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Turbulence in Fluids

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Abstract

The seminar gives an introduction to turbulent motions of incompressible fluids. Due to the nature of the subject, most of the explanations are qualitative, although references to the actual fundamental equations underlying the processes are also made. Special emphasis is made on the dynamics of vorticity in turbulent flows and the main results of the celebrated Kolmogorov 1941 theory.

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1 Introduction

Turbulence is the most common property of fluid motion as opposed to laminar flows. It may be observed in a wide variety of situations. Some of the obvious everyday examples include the flow of water in rivers, convection of air inside a room, smoke rising from a chimney and the motion of cumulus clouds. We may also find turbulence in less pleasant environments such as the Earth's liquid core, interstellar gas clouds and the photosphere of the sun.

The equation prescribing the motion of fluids is the Navier-Stokes equation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2 \mathbf{v}, \tag{1}$$

where $\mathbf{v}(\mathbf{r}, t)$, p, and ρ are the fluid velocity field, pressure and density, respectively. The ratio $\nu = \mu/\rho$ is called kinematic viscosity, whereas the coefficient μ is known as dynamic viscosity. Equation (1) is valid for the case of an incompressible fluid where $\nabla \cdot \mathbf{v} = 0$. Turbulence is associated with solutions for $\mathbf{v}$, which display random fluctuations both in space and time.

It may be very helpful to rewrite equation (1) in a dimensionless form. Let us consider a flow with characteristic length L, velocity u and kinematic viscosity ν. We can put equation (1) in a dimensionless form by making the following substitutions

$$t' = tu/L, \qquad p' = p/\rho u^2, \qquad \mathbf{r}' = \mathbf{r}/L, \qquad \mathbf{v}' = \mathbf{v}/u. \tag{2}$$

The Navier-Stokes equation then becomes

$$\frac{\partial \mathbf{v}'}{\partial t'} + (\mathbf{v}' \cdot \nabla')\mathbf{v}' = -\nabla' p' + \frac{1}{\mathrm{Re}}\nabla'^2 \mathbf{v}'. \tag{3}$$

The parameter Re is known as the *Reynolds number* and is given by

$$\mathrm{Re} = \frac{Lu}{\nu}. \tag{4}$$

This is the only dimensionless parameter we can construct out of the quantities L, u and ν. It is now evident, that any solution of (1) must be of the form $\mathbf{v} = u\mathbf{f}(\mathbf{r}', t', \mathrm{Re})$ [1]. This means that any dimensionless solution for a flow of certain type gives a whole class of solutions for $\mathbf{v}$ with the same Reynolds number, but different characteristic lengths, velocities and viscosities. The property of flows with the same Reynolds number to display a similar type of motion on various scales is known as the *law of similarity.*

It is generally believed that the occurrence of turbulence is closely related with a critical value[1] of Reynolds number, beyond which the flow becomes turbulent. This phenomenon was first observed by O. Reynolds in 1883 [2]. In his paper, Reynolds studied the flow through a pipe with circular cross-section driven by a pressure gradient. He measured the mean velocities of water in tubes with different diameters

[1]The critical value is not a universal constant. It may for example depend on the geometry of the problem [1].

and at different viscosities, i.e. temperatures of water. His results showed that the flow displays a time-independent laminar motion, which is the obvious solution of equation (1) for this case, only below below a critical value of the ratio Lu/ν, which later became known as the Reynolds parameter. When the value of Re was sufficiently increased, the streamlines began to show an irregular and rapidly changing pattern.

