University *of Ljubljana*
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DEPARTMENT OF PHYSICS

Lifshitz interactions and stochastic electrodynamics

Third Year Seminar

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Abstract

Evgenii Lifshitz's solution of the various Casimir problems is presented, with a focus on Rytov's theory of thermal fluctuations of the zero-point field, upon which Lifshitz's calculation is based. We start with a historical review of the reasons that led to the formulation of the problem of zero-point field, from Planck's solution of the black body issue to van der Waals's discovery of anomalous interactions between molecules. Finally an explanation of this approach is outlined in light of its importance for the birth of the alternative theory of stochastic electrodynamics, briefly indicating its effectivenesses and difficulties.

The seminar is written without any reference to quantum field theory and thus is suitable for Bachelor students to help them understand the problem of the interplay of microscopic and macroscopic worlds in electromagnetic field theory.

Contents

1 Prologue

1.1 The discovery of the quantum nature of light

It's a shared opinion among scholars that the beginning of the quantum era is marked by the discovery of the correct blackbody spectrum. In 1900 Planck[1] was the first able to derive a formula which gives as limiting cases the previously known and experimentally verified Rayleigh-Jeans[2] and Wien[3] formulas. His derivation was made by means of classical statistical theory applied to thermodynamics and, driven by the aim to get the desired result, he was forced to use a mathematical expedient that later revealed completely new physics behind it.

In particular, given the classical oscillator model of an atom, by equalizing the absorption and emission rates obtained in this way[1], he got an equation for the frequency dependent oscillator energy density $\rho(\omega)$ (where we have the overall energy of the system of oscillators defined as $E = \int_0^{\omega_{max}} \rho(\omega) d\omega$):

$$\rho(\omega) = \frac{\omega^2}{\pi^2 c^3} U(\omega), \tag{1}$$

where U(ω) is the total energy of the oscillator. In computing the latter, he assumed that a system of oscillators had an overall energy of $NU = P\epsilon$, where P is the number of the discrete amounts of energy ϵ the total is made of. Then by computing the entropy of the P-N system (according to Boltzmann[4] this is $S = k_B \log W$ where W is the number of ways in which the P energy elements can be distributed among the N radiators)[1] he came to the result

$$U(\omega, T) = \epsilon \cdot \frac{1}{e^{\frac{\epsilon}{k_B T}} - 1} = \hbar\omega \cdot \frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1}, \tag{2}$$

where the value for ϵ was established because it describes very well the final distribution obtained putting (2) in (1) (Planck's law).

We remark here that this is the first time in history when the "quantum" concept is used. Additionally, in the computation of entropy, it was crucial to consider the discrete amounts of energy as indistinguishable, thus introducing one of the main axioms of the quantum theory. The idea of the photon was later confirmed by Einstein with his photoelectric effect explanation. Now we know that the two formulae above are correct because we can compute them from statistical theory in a simpler fashion if we assume the quantum concept: the coefficient in equation (1) is the photon 3D density of states (remember to consider that the photon has two polarizations), while the second function in the product (2) is the Bose-Einstein[5] distribution which is valid for photons.

However the $\rho(\omega)$ we have still isn't correct. If we take the limit $k_B T \gg \hbar\omega$ we get

$$U = \frac{\hbar\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} \backsimeq \frac{\hbar\omega}{1 + \frac{\hbar\omega}{k_B T} + \frac{1}{2}(\frac{\hbar\omega}{k_B T})^2 - 1} = \frac{k_B T}{1 + \frac{1}{2}\frac{\hbar\omega}{k_B T}} \backsimeq k_B T - \frac{1}{2}\hbar\omega \tag{3}$$

[1]Max Planck (1858-1947), German theoretical physicist

[2]John Rayleigh (1842-1919), English physicist; James Jeans (1877-1946), English physicist, astronomer and mathematician

[3]Wilhelm Wien (1864-1928), German physicist

[4]Ludwig Boltzmann (1844-1906), Austrian physicist and philosopher

[5]Satyendra Nath Bose (1894-1974), Indian physicist; Albert Einstein (1879-1955), German-born theoretical physicist

i.e. a first order correction to the classical limit. This means that in order to have a clean equipartition theorem result for high temperatures we need to add a $\frac{1}{2}\hbar\omega$ term. The present work wishes to discuss about the nature of this term focusing on an unusual interpretation of it.

1.2 Quantum electrodynamics

When he wrote his first blackbody theory, Planck was aware of this and tried to find another way to derive the expression that included the missing part, still within the domain of statistical thermodynamics. After some time he succeeded in this, however, the physical significance of some mathematical assumptions had already marked this "second theory" as unlikely to correspond to truth (for instance he postulated that absorption proceeded in the classical fashion, while emission was composed of the outgoing quanta[1]).

