)} \ .$$

Regular lattices however do not show the small-world effect of vertex-to-vertex distances which increase only slowly with system size. It is easy to show that for a regular lattice in *d* dimensions with the shape of a square or a hypercube of side *L* with $N = L^d$ vertices, the average vertex-to-vertex distance increases as *L*, or equivalently as $N^{1/d}$. In one dimension, it means that the average distances increases linearly with system size, this is not the typical small-world behaviour.

So, if random graphs and ordered lattices do not match well the properties of real-world networks, is there an alternative model that does?

2.3 The small-world model of Watts and Strogatz

Watts and Strogatz have proposed a model for small-world networks, which fits well with our intuitions about the nature of social networks [2]. Their model is essentially a regular lattice with some degree of randomness in it to produce the small-world effect. They suggested a specific procedure to achieve this. Each edge of a regular lattice from fig. 2b is randomly rewired with some probability *p*, meaning that one of its ends is moved to a new randomly chosen position in the lattice. For small *p* this produces a mostly regular graph but a few connections stretch long distances across the lattice (fig. 3). The coordination number of the lattice is still z on average as it was before, although any particular vertex can have more or less than *z* neighbours.

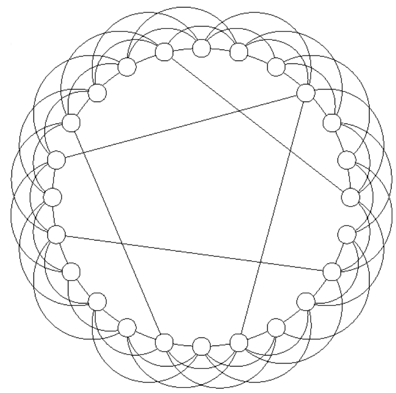

Figure 3. The Watts-Strogatz model is created by rewiring a small fraction of the links to new sites chosen at random.

This model can be justified by saying that, while most people are friends with their immediate neighbours, some are also friends with a few people who are a long way away, in a social or geographical way. These acquaintances are presented by the long-range links in the model.

Structural properties of the model are quantified by the characteristic path length $L(p)$ and clustering coefficient $C(p)$ as functions of the rewiring probability p. Characteristic path length L is defined as the number of edges in the shortest path between two vertices, averaged over all pairs of vertices.
We know that the regular lattice at $p = 0$ is a highly clustered, large world where L grows linearly with N, whereas the random graph at $p = 1$ is a poorly clustered, small world where L grows only logarithmically with N. These limiting cases might lead one to suspect that large C is always associated with large L, and small C with small L.

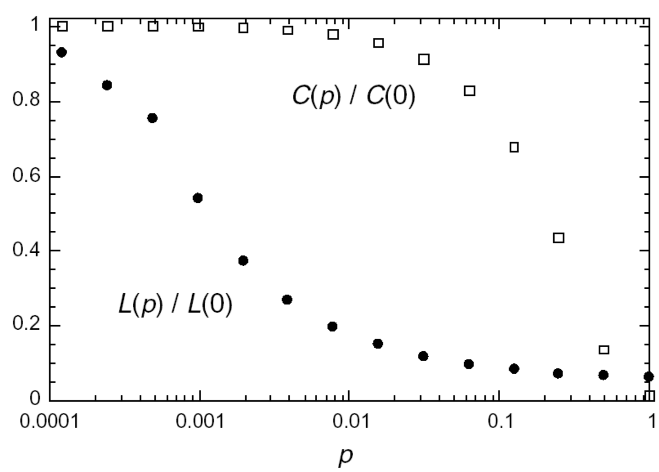

Figure 4. Characteristic path length $L(p)$ and clustering coefficient $C(p)$ for the family of randomly rewired graphs. The data shown in the figure are averages over 20 random realizations of the rewiring process and have been normalized by the values $L(0)$, $C(0)$ for a

regular lattice. All the graphs have $N = 1000$ vertices and an average of $z = 10$ edges per vertex. A logarithmic horizontal scale has been used to resolve the rapid drop in $L(p)$, corresponding to the onset of small-world phenomenon. During this drop, $C(p)$ remains almost constant at its value for the regular lattice, indicating that the transition to a small world is almost undetectable at the local level.