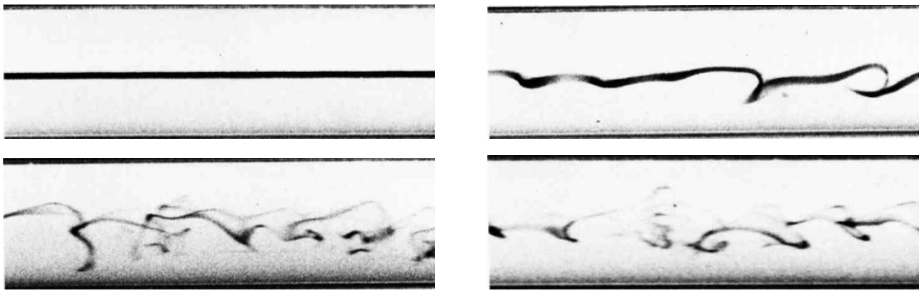

Figure 1: Repetition of Reynolds' experiment a century later [3]. The pictures show the trace of a colored filament injected into the tube at different velocities. The flow develops an instability on the top right picture. The last two pictures show a fully turbulent regime.

Reynolds also found out that he was able to push the critical limit of Re higher, if he tried to minimize any outside disturbances acting on his experiment. This fact demonstrates that the development of turbulence under steady-state conditions is caused by small perturbations acting on the flow, which begin to grow exponentially above some value of Re. Turbulence may therefore be associated in a mathematical sense with a set of solutions to equation (1), which acquire their higher stability only for the high values of Reynolds number [4].

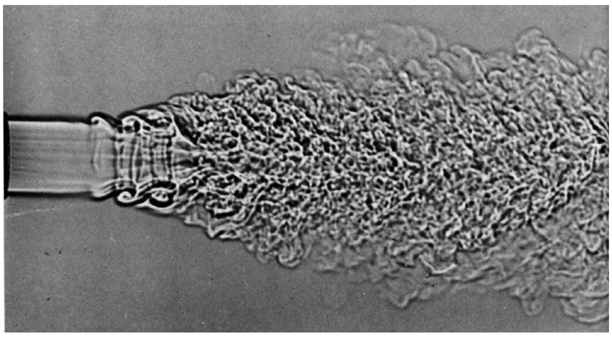

Figure 2: Instability of a round jet [3]. The jet of carbon dioxide is laminar as it leaves the nozzle, but it quickly develops and instability, which causes the formation of vortex rings and a transition to turbulence.

The difficulties in understanding turbulence from a physical point of view are in fact also associated with a lack of a general mathematical theory for the Navier-Stokes equations. For the 3-dimensional case mathematicians were so far unable to prove the existence and uniqueness of general solutions to these equations [5]. In 2 dimensions, existence and uniqueness of solutions has been demonstrated by Leray (1933). Leray also investigated solutions to the 3 dimensional problem [6]. In this case, he was only able to show the existence of at least one 'turbulent' solution, which satisfies equation (1), except at certain times of irregularity, when the solution is subject only to a very weak continuity condition.

Leray's results suggest a fundamental difference between turbulence in 2 and 3 dimensions and indeed it really is so. We should be however a little bit careful, when we talk about turbulence in 2 dimensions. Real turbulence in nature is in principle always a 3-dimensional phenomenon, but there are certain situations[2] where turbulent flows are 'almost' 2-dimensional [7]. The main physical reason for the difference between turbulence in 2 and 3 dimensions lies in the nature of vorticity dynamics. In particular, there is no vortex

[2]Some of the most important examples include the motions of oceanic currents, tropical cyclones and laboratory systems under the influence of strong rotations.

stretching mechanism available in 2 dimensions, so that vorticity is conserved (it is only created at the boundaries), whereas in 3 dimensions vortex filaments may be deformed and strengthened [4]. More about vorticity dynamics and it's relation to turbulence will be said in the next section.