Anyway, some years later, the formal development of quantum formalism allowed to compute the Planck spectrum from this perspective and showed that the original result by Planck together with the constant correction yielded the exact result for blackbody emission energy density, which finally turns out to be

$$\rho(\omega, T) = \frac{\hbar\omega^3}{\pi^2 c^3}\left(\frac{1}{e^{\frac{\hbar\omega}{k_B T}} - 1} + \frac{1}{2}\right). \tag{4}$$

So which is the correct way to interpretate Planck's result? Actually only the first term in the sum is referring directly to matter, i.e. to the emitting body, while the second term indicates that at the same time we have another amount of energy which is contributing to the spectrum. In particular, if we put temperature to be zero, we still have a nonzero spectrum. This spectrum is due to the so-called zero-point energy (the energy of the ground/zero state of the oscillators). It's important here to remark, that the first Planck's calculation, the formally consistent one, couldn't account for this term in the limiting case of T=0; but the same calculation in the opposite limit ($T = \infty$) suggested the need to add it.

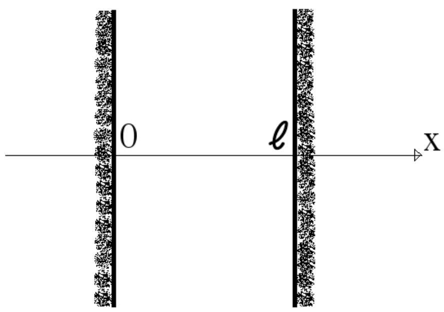
In other words, we can say that a classical attempt to explain emission and absorption of electromagnetic radiation of bodies over the whole temperature scale without taking into account collateral phenomena is impossible, and that the sole fact of using some quantum assumptions in the derivation (i.e. the quantization of energy) shows automatically where this interpretation is lacking, this is in the presence of a zero-point field responsible for the above mentioned zero-point energy (we call this field that way in analogy with the given definition of zero-point energy). This zero-point field in fact is showing to have a quantum nature, and in modern quantum electrodynamics has its own role. However, not being able at this point to explain everything in those terms, we shall see that another interpretation of the zero-point energy is possible, because it can give the same calculation results when applied to some concrete physical problems which are experimentally verifiable.

2 Casimir's problem

2.1 Van der Waals microscopic forces

In 1873 van der Waals[6] reformulated the ideal gas law in order to take into account the intermolecular forces that were influencing gas under certain conditions. During the next years these forces were found out to be consequence of permanent (Keesom[7]) or induced (Debye[8]) dipoles on molecules. However, it immediately became clear that there must be another contribution to match experimental data, and that this contribution might be found in terms of the just born quantum theory. The problem was finally resolved in 1930 by London[9], who employing fourth-order quantum mechanical perturbation theory derived an attraction potential between polarizable molecules inversely proportional to the sixth power of their distance ($\propto R^{-6}$). This theory was then further perfectioned by the work of Casimir[10] and Polder[11], who proposed that for large distances the potential of attraction should fall even faster ($\propto R^{-7}$) because of taking into account retardation effects.

Once this "elementary" problem had been solved, the question of the generalization of this theory for macrostructures was set. Casimir and Polder first calculated, still by means of quantum field theory, the interaction between an atom (molecule) and a perfectly conducting wall, simplifying the problem to the interaction of that atom with its image on the surface. Next a further generalization had to be done, trying to evaluate the force between two perfectly conducting plates. Such a step would have a great importance since it could give predictions that might be experimentally confirmed.



2.2 Casimir's macroscopic computation

The problem above was first tackled independently by Casimir in 1948. At the beginning he understood that an alternative approach to the perturbative method could be used in this case. He referred directly to the concept of zero-point energy, and calculated the difference between the amounts of energy of the considered system (due to the ground state of the electromagnetic vacuum) when the plates are close

[6]Johannes Diderik van der Waals (1837-1923), Dutch theoretical physicist
[7]Willem Hendrik Keesom (1876-1956), Dutch physicist
[8]Peter Debye (1884-1966), Dutch-born physicist and physical chemist
[9]Fritz London (1900-1954), German-born physicist
[10]Hendrik Casimir (1909-2000), Dutch physicist
[11]Dirk Polder (1919-2001), Dutch physicist

to each other at a distance of ℓ and when they are far apart, given that for a single frequency this energy is

$$E_i = \frac{1}{2}\hbar\omega_i. \tag{5}$$

In two directions of the space (y and z) all frequencies are available, while in the chosen x direction for a finite separation the modes are discrete:

$$k_x = \frac{\pi}{\ell}n. \tag{6}$$

We shall therefore integrate over the formers and sum up the latter ones:

$$\begin{aligned}\Delta E = 2\frac{1}{2}\hbar c(\frac{L}{\pi})^2[\sum_n{}' \int_0^\infty dk_y \int_0^\infty dk_z((\frac{\pi}{\ell}n)^2 + k_y^2 + k_z^2)^{\frac{1}{2}} \\ -\frac{d}{\pi}\int_0^\infty dk_x \int_0^\infty dk_y \int_0^\infty dk_z(k_x^2 + k_y^2 + k_z^2)^{\frac{1}{2}}],\end{aligned} \tag{7}$$

where L^2 is the area of the plates. Note that each state has two polarizations, except when n=0 (for this reason there's a prime on the summation symbol).