On the contrary, we find that there is a broad interval of p over which $L(p)$ is almost as small as L_{random} yet $C(p) >> C_{random}$ (fig. 4). Introduction of a few long-range edges results in an immediate drop in $L(p)$. Such short-cuts connect vertices that would otherwise be much farther apart than L_{random}. For small p, each short-cut has a highly nonlinear effect on L, contracting the distance not just between the pair of vertices that it connects, but between their immediate neighbourhoods, neighbourhoods of neighbourhoods and so on. By contrast, an edge removed from a clustered neighbourhood to make a short-cut has, at most, a linear effect on C. Hence $C(p)$ remains practically unchanged for small p even though $L(p)$ drops rapidly. The important implication is that at the local level – as reflected by $C(p)$ – the transition to a small world is almost undetectable.

Characteristic path length L is comparable with that for a true random graph, even for small values of p. For example, for a random graph $N = 1000$ and $z = 10$, the average distance is about $L = 3.2$ between two vertices chosen at random. For the rewiring model, L was only slightly greater, 3.6, at $p = ¼$, compared with $L = 50$ for a perfectly ordered lattice with no rewired links. And even for $p = 1/64 = 0.0156$, $L = 7.4$, about twice the value for the random graph. Thus the model appears to show both the clustering and the small-world properties simultaneously.

3 Empirical examples of small-world networks

To test the above ideas, L and C have been computed for three different real-world networks, for which all the data necessary is available [2]. The first system is a collaboration graph of feature film actors (generated from data available at http://www.imdb.com), the second one is the electrical power grid of the western United States and the third is the neural network of the nematode worm *Caenorhabditis elegans*.

These examples were not hand-picked; they were chosen because of their inherent interest and because complete wiring diagrams were available. The graph of film actors is a surrogate for a social network, with the advantage of being much more easily specified. The graph of the power grid is relevant to the efficiency and robustness of power networks. And *C. elegans* is the sole example of a completely mapped neural network.

The graphs are defined as follows: Two actors are joined by an edge if they have acted in a film together. For the power grid, vertices represent generators, transformers and substations, and edges represent high-voltage transmission lines between them. For *C. elegans*, an edge joins two neurons if they are connected by either a synapse or a gap function.

Table 1 shows characteristic path lengths and clustering coefficients for the three networks, compared to random graphs with the same number of vertices N and average number of edges per vertex z. All three networks show the small-world phenomenon: $L \leq L_{random}$ but $C >> C_{random}$.

	L_{actual}	L_{random}	C_{actual}	C_{random}
Film actors	3.65	2.99	0.79	0.00027
Power grid	18.7	12.4	0.080	0.005
C. elegans	2.65	2.25	0.28	0.05

Table 1. Characteristic path length L and clustering coefficient C for three real networks.

4 Diameter of the World-Wide Web

World-Wide Web is another system that has been found to display small-world properties. Although obtaining a complete topological map of WWW is an impossible task, its large-scale properties can be characterized by local connectivity measurements [3].

To determine the local connectivity of the web, the robot was constructed that added to its database all URLs found on a document and recursively followed these to retrieve the related documents and URLs.

The data collected was used to determine the probability P_k that a document has k links. P_k follows a power law over several orders of magnitude, remarkably different from the Poisson distribution predicted by the classical theory of random graphs (Fig. 7). The power-law tail indicates that the probability of finding documents with a large number of links is significant as the network connectivity is dominated by highly connected web pages.

The shortest path between two documents was found to follow the form

$$L = 0.35 + 2.06 \log N$$

At the time of the experiment (1999), the total number of documents N on WWW was estimated at around 8×10^8, which yields $L = 18.59$, meaning that any two documents on the Web are just 19 clicks away. But the logarithmic dependence on N means that even a 1000% increase in the size of the Web changes L very little, from 19 to only 21.