2 The dynamics of turbulence

We have already introduced the equation, which is of central interest in the study of turbulence – the Navier-Stokes equation. It is generally believed that equation (1) correctly describes turbulence in Newtonian[3] fluids [5, 4, 8]. Yet, this is not the only equation worth mentioning when we study turbulence. At first, we may notice that the system of equations (1) is incomplete, since the pressure is also one of the unknowns to be determined. A common way of eliminating the pressure is by taking the curl of (1) to obtain an equation for the vorticity $\boldsymbol{\omega} = \nabla \times \mathbf{v}$. In it's final form, this equation is given by [9]

$$\frac{D\boldsymbol{\omega}}{Dt} = \mathsf{S}\cdot\boldsymbol{\omega} + \nu\nabla^2\boldsymbol{\omega}, \tag{5}$$

where $S_{ij} = \frac{1}{2}\big(\partial v_i/\partial x_j + \partial v_j/\partial x_i\big)$ is the symmetrized velocity gradient or *strain-rate tensor*. The convective derivative $D/Dt = \partial/\partial t + (\mathbf{v}\cdot\nabla)$ prescribes the change of vorticity in a fluid element traveling along the flow. Each of the terms on the right therefore represents a different mechanism for the change of vorticity in a fluid element.

Let us first examine the second term in equation (5). If we omit the term $\mathsf{S}\cdot\boldsymbol{\omega}$, we may notice that equation (5) represents a form of diffusion equation. Thus, the term $\nu\nabla^2\boldsymbol{\omega}$ cannot amplify the vorticity in a fluid element. It can only spread localized distributions of vorticity, which is essentially created at fixed solid surfaces due to the presence of large velocity gradients in these regions[4]. We already now that turbulence corresponds to the limit of small viscosity. Since the localized eddies (blobs of vorticity) will spread very slowly when ν is small, they should probably deform or break up at some point, if we expect the fluid motion to be chaotic. In most types of turbulence this indeed happens and the mechanism for the distortion of eddy shapes provides the first term on the right-hand side of (5).

The action of the strain rate S_{ij} on $\boldsymbol{\omega}$ results in an amplification or rotation of vorticity inside a fluid element over time. This term does not occur in 2 dimensions, since $\mathbf{v}$ does not vary along the direction of $\boldsymbol{\omega}$ in a 2-dimensional flow. If we observe the product $S_{ij}\omega_j$ in a coordinate system aligned with the principal axes of S_{ij}, we may also notice that vorticity is amplified in the directions of positive strain rate and attenuated in directions of negative strain rate (figure 3). The magnitude of $\boldsymbol{\omega}$ inside a fluid element changes because vortex filaments are being stretched or squeezed under the presence of a strain field [9]. The stretching or squeezing results in a change of the fluid element's moment of inertia. Because the angular momentum of the flow is (approximately) conserved at large Reynolds numbers [7], each change in the moment of inertia necessarily implies a change of angular velocity. It may be also demonstrated that there is on average more vortex stretching than squeezing at high Re [9].

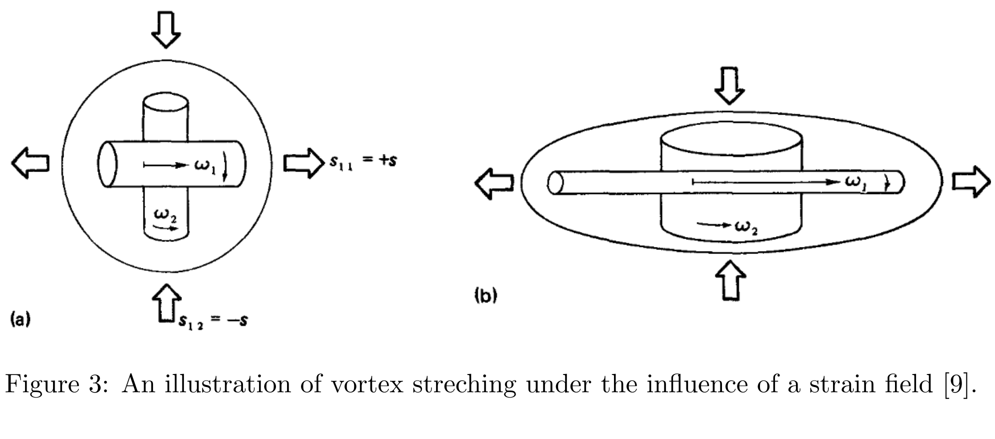

Figure 3: An illustration of vortex streching under the influence of a strain field [9].