Both parts of this expression are of course an infinite quantity, however it is possible to show that their difference is finite[1].

Deriving with respect to ℓ the result divided by L^2 gives the attractive force F per unit area between the plates:

$$F = \frac{1}{L^2}\frac{d\Delta E}{d\ell} = \frac{\pi^2}{240}\frac{\hbar c}{\ell^4}. \tag{8}$$

However that's not enough: the idealization of perfect conductivity, which determined the discrete values in the above calculation (on the surface there are no charges so the tangential components of the field have to be zero) is too far from reality. In order to consider this model for experimental validation it is necessary to include an approximation that should account for the properties of matter, i.e. for its polarizability.

Hamaker[12] suggested obtaining the interaction between macroscopic bodies in a pairwise summation over molecules inside both plates, according to the Casimir-Polder model of the interaction of molecules facing themselves through a conducting mirror. Unfortunately experiment gave evidence for much a stronger interaction between the plates than what was predicted in this way[1]. This confirmed the hypothesis that in non-rarefied media van der Waals forces cannot be accounted for in a linear pairwise approximation. Consequently the calculation of the overall force between dielectric bodies had to be performed inside the zero-point field framework, this is the framework already used by Casimir in the case of conductors. Then, if such a calculation proved to be true, Van der Waals forces would look to arise directly from the zero-point field, and from its energy variations due to the presence of (polarizable) bodies.

[12]Hugo Christiaan Hamaker (1905-1993), Dutch scientist

3 Lifshitz's solution

3.1 Macroscopic approach

Perhaps because of its complexity, a QED computation of the interaction forces between real (dielectric) macroscopic bodies caused by their microscopic components had taken a long time (Kampen-Nijboer-Schram, 1968) to be performed. As alluded, however, surprisingly this was made possible much earlier by the work of Lifshitz[13] (1955)[2]. The most interesting feature of his theory is that it's performed on an apparently pure classical basis, and correctly anticipated all the later QED results on this topic.

This Lifshitz's theory is in fact "classical" first of all beacuse it approaches the problem considering the bodies as macroscopic structures characterized by the parameters of matter that are used in Maxwellian[14] formulation of electrodynamics, this is notably the complex dielectric constant $\varepsilon = \varepsilon(\omega)$. Provided that at the quantum scale close to the absolute zero there is a non vanishing energy field (as suggested by Planck) and that this field itself is responsible for attraction forces between bodies (as predicted by Casimir), it was unavoidable that an alternative theoretical picture of the problem should account for this field, too.

3.2 Rytov's theory

What is today commonly referred to as the Lifshitz theory of zero-point field interactions (from this the denomination "Lifshitz interactions") originates from the arguments written by Lifshitz in his article[2] used to deal with the Casimir's problem. Actually, like Lifshitz himself wrote in the introduction, his calculations stem from the application of the general theory of electromagnetic field fluctuations discussed by Rytov[15] in his monograph[3]. In this work published a few years before (1953) Rytov makes an extension of Maxwellian macroscopic electrodynamics with classical physical-statistical methods, primarly to explain the origin of electrical noise in circuits by means of thermal radiation "in the form of a unified theory" (these words marking his advance compared to Nyquist's[16] work on the same topic).

The main innovative argument of Rytov's research is to consider that during the process of absorption of light (note: always classically speaking) at least a part of gained energy goes back to vacuum, where macroscopic matter "floats". Because of this, energy is deposited into the fluctuating field in vacuo. In this way we consider the fluctuating field in the same way as fluctuation is defined in statistical physics; this means that the field has a random value over time, but when time averaged it is reduced to zero. This field, which Rytov in his work calls "chaotic lateral field", - directly citing:

> distributed over the volume of the bodies under consideration constitutes the basis for the application of statistics to general electrodynamics. The action of this field determines the fluctuation of the charge and of the current in bodies, and thereby also the corresponding fluctuation radiation (i.e. electrical noise, ed.).