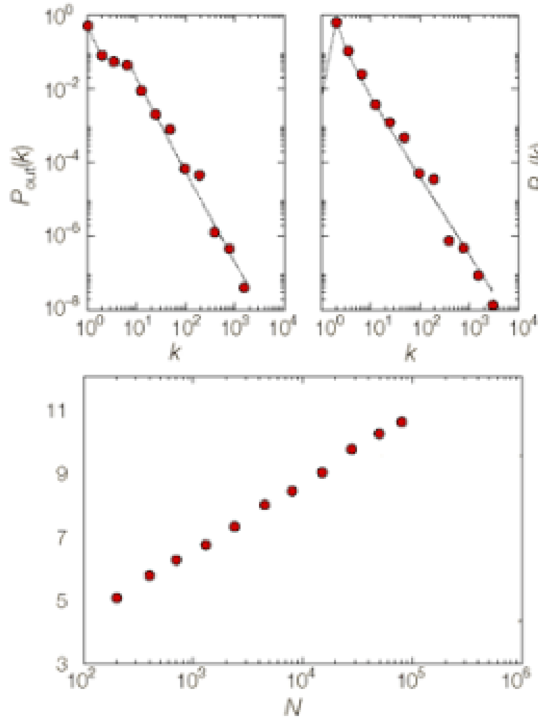

Figure 7. Distribution of links on the World-Wide Web, separated between a) outgoing links (URLs found on an HTML document) and b) incoming links (URLs pointing to a certain document). Data were obtained from a complete map of the domain nd.edu, which contains 325,729 documents and 1,469,680 links. C) Average of the shortest path between documents as a function of system siz, as predicted by the model. To check the validity of the predictions, L was determined for documents in the domain nd.edu. The measured L = 11.2 agrees well with the prediction L = 11.6 obtained form the model.

5 Scaling properties of real world networks

We have seen that the connectivity distribution P_k for WWW decays as a power law, following $P_k = k^{-\gamma}$. It is a feature unpredicted by existing random graph models, where P_k has a Poisson distribution with a fast decaying tail. In fact, many real world networks have connectivity distributions obeying a power law [4] with different exponents γ (Fig. 8). Power law distributions are said to have no scale, to be *scale-free*. In scale-free distributions very large values – i.e. much larger than the mean – can be observed. In contrast, bell-shaped distributions such as Poisson or Gaussian are called *single scale* as they are characterized by a single scale (mean). Such is, for example, the distribution of human height. It is the fact that we associate a scale to the height of humans that would lead us to reject as false any report of a human even twice as tall as the mean. For power law distributions values even 10 times larger than the mean can be observed for quite small samples.

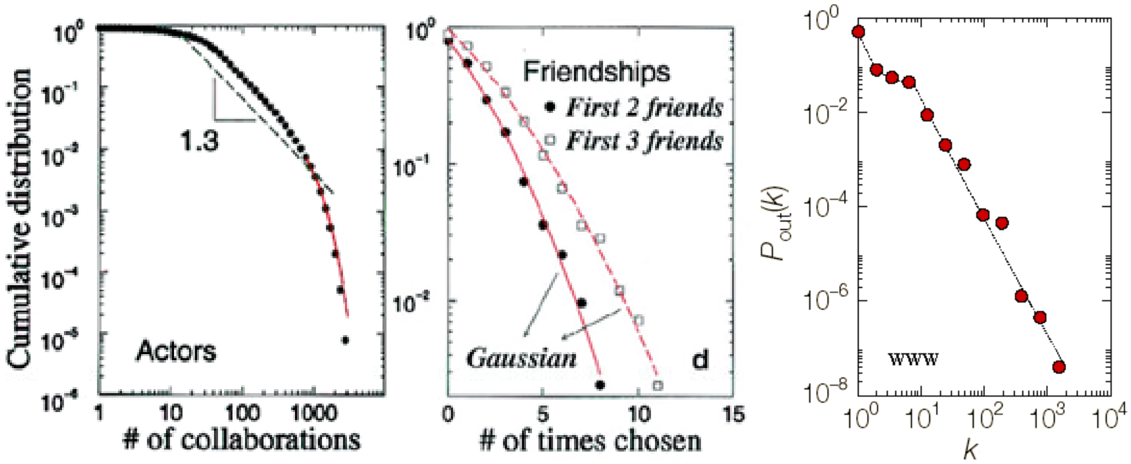

Figure 8. a) Log-log plot of the cumulative distribution of connectivities for the network of movie actors. This plot suggests that, for values of number of collaborations between 30 and 300, the data are consistent with a power law decay. The apparent exponent of this cumulative distribution is γ = 2.3. For larger number of collaborations, the power law decay is truncated. b) Linear-log plot of the cumulative distribution of connectivities for the friendship network of 417 high-school students. The number of links is the number of times a student is chosen by another student as one of his/her two best friends. The lines are Gausssian fits to the empirical distributions. c) Distribution of links on WWW. Dotted lines represent analytical fits used as input distributions in constructing the topological model of the Web. The tail of distributions follows $P_k = k^{-\gamma}$ with $\gamma = 2.45$

Scale-free networks emerge in the context of a growing network in which new vertices connect preferentially to the more highly connected vertices in the network. By incorporating growth and preferential attachment into existing models, the power law scaling is obtained.