We are now ready to draw a very important conclusion based on our qualitative discussion. The main remark to be made here is that vortex stretching includes a transition of energy from larger to smaller

[3]Newtonian fluids are characterized by a linear relation between the viscous stress tensor and velocity gradient.
[4]This is a direct consequence of the fact that velocity should go to zero at any fixed surface.

eddies. We may imagine the larger eddies as sources of the 'mean' field S_{ij}, which deforms the shape of smaller eddies. The exchange of energy is a natural consequence of the whole process, because the strain rate performs deformation work on the vortices that are being stretched [9]. This kind of energy cascade from larger to smaller scales is effective as long as the characteristic size of the eddies is large enough for the viscous dissipation effects to play a negligible role [7]. To obtain an estimate for the dissipation scale in a turbulent flow we first need to derive an expression for energy balance.

To obtain an equation for energy balance we calculate the time derivative of kinetic energy density by substituting for $\partial \mathbf{v}/\partial t$ the expression for it given by the Navier-Stokes equation. For our purpose, it is more convenient to write (1) in an alternative form as [1]

$$\frac{\partial v_i}{\partial t} = -\frac{1}{\rho}\frac{\partial \Pi_{ij}}{\partial x_j}, \tag{6}$$

where the momentum flux density tensor Π_{ij} is given by [1]

$$\Pi_{ij} = p\delta_{ij} + \rho v_i v_j - \rho\nu\Big(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\Big). \tag{7}$$

After we multiply equation (6) by u_i and sum over all indices i, we finally obtain an expression for the change of kinetic energy (per unit mass) [7]

$$\frac{\partial}{\partial t}\left(\tfrac{1}{2}v^2\right) = -\frac{\partial}{\partial x_j}\left[v_j\left(\tfrac{1}{2}v^2 + \frac{p}{\rho}\right) \; - \; 2\nu v_i S_{ij}\right] - 2\nu S_{ij}S_{ij}. \tag{8}$$

If we integrate the above expression over a finite volume, we may transform the divergence term using Gauss' theorem to a surface integral. The first term on the right therefore represents the energy flux density inside the fluid. The term $v_j\big(\frac{1}{2}v^2 + \frac{p}{\rho}\big)$ is the actual transfer of fluid mass, whereas $2\nu v_i S_{ij}$ corresponds to energy flux due to processes of internal friction [1]. The second term on the right-hand side of equation (8) has to represent the conversion of kinetic energy to heat. Thus, the decrease of kinetic energy density per unit mass ϵ is given by the formula [7]

$$\epsilon = 2\nu S_{ij}S_{ij}. \tag{9}$$

We have already discussed the picture of turbulence as a distribution of eddies in which the energy is passed from larger to smaller scales by means of vortex stretching and is finally dissipated into heat at small scales. We would now like to obtain an estimate for the dissipation scale η and characteristic velocity in the small eddies v_0, based on the size of largest eddies l_0 and their typical magnitude of velocity fluctuations u. The parameter l_0 is called the *integral scale.* Under steady conditions in a statistical sense the rate of energy dissipation ϵ should balance the injection of energy at the largest scales [7]. Experimental data (see for example [8]) suggests, that the mean energy of large scale flow, which is of the order of u^2, is passed down the cascade with a characteristic time l_0/u, so that the rate of energy transfer Π is approximately $\Pi \sim u^3/l_0$. The rate of dissipation ϵ is of the order $\epsilon \sim \nu v_0^2/\eta^2$. Under steady conditions Π and ϵ should balance

$$u^3/l_0 \sim \nu v_0^2/\eta^2. \tag{10}$$

Since Re approximately represents the ratio of inertial[5] to viscous forces [7], we should also have Re ~ 1 at the dissipation scale and therefore

$$v_0\eta/\nu \sim 1. \tag{11}$$

Combining conditions (10) and (11) we get [7]

$$\eta \sim l_0 \mathrm{Re}^{-3/4} \sim \big(\nu^3/\epsilon\big)^{1/4}, \tag{12}$$

$$v_0 \sim u \mathrm{Re}^{-1/4} \sim \big(\nu\epsilon\big)^{1/4}. \tag{13}$$

The dissipation scales η and v_0 are known as the *Kolmogorov microscales* of turbulence. In a typical wind tunnel experiment we might have Re $\sim 10^3$ and $l_0 \sim 1$ cm [7]. This gives us as estimate for η the value $\eta \sim 0.06$ mm, which is more than two orders in magnitude smaller than l_0.