[13] Evgenii Mikhailovich Lifshitz (1915–1985), Soviet physicist
[14] James Clerk Maxwell (1831-1879), Scottish theoretical physicist
[15] Sergei Mikhailovich Rytov (1908-1996), Soviet physicist
[16] Harry Nyquist (1889-1976), Swedish-born electronic engineer

We remark from the quotation that the field is called "lateral" because it is a collateral phenomenon to the presence *inside vacuum* of matter which is absorbing light. It is attributed to vacuum but it cannot exist in vacuum alone.

The result of this idea is that to solve Maxwell equations they have to be properly amended, to account for the presence of a field that is not caused by real charges or currents:

$$\varepsilon(\omega)\varepsilon_0\nabla\cdot\mathbf{E}=\rho, \qquad \nabla\cdot\mathbf{B}=0, \tag{9}$$

$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}, \qquad \nabla\times\mathbf{H}=\mathbf{j}+\varepsilon(\omega)\varepsilon_0\frac{\partial\mathbf{E}}{\partial t}+\frac{\partial\mathbf{K}}{\partial t}. \tag{10}$$

Here $\varepsilon(\omega)=\varepsilon'(\omega)+i\varepsilon''(\omega)$ is the complex dielectric constant and $\mathbf{K}$ is a random field describing the polarization fluctuations (for a nonmagnetic medium we set $\mu=1$).

Another fundamental step made by Rytov in his work is to define a necessary property of the fluctuating field to allow further calculation. Given that the field is stochastic in time, its space correlation function is given by:

$$< K_i(x,y,z,t)K_j(x',y',z',t') >= A\varepsilon''(\omega)\varepsilon_0\delta_{ij}\delta(x-x')\delta(y-y')\delta(z-z')\delta(t-t'), \tag{11}$$

$$A=4\hbar\left(\frac{1}{e^{\frac{\hbar\omega}{k_BT}}-1}+\frac{1}{2}\right).$$

This is the average value of the product of components $(i,j)\in(x,y,z)$ of the field at two different points in space; we see that the field is in this respect homogenous because of our macroscopic assumptions. A more compact notation for this is possible if we consider that

$$2\left(\frac{1}{e^x+\frac{1}{2}}\right)=\coth\frac{x}{2} \tag{12}$$

(later we will use this form). Rytov's derivation of this formula is quite long and complex, but for us it is enough to note that it includes the same temperature dependence of Planck's law. It is interesting that the fluctuating field, though being not properly a body made of oscillators which re-emits absorbed light, must be such as to include the zero-point field contribution and the "matter" contribution (remember, the fluctuating field is regarded as external to matter, and for this reason is said to belong to vacuum, but at the same time it cannot exist without the presence of matter). Furthermore it is relevant that, as expected, the formula includes as an indispensable factor the imaginary part of the dielectric constant, the part responsible for absorption; absorption itself namely implies the existence of the fluctuating field.

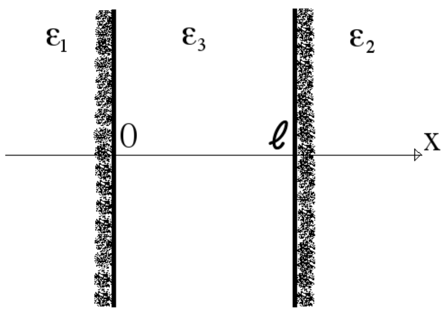
What Rytov had just derived is in fact what today we call the "fluctuation-dissipation theorem" (where dissipation is a synonym for absorption). This is a general physical-statistical tool that had already been used by Nyquist (to explain Johnson[17] noise), but it got a rigorous formulation only much later with Kubo[18] (1967).

3.3 Calculation

Let's go back to our problem and follow Lifshitz's solution. The system is made of just two infinite plates, so there are no free charges and currents ($\rho=0$ and $\mathbf{j}=0$).

[17]John Bertrand Johnson (1887-1970), Swedish-born electrical engineer and physicist

[18]Ryogo Kubo (1920-1995), Japanese mathematical physicist



We solve it for monochromatic fields with time factor $e^{-i\omega t}$. For an easier calculation we rewrite so obtained Maxwell equations in Gauss[19] notation ($H \equiv \sqrt{4\pi\mu_0}H$, $E \equiv \sqrt{4\pi\varepsilon_0}E$):

$$\nabla \cdot \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{H} = 0, \tag{13}$$

$$\nabla \times \mathbf{E} = i\frac{\omega}{c}\mathbf{H}, \qquad \nabla \times \mathbf{H} = -i\varepsilon(\omega)\frac{\omega}{c}\mathbf{E} - i\frac{\omega}{c}\mathbf{K}. \tag{14}$$