Most real networks are open and they form by continuous addition of new vertices to the system. For example, the actor network grows by the addition of new actors, WWW grows exponentially over time by the addition of new Web pages…

In contrast to random network models, most real networks exhibit preferential connectivity. For example, a new actor is most likely to be cast in a supporting role with more established and better-known actors. Similarly, a newly created Web page will be more likely to include links to well-known popular documents with already high connectivity. These examples indicate that the probability with which a new vertex connects to existing vertices is not uniform; there is a higher probability that it will be linked to a vertex that already has a large number of connections. Preferential attachment is taken into account by assuming that the probability Π that a new vertex will be connected to vertex i depends on the connectivity k_i of that vertex, so that

$$\Pi(k_i) = \frac{k_i}{\sum_j k_j}$$

Such a network will evolve into a scale-invariant state with the probability that a vertex has *k* edges following a power law [5].

This "rich-get-richer" phenomenon is easily detected in other real networks such as business, social and transportation networks.

6 Spread of Infectious Disease

To investigate the dynamical behaviour of a small-world network, we take a deliberately simplified model for the spread of an infectious disease [2]. The population structure is modeled by the small-world graphs in Fig. 3. At time $t = 0$, a single infective individual is introduced into an otherwise healthy population. Infective individuals are removed permanently (by immunity or death) after a period of sickness that lasts one unit of dimensionless time. During this time, each infective individual can infect each of its healthy neighbours with probability *r*. On subsequent time steps, the disease spreads along the edges of the graph until it either infects the entire population, or it dies out, having infected some fraction of the population in the process.

Two results emerge. First, the critical infectiousness r_{half}, at which the disease infects half the population, decreases rapidly with *p* (Fig. 5). There is a clear correlation between critical infectiousness and the amount of randomness in the network.

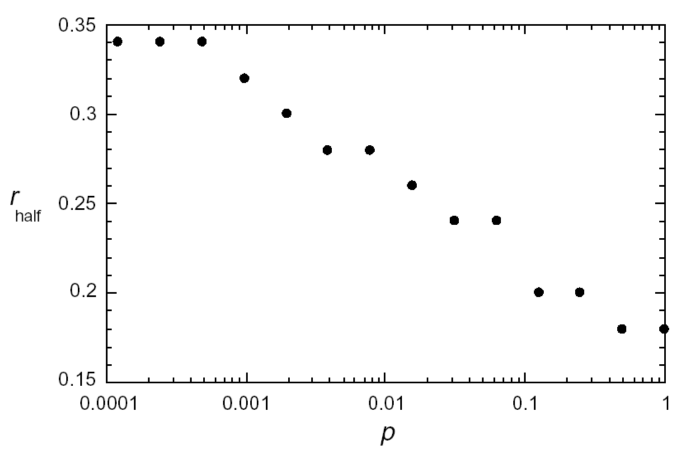

Figure 5. Simulation results for a simple model of disease spreading. The community structure is given by one realization of a randomly rewired graph. Critical infectiousness r_{half} at which the disease infects half of the population, decreases with *p*.

Second, for a disease that is sufficiently infectious to infect the entire population regardless of its connective topology, the time $T(p)$ required for global infection has essentially the same functional form as the characteristic path length $L(p)$ (Fig. 6). Even if only a few percent of the edges in the original lattice are randomly rewired, the time to global infection is nearly as short as for a random graph. Thus, infectious diseases are predicted to spread much more easily and quickly in a small world.