[5] By inertial forces we refer to the effects of all terms in the Navier-Stokes equation, except the viscous term.

Based on the above analysis we may conclude that turbulence evolves through an interaction of eddies of different sizes, which range from the integral scale l_0 down to the Kolmogorov scale η. The interaction includes a cascade of energy from larger to smaller scales. This kind of dynamics in a turbulent flow is called the *Richardson cascade* (figure 4). The idea of Richardson is generally believed to be mostly correct with a few important exceptions [7]. The most obvious exception to this is 2-dimensional turbulence, since the vortex stretching mechanism is absent there. As it turns out, the energy is in 2-dimensional turbulence actually transfered in the opposite way from smaller to larger scales [7].

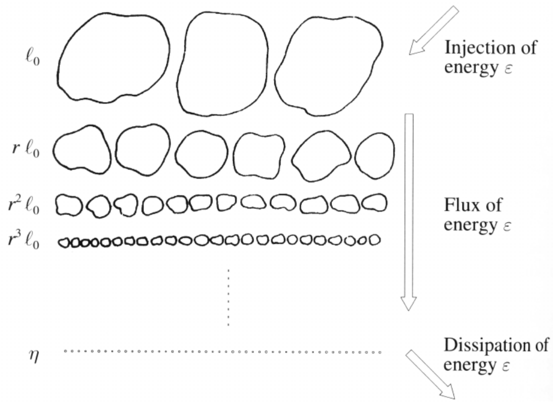

Figure 4: An illustration of the Richardson cascade [5].

3 Statistical properties of homogeneous turbulence

Turbulent motion of fluids is random and unpredictable in a sense that we obtain slightly different measurements each time we repeat an experiment under seemingly identical conditions. Turbulence is therefore often studied in a statistical sense by considering the mean values of some measurable quantities assigned to the flow. The mean values are formally obtained as averages over the probability distribution of the velocity field. Under steady conditions this kind of averages may be evaluated as time averages of a particular realization of the flow under the assumption of ergodicity [8]. In our succeeding discussion we shall limit ourselves to the case of homogeneous turbulence. This assumption requires (at least) that the flow be not considered too close to the boundaries [4]. A good experimental approximation to homogeneous turbulence is the so called grid turbulence (figure 5) [8]. Grid turbulence is created by a uniform stream of fluid passing through a regular array of holes in a rigid sheet or a regular array of bars. The motion downstream consists of the same uniform velocity together with a superimposed random distribution of velocity. This random motion dies away with distance from the grid, and to that extent it is not statistically homogeneous, but the rate of decay is found to be so small that the assumption of homogenity is valid for all purposes [8].

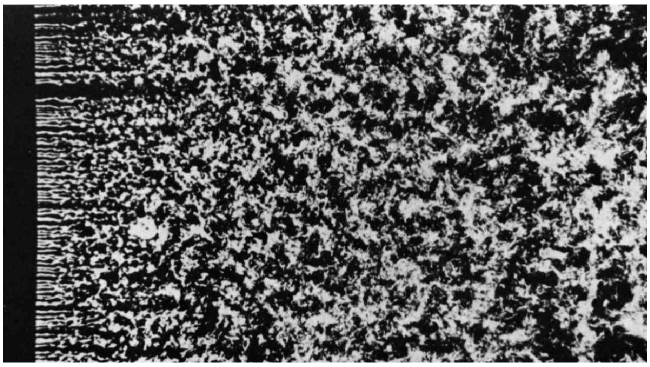

Figure 5: Homogeneous turbulence behind a grid [3]. The pattern was produced by a laminar stream of smoke wires passing through a regular array of holes.