These are the equations for the electric field E and magnetic field H that differ from the ordinary empty space Maxwell equations becasue of the presence of stochastic sources proportional to the time derivative of K. The first two just require, as usual in vacuum, that the fields must be transversal to the waves ((24) and (28)), so we actually have a set of two partial differential vector equations for two vector unknowns; we try to find solutions in the form of Fourier[20] integrals. Since we can express also the random field $\mathbf{K}$ in terms of its known Fourier components $\mathbf{g}(\mathbf{k})$ (so that $\mathbf{K}(\mathbf{r}) = \int_{-\infty}^{\infty} \mathbf{g}(\mathbf{k})e^{i\mathbf{kr}}d\mathbf{k}$), we get

$$< g_i(\mathbf{k})g_j(\mathbf{k}') >= \frac{1}{4\pi^3}A\varepsilon''(\omega)\delta_{ij}\delta(k-k'). \tag{15}$$

The system of equations has to be solved separately for each of the three parts in which the space is divided along the x axis. Remember, we want to calculate the force between the plates, and this will be done via the Maxwell stress tensor. However in order to calculate the required fields of the central region we have to know the fields of the lateral regions, which define the central fields by the well-known four boundary conditions for normal and tangential components of the electric and magnetic field on each surface.

In short, what we do is to solve the inhomogeneous Maxwell equations in the presence of stochastic sources $\mathbf{K}$ and the boundary conditions which are of the standard type

$$E_{1t} = E_{3t}, \qquad H_{1t} = H_{3t}, \qquad \varepsilon_1 E_{1n} = \varepsilon_3 E_{3n}, \qquad H_{1n} = H_{3n}; \tag{16}$$

$$E_{3t} = E_{2t}, \qquad H_{3t} = H_{2t}, \qquad \varepsilon_3 E_{3n} = \varepsilon_2 E_{2n}, \qquad H_{3n} = H_{2n}. \tag{17}$$

The sources satisfy the fluctuation-dissipation theorem which we take into account when we calculate the average of the square of the field in the expression for the average Maxwell stress tensor. At this point the integration over the positive frequencies can be extended to the whole axis if one takes into account the parity of the dielectric response function in the frequency domain.

[19]Johann Carl Friedrich Gauss (1777-1855), German mathematician
[20]Jean-Baptiste Joseph Fourier (1768-1830), French mathematician

To show how all this works we perform calculations for the first part (with index 1, i.e. for the medium with ε_1). There the random field $\mathbf{K}$ takes the form of

$$\mathbf{K_1}(x,y,z) = \int_{-\infty}^{\infty} \mathbf{g(k)} e^{i\mathbf{qr}} \cos(k_x x) d\mathbf{k} \tag{18}$$

where we have defined $\mathbf{q} = (0, k_y, k_z)$ and $\mathbf{r} = (0, y, z)$. This is an elementary Fourier transform definition. Similarly we write the solutions for fields $\mathbf{E_1}$ and $\mathbf{H_1}$ in the first medium (note $\mathbf{n} = (1, 0, 0)$):

$$\mathbf{E_1} = \int_{-\infty}^{\infty} \{\mathbf{a_1(k)} \cos(k_x x) + i\mathbf{b_1(k)} \sin(k_x x)\} e^{i\mathbf{qr}} d\mathbf{k} + \int_{-\infty}^{\infty} \mathbf{u_1(q)} e^{i\mathbf{qr} - is_1 x} d\mathbf{q}, \tag{19}$$

$$\mathbf{H_1} = \frac{c}{\omega} \int_{-\infty}^{\infty} \{(\mathbf{q} \times \mathbf{a_1} + k_x \mathbf{n} \times \mathbf{b_1}) \cos(k_x x) + i(\mathbf{q} \times \mathbf{b_1} + k_x \mathbf{n} \times \mathbf{a_1}) \sin(k_x x)\} e^{i\mathbf{qr}} d\mathbf{k} + \frac{c}{\omega} \int_{-\infty}^{\infty} (\mathbf{q} \times \mathbf{u_1} - s_1 \mathbf{n} \times \mathbf{u_1}) e^{i\mathbf{qr} - is_1 x} d\mathbf{q} \tag{20}$$

with

$$s_1 = \sqrt{\frac{\omega^2}{c^2}\varepsilon_1 - q^2}, \tag{21}$$

where the first term is the solution of the inhomogeneous equations and the second of the homogeneous (i.e. for $\mathbf{K}$=0). The coefficients of the inhomogeneous equation are calculated from the second of the two considered Maxwell equations (rot$\mathbf{H}$) when we define $\mathbf{K}$ as (18):

$$\mathbf{a_1} = \frac{1}{\varepsilon_1(k^2 - \frac{\omega^2}{c^2}\varepsilon_1)} \left[\frac{\omega^2}{c^2} \varepsilon_1 \mathbf{g_1} - \mathbf{q}(\mathbf{q}\mathbf{g_{1r}}) - k_x^2 g_{1x} \mathbf{n} \right], \tag{22}$$

$$\mathbf{b_1} = \frac{k_x}{\varepsilon_1(k^2 - \frac{\omega^2}{c^2}\varepsilon_1)} \left[\mathbf{n}(\mathbf{q}\mathbf{g_{1r}}) + \mathbf{q} g_{1x} \right]. \tag{23}$$