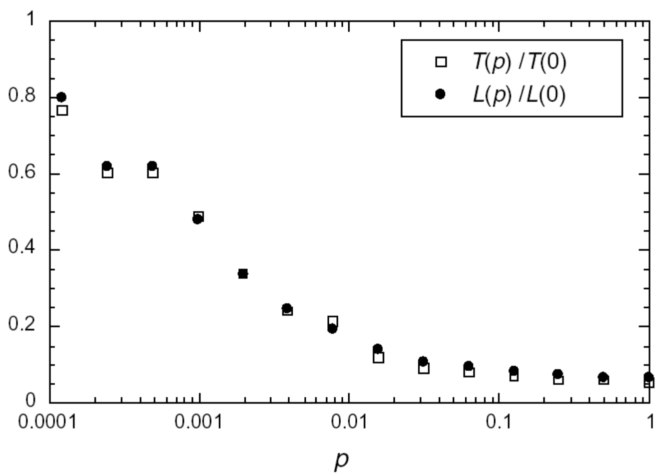

Figure 6. Simulation results for a simple model of disease spreading. The time $T(p)$ required for a maximally infectious disease ($r = 1$) to spread throughout the entire population resembles the $L(p)$ curve.

7 Other models of the small world

Although most of the work on the small world phenomena is based on the Watt-Strogatz model, a number of other models of social networks have been proposed.

One alternative to the view put forward by Watts and Strogatz is that the small-world phenomenon arises not because there are a few long-range connections in the otherwise short-range structure of a social network, but because there are a few nodes in the network which have unusually high coordination numbers or which are linked to a widely distributed set of neighbours. Perhaps the "six degrees of separation" effect is due to a few people who are particularly well connected. A simple model is depicted in Fig. 8, in which the starting point is again a one-dimensional lattice, but instead of adding extra links between pairs of sites, a number of extra vertices are added in the middle which are connected to a large number of sites on the main lattice, chosen at random. This model is similar to the Watts-Strogatz model in that the addition of extra sites effectively introduces shortcuts between randomly chosen positions on the lattice. Even in the case where only one extra site is added, the model shows the small-world effect if that site is sufficiently highly connected.

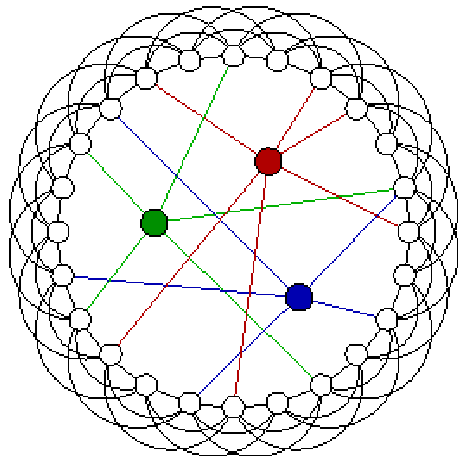

Figure 8. An alternative model of a small world, in which there are a small number of individuals who are connected to many widely-distributed acquaintances.

Another suggestion has argued that a model such as that of Watts and Strogatz, where shortcuts connect vertices arbitrarily far apart with uniform probability, is a poor representation of at least some real-world situations. In the real world, people are surprisingly good at finding short paths between pairs of individuals given only local information about the structure of the network (Milgram's letter experiment is a good example). On the other hand, no algorithm exists which is capable of finding such paths on networks of the Watts-Strogatz type, again given only local information. There must be some additional properties of real-world networks which make it possible to find short paths with ease. The corresponding model has shortcuts added between pairs of vertices i, j with probability which falls off as a power law d_{ij}^{-r} of the distance between them.

8 Conclusions

The work by Watts and Strogatz has set off a small avalanche among researchers in both the natural and social sciences to explore the implications of the small-world phenomenon. Small-world graphs – those possessing both short average person-to-person distances and “clustering” of acquaintances – show behaviours very different from either regular lattices or random graphs. Some of the more interesting observations are:

- These graphs show a transition from a large-world regime in which the average distance between two people increases linearly with system size, to a small-world one in which it increases logarithmically.
- Disease models which incorporate a measure of susceptibility to infection have a transition point at which an epidemic sets in, whose position is influenced strongly by the small-world nature of the network.
- Some real-world networks appear to have a scale-free distribution of the coordination number of vertices as a result of growth and preferential attachment. An example is the World-Wide Web.

Empirical work to determine the exact structure of real networks is underway in a number of groups, as well as theoretical work to determine the properties of the proposed models. As Watts notes in one of his early papers on the subject, the notion of small-world connectivity "may have implications in fields as diverse as public health, organizational behavior, and design." The work has just begun.

9 References

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