The fundamental quantity to be considered in the statistical description of turbulence is the *velocity correlation tensor* [8]

$$R_{ij}(\mathbf{r}) = \langle v_i(\mathbf{x}) v_j(\mathbf{x}+\mathbf{r}) \rangle \,. \tag{14}$$

The symbol $\langle \sim \rangle$ denotes averaging over the probability distribution, which is the same as taking the space average, since we consider homogeneous turbulence. The velocity correlation tensor is closely related to the energy spectrum of turbulence. This can be seen by considering the Fourier transform of R_{ij}

$$\Phi_{ij}(\mathbf{k}) = \frac{1}{(2\pi)^3} \int R_{ij}(\mathbf{r}) \exp(-\mathrm{i}\mathbf{k}\cdot\mathbf{r}) \, \mathrm{d}^3\mathbf{r}. \tag{15}$$

The spectral function $\Phi_{ij}(\mathbf{k})$ depends in general on the direction of wave vector $\mathbf{k}$. To obtain a function of a single scalar variable $\Psi_{ij}(k)$, we average the expression for $\Phi_{ij}(\mathbf{k})$ over a sphere with radious k [8]

$$\Psi_{ij}(k) = \int \Phi_{ij}(\mathbf{k}) \, \mathrm{d}S_k. \tag{16}$$

The energy per unit mass is given by $\frac{1}{2}\langle \mathbf{v}^2 \rangle = R_{ii}(\mathbf{r}{=}0)$ and therefore

$$\frac{1}{2}\langle \mathbf{v}^2 \rangle = \frac{1}{2} \int \Phi_{ii}(\mathbf{k}) \, \mathrm{d}^3\mathbf{k} = \int_0^\infty \frac{1}{2} \Psi_{ii}(k) \, \mathrm{d}k. \tag{17}$$

The expression under the integral on the right side has to represent the spectral energy density $E(k) = \frac{1}{2}\Psi_{ii}(k)$. For the case of isotropic turbulence $E(k)$ may also be expressed as [7]

$$E(k) = 2\pi k^2 \Phi_{ii}(k). \tag{18}$$

The main conclusion to be made out of this result is that measurements of $\langle v_i(\mathbf{x}) v_j(\mathbf{x}+\mathbf{r}) \rangle$ are sufficient to completely determine the energy spectrum of turbulence [8]. However, in real experiments it is usually more common to measure only the one-dimensional energy spectra [8]

$$E_{ij}(k_1) = \frac{1}{2\pi} \int R_{ij}(x_1, 0, 0) \exp(-\mathrm{i}k_1 x_1) \, \mathrm{d}x_1. \tag{19}$$

In the case $i = j = 1$, $E_{ij}(k_1)$ becomes the *longitudinal spectrum*, while $i = j = 2$ or 3 gives the *lateral spectrum*. If the turbulence is isotropic $E(k)$ is uniquely determined by the form of E_{11} or E_{22} [7].

Let us now focus on one particular theory from this field, namely the Kolmogorov 1941 theory. Kolmogorov's results represent one of the most important findings on turbulence even today, mainly because they indicate some general properties of turbulent flows. English translations of Kolmogorov's original papers may be found in [11, 12]. The focus of his work is devoted to the statistics of small eddies with linear size l much smaller then the integral scale l_0.