Finally transversality condition ($\nabla \cdot \mathbf{E} = 0$) sets

$$\mathbf{u_{1r}} \mathbf{q} - u_{1x} s_1 = 0. \tag{24}$$

In the central part (with index 3, i.e. vacuum-no matter) for $\mathbf{K}$=0 we have only homogeneous solutions:

$$\mathbf{E_3} = \int_{-\infty}^{\infty} \{\mathbf{v(q)} e^{ipx} + \mathbf{w(q)} e^{-ipx}\} e^{i\mathbf{qr}} d\mathbf{q}, \tag{25}$$

$$\mathbf{H_3} = \frac{c}{\omega} \int_{-\infty}^{\infty} \{(\mathbf{q} \times \mathbf{v} + p\mathbf{n} \times \mathbf{v}) e^{ipx} + (\mathbf{q} \times \mathbf{w} - p\mathbf{n} \times \mathbf{w}) e^{-ipx}\} e^{i\mathbf{qr}} d\mathbf{q} \tag{26}$$

with

$$p = \sqrt{\frac{\omega^2}{c^2} - q^2} \tag{27}$$

and transversality conditions

$$\mathbf{v_r}\mathbf{q} + v_x p = 0, \qquad \mathbf{w_r}\mathbf{q} - w_x p = 0. \tag{28}$$

Next we consider the boundary conditions, given that $\varepsilon_3 = 1$:

$$E_{1t} = E_{3t}, \qquad H_{1t} = H_{3t}, \qquad \varepsilon_1 E_{1n} = E_{3n}, \qquad H_{1n} = H_{3n}. \tag{29}$$

The missing equations for calculating all field Fourier amplitudes are found by solving the same problem on the other plate i.e. at $x = \ell$. There the formulation is the same with appropriate indices, except for setting $k_x(x - \ell)$ in the arguments of harmonic functions instead of $k_x x$. Solving the whole set of equations gives formulas for central fields amplitudes in terms of the amplitudes g of the random field.

Now we can calculate the xx component of Maxwell stress tensor, perpendicular to the surfaces, by integrating over all frequencies (positive and negative); since the integrated function is even, we get:

$$F_{xx} = 2\int_0^\infty F_\omega d\omega = \frac{1}{4\pi}2\int_0^\infty (E_{3x}^2 - \frac{1}{2}E_3^2 + H_{3x}^2 - \frac{1}{2}H_3^2)d\omega \\ = \frac{1}{4\pi}\int_0^\infty (E_{3r}^2 - E_{3x}^2 + H_{3r}^2 - H_{3x}^2)d\omega. \tag{30}$$

The fields appearing above are intended to be statistically averaged ($< F_{xx} >= ...$). Note that the squares of both E and H contain double integrals over **k** and a series of products of g components whose result is given by (15) carrying out one of these integrations. In this regard remember that quantities g referring to different media are statistically independent, too, so their product is 0.

Next we perform separately another integration over dk_x and substitute the integration over dq with dp. The whole expression has to be simplified, neglecting the terms appearing without ℓ-dependence, which are not relevant for the problem. If we make the p variable dimensionless dividing it by $\frac{\omega}{c}$ the final result is obtained as:

$$< F_{xx} >\equiv F = \frac{\hbar}{2\pi^2 c^3}\int_0^\infty \coth\frac{\hbar\omega}{2k_BT}\omega^3 d\omega \int p^2 dp\{\left[\frac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)}e^{-2ipl\frac{\omega}{c}} - 1\right]^{-1} \\ + \left[\frac{(s_1+\varepsilon_1 p)(s_2+\varepsilon_2 p)}{(s_1-\varepsilon_1 p)(s_2-\varepsilon_2 p)}e^{-2ipl\frac{\omega}{c}} - 1\right]^{-1}\} \tag{31}$$

with

$$s_1 = \sqrt{\varepsilon_1(\omega) - 1 + p^2}, \qquad s_2 = \sqrt{\varepsilon_2(\omega) - 1 + p^2} \tag{32}$$

for

$$p = \sqrt{1 - \frac{c^2}{\omega^2}q^2} \tag{33}$$

defined on the real segment [0,1] and the whole upper half of the imaginary axis. The force between the plates would be the real part of this complicated analitic integral.