The starting point of our discussion are the statistics of velocity increments

$$\delta\mathbf{v}(\mathbf{l}) = \mathbf{v}(\mathbf{x}+\mathbf{l}) - \mathbf{v}(\mathbf{x}), \tag{20}$$

for which we may define the *longitudinal structure function of order p* [5]

$$S_p(l) = \left\langle \left(\delta v_{\|}(l)\right)^p \right\rangle = \left\langle \left[\delta\mathbf{v}(\mathbf{l}) \cdot \mathbf{l}/l\right]^p \right\rangle. \tag{21}$$

At this point Kolmogorov supposed that, for large Re, the small vortices are statistically isotropic, in equilibrium and of universal form [7]. This means that the eddies small compared to l_0 interact with the larger eddies only through the exchange of kinetic energy, while the (possibly) anistropic and unsteady effects of large scales become negligible. In addition, this hypothesis also implies that we do not longer require the turbulence to be accurately homogeneous in terms of the large scales [8]. The regime $l \ll l_0$ is known as the *universal equilibrium range* [7].

Under the above assumptions, the statistical properties of $\delta\mathbf{v}(\mathbf{l})$ may only depend on l, $\langle \epsilon \rangle = \langle 2\nu S_{ij} S_{ij} \rangle$ and ν [7]. If we apply these considerations to the second order structure function $S_2(l)$, we find that it has to be of the form [7]

$$S_2(l) = v_0^2 F(l/\eta) \qquad\qquad (l \ll l_0), \tag{22}$$

where η and v_0 are the Kolmogorov scales, given by (12) and (13), and $F(\cdot)$ is a universal function of the dimensionless argument l/η. When the Reynolds number is large, there may exist a considerable subrange, in which negligible dissipation occurs. Within this *inertial subrange*, the transfer of energy by inertial forces should be the dominant process. This idea led Kolmogorov to the formulation of an additional hypothesis, which states that for values of l in the intermediate range $\eta \ll l \ll l_0$, the statistical properties are completely determined by l and $\langle \epsilon \rangle$ only. If we are to obtain an inertial range of considerable size we need $\mathrm{Re}^{3/8} \gg 1$ [7]. The only possibility of eliminating ν from (22) is if $F(x) \sim x^{2/3}$ [7], which gives for $S_2(l)$ the expression

$$S_2(l) = C \langle \epsilon \rangle^{2/3} l^{2/3} \qquad (\eta \ll l \ll l_0). \tag{23}$$

The above formula is known as the *two-thirds law*. Within the limits of experimental accuracy the $l^{2/3}$ law appears to be correct [7]. On the basis of the same arguments as the ones above, we may also obtain a relation for the energy power spectrum in the inertial range

$$E(k) = A \langle \epsilon \rangle^{2/3} k^{-5/3} \qquad (\eta \ll k^{-1} \ll l_0). \tag{24}$$

This is known as the *Kolmogorov's five-thirds law* and has been also verified experimentally (figure 6). The one-dimensional power spectra E_{11}, E_{22} and E_{33} obey the same power law as $E(k)$ [8].

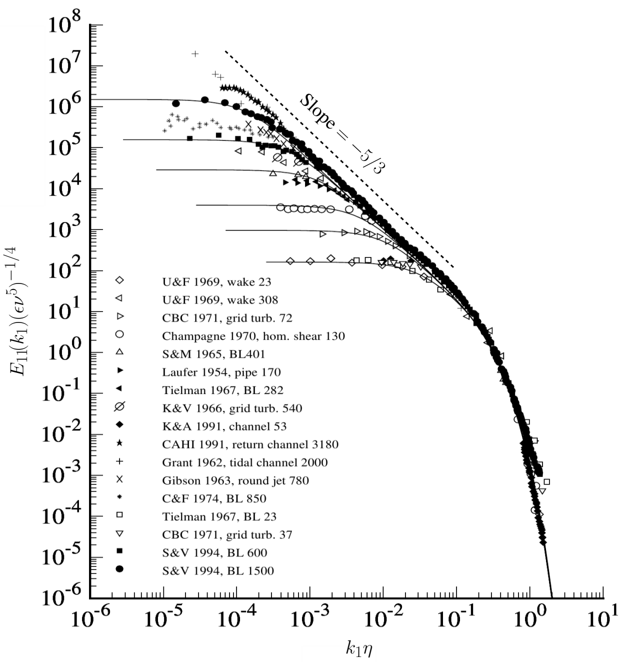

Figure 6: Longitudinal power spectra normalized by the Kolmogorov scales [10]. Experimental data was taken from [13] and incorporates measurements from many experiments including measurements made in boundary layers, wakes, grids, pipes and jets.