However is possible to compute this integral in an easier way. If we do the transformation $\omega = i\xi$ we redefine the domain of p (33) over only real values from 1 to infinity and we make the exponents in (31) real. Straightforwardly we see that even the integration by ξ over the same domain as ω gives a real result. In fact we note that the function $\coth\frac{\hbar\omega}{2k_BT}$ has an infinite number of poles on the imaginary axis at

$$\omega_n = i\xi_n = i\frac{2T}{\hbar}\pi n. \tag{34}$$

The standard procedure that we use for evaluating this integral, taken from the theory of analytic functions, is to first extend the integration domain of ξ to the upper right quadrant of the complex plane and then use the Cauchy theorem when we close the frequency integration path at infinity. Since the coth function has poles in this domain we then get as a result of the integration the sum of the residua of the poles of the integrand times $i\pi$ (except for the pole at n=0 where we shall account just $i\frac{\pi}{2}$ at the right angle). It's easy to verify that all the residua of these poles are the same i.e. $-i\frac{2T}{\hbar}$ so the overall expression takes the form of

$$F = \frac{2T\pi}{\hbar} \cdot \frac{\hbar}{2\pi^2 c^3} \sum_n{}' \xi_n^3 \int_1^\infty p^2 dp \{ \left[\frac{(s_{1n}+p)(s_{2n}+p)}{(s_{1n}-p)(s_{2n}-p)} e^{2pl\frac{\xi_n}{c}} - 1 \right]^{-1} + \left[\frac{(s_{1n}+\varepsilon_{1n}p)(s_{2n}+\varepsilon_{2n}p)}{(s_{1n}-\varepsilon_{1n}p)(s_{2n}-\varepsilon_{2n}p)} e^{2pl\frac{\xi_n}{c}} - 1 \right]^{-1} \}, \tag{35}$$

where the prime on the summation symbol reminds the exception at n=0, with

$$s_{1n}(i\xi_n) = \sqrt{\varepsilon_{1n}(i\xi_n) - 1 + p^2}, \qquad s_{2n}(i\xi_n) = \sqrt{\varepsilon_{2n}(i\xi_n) - 1 + p^2}, \tag{36}$$

$$\varepsilon_{1n} = \varepsilon_1(i\xi_n), \qquad \varepsilon_{2n} = \varepsilon_2(i\xi_n), \tag{37}$$

for n from 0 to infinity. This formula enables us to calculate the force F for any value of ℓ and T, provided we know the values of the functions $\varepsilon(i\xi)$. These can be directly obtained from the already required imaginary part of the complex dielectric function with the formula

$$\varepsilon(i\xi) = 1 + \frac{2}{\pi} \int_0^\infty \frac{\omega \varepsilon''(\omega)}{\omega^2 + \xi^2} d\omega \tag{38}$$

derived from Kramers-Kronig relations[4]. So to determine the force of interaction between bodies only the absorption part of the dielectric constant is needed.

If we want to get to Casimir's result from section 2.2 it's necessary to put T=0, in order to consider just the zero-point field contribution. In this case, we see that the distances between the poles tend to zero, so we can substitute the summation with an integration over $d\xi = \frac{2T\pi}{\hbar}$ deriving the formula

$$F = \frac{\hbar}{2\pi^2 c^3} \int_0^\infty \xi^3 d\xi \int_1^\infty p^2 dp \{ \left[\frac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)} e^{2pl\frac{\xi}{c}} - 1 \right]^{-1} + \left[\frac{(s_1+\varepsilon_1 p)(s_2+\varepsilon_2 p)}{(s_1-\varepsilon_1 p)(s_2-\varepsilon_2 p)} e^{2pl\frac{\xi}{c}} - 1 \right]^{-1} \}. \tag{39}$$

To allow further calculations we introduce a new variable $x = 2pl\frac{\xi}{c}$ instead of ξ and we get

$$F = \frac{\hbar c}{32\pi^2 \ell^4} \int_0^\infty x^3 dx \int_1^\infty \frac{1}{p^2} dp \{ \left[\frac{(s_1+p)(s_2+p)}{(s_1-p)(s_2-p)} e^{x} - 1 \right]^{-1} + \left[\frac{(s_1+\varepsilon_1 p)(s_2+\varepsilon_2 p)}{(s_1-\varepsilon_1 p)(s_2-\varepsilon_2 p)} e^{x} - 1 \right]^{-1} \}. \tag{40}$$

with

$$\varepsilon = \varepsilon(i\frac{xc}{2p\ell}). \tag{41}$$

The main contributions to the x integral come from small values of x: this implies, for large ℓ as assumed in Casimir's calculation, small ξ and $p \backsimeq 1$. Under those circumstances the argument of ε function is close to zero so we can replace ε function with its static value $\varepsilon_0 = \varepsilon_{\omega=0}$, which for metals assumes the value $\varepsilon_{\omega\to 0} \to \infty$. Thus

$$F = \frac{\hbar c}{32\pi^2 \ell^4} \int_0^\infty \int_1^\infty \frac{2x^3 dp dx}{p^2(e^x - 1)} = \frac{\pi^2}{240} \frac{\hbar c}{\ell^4}, \tag{42}$$

that is exactly the Casimir result.

3.4 Remarks

We have at last obtained the same result as in (8). But how close may our plates stand for, Casimir's perfect metal approximation to remain still valid? To answer to such a question we need to compute the next term in the expansion of our formula, and require it to be much smaller than the first term. This can be done by replacing the limit $\varepsilon = \infty$ with its proper frequency dependence in the surroundings. Lifshitz found out that the exact metal equation $\varepsilon(\omega) = 1 + \frac{i\sigma}{\varepsilon_0\omega}$[4] gives a very tiny correction, so for this purpose he had to take the plasma limit for metals $\varepsilon(\omega) = 1 - \frac{e^2N}{\varepsilon_0 m\omega^2}$[4] (where unity can be neglected for low frequencies), which, for typical metal (e.g. silver) free-electron density N, requested the distance to be much greater than $0.6\mu m$. Similarly, for temperatures close to zero he used the expansion of the integral over ξ in (39) towards summation of discrete frequencies in (35), given by the Euler sum formula, to discover that this limit is allowed whenever $\ell \ll \frac{\hbar c}{k_B T}$. Surprisingly, in the case of silver, as long as $\ell < 5\mu m$ (but still well over $0.6\mu m$) we can regard it as a perfect metal with T=0 even at room temperature[2].

As Lifshitz points out in his article, Casimir in his computation couldn't account for all this. In fact equation (35) for the geometry of this problem (and the method here explained for any other one) has full generality and can be used to investigate all limiting cases in both distance and temperature for bodies of any kind. In this way, retardation effects associated with the finite value of light speed are automatically taken into account, too; it's even possible to reduce the problem to the interaction between individual atoms with the limit of rarefied media ($\varepsilon - 1 \ll 1$).

4 Epilogue. Stochastic electrodynamics

With his work Lifshitz confirmed previous calculations of the van der Waals force, and at the same time indicated how to solve a whole spectrum of problems related to it, showing that the various Casimir effects are all caused by that same force, which is the expression of a zero-point field for T=0, and thermally excited electro-magnetic field for $T \neq 0$. Later, as anticipated in section 3.1, the missing macroscopic results were achieved within QED formalism. However, these results were not as systematic as Lifshitz's approach and needed to be perfected through several steps to agree with Lifshitz's physical conclusions, which in the meantime had been accepted, especially regarding the role of absorption in creating the zero-point field (a detailed review of this attempts is found in Milonni[1]).

Anyway the fact that a phenomenon occurring at very small scales, which have been used to be described by the quantum theory, could be computed without explicit reference to quantum axioms at the time gave a boost to those who supported the non-necessity of quantum mechanics and its "paradoxes" (notably the violation of local realism) to describe nature in a complete way. For this reason an alterative theory was thereafter developed, called "Stochastic electrodynamics" (SED). In this theory the $\hbar$ constant is just one among the several constants in nature to be experimentally determined, without any relation to the theory of quantum measurement. In this case the constant appears to arise to quantify the black-body function dependency, and in particular is proper to the "residual" energy we see approaching the thermal zero, i.e. the factor $\frac{1}{2}\hbar\omega$. This energy is attributed then to a classical-behaving field, whom we give reality, and whose features are described by Rytov.

In the years after Lifshitz's article had been published, many scientists had been working actively to give this theory more solid and complete foundations, and numerous physical effects were successfully explained within it (a review of this developments is presented in the reference [5]). Nowadays there are claims that some phenomena still unaccounted by QED are intelligible in SED, while others, like the more common particle spin and the Stern-Gerlach experiment, remain unsolved. For this reason, as Milonni points out, SED cannot yet be regarded as a fundamental theory of the electromagnetic field. Moreover many SED calculations have the same mathematical nature as QED (for example the discrete frequencies of equation (34) are the so-called Matsubara frequencies known in quantum field theory), and even the Planck idea about the energy of a light plane wave ($\epsilon = \hbar\omega$) is kept as valid. Because of this Lifshitz's stochastic theory could be simply a more comprehensive and intuitive way, or in other words a convenient physics approach, available to explain a macroscopic phenomenon whose quantum component is non-removable but hidden in the statistical nature of quantum mechanics.

The real advantage of the Lifshitz formalism, however, becomes apparent whenever one tries to calculate non-equilibrium Casimir interactions (e.g. two bodies at different T), where QED is inapplicable.

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