Since we obtained the expression for $S_2(l)$ in the inertial range only with the help of dimensional arguments, there is no reason, which would be inconsistent with Kolmogorov's hypotheses, not to go further and write a general relation for $S_p(l)$ as [5]

$$S_p(l) = C_p \big(\langle \epsilon \rangle l \big)^{p/3}. \tag{25}$$

For $p = 3$ we have $S_3(l) = C_3 \langle \epsilon \rangle l$ and in this case Kolmogorov managed to show that this result is exact (with $C_p = -4/5$) in the limit of infinite Reynolds numbers [5]. Like the two-thirds law, this is well supported by the experimental data. However, the consistency with experimental data becomes questionable for $p > 3$ [5]. The reason for the discrepancy lies in the method of averaging the local dissipation $\epsilon = 2\nu S_{ij} S_{ij}$ to obtain an averaged value $\langle \epsilon \rangle$. The energy dissipation is extremely intermittent in space, with regions of large and regions of small dissipation. Therefore, for scales of size l, the energy flux is equal to the local average of ϵ taken over a volume of linear size l [7]. As suggested by L. Landau in a footnote, which appeared in the first edition of Landau and Lifshitz's *Fluid Mechanics*, the result of the global average of ϵ cannot be universal, since an important part is played by the varitions of ϵ on scales comparable to the integral scale [5]. One way to avoid Landau's dilema is to take a rather different global average of ϵ as [7]

$$S_p(l) = C_p \left\langle \epsilon_{AV}^{p/3}(l, \mathbf{x}) \right\rangle l^{p/3}. \tag{26}$$

where $\epsilon_{AV}(l, \mathbf{x})$ is the local average over a sphere with radious l centered at $\mathbf{x}$. Only for the case $p = 3$, where the formula (25) is exact in the limit $\mathrm{Re} \to \infty$, equations (25) and (26) give the same result. For $p = 2$, most statistical models based on (26) predict small corrections to the two-thirds law [7].

4 Direct numerical simulations

Numerical simulations of turbulence have become increasingly popular over the last few decades, due to the a rapid growth of computational power. These simulations have revealed many new insights into the properties of turbulent flows, although it is still relatively difficult to achieve high values of Re (above Re $\sim 10^4$), if we insist on explict calculations for all turbulent scales [7]. This kind of numerical experiments with a detailed treatment of the small scale dynamics are known as *direct numerical simulations* (DNS). DNS computations are often performed in boxes with periodic boundary conditions, since in this case, the geometry permits the use of efficient algorthms based on fourier transforms of the velocity field

$$\mathbf{v}(\mathbf{r}, t) = \sum_{\mathbf{k}} \mathbf{v}_{\mathbf{k}}(t) \mathrm{e}^{\mathrm{i}\mathbf{k}\cdot\mathbf{r}}. \tag{27}$$

The main limitation to these simulations is the rapid increase of computing time with the Reynolds number. To obtain an estimate for the required computational time, we need estimates for minimal number of data points in space domain N_x and number of required time steps N_t. The number of data points N_x is equal to $N_x = (L_{\mathrm{BOX}}/\Delta x)^3$, where Δx is the separation between mesh points and L_{BOX} is the length of one edge of the periodic cube. To include the dissipation scales we should have $\Delta x \sim \eta$ and therefore [7]

$$N_x \sim \left(\frac{L_{\mathrm{BOX}}}{l_0} \right)^3 \mathrm{Re}^{9/4}. \tag{28}$$

Similarly, the simulation time step should be of the order $\Delta t \sim \eta/u$. This gives us for the number of required time steps the estimate [7]

$$N_t \sim \frac{T}{\Delta t} \sim \frac{T}{l_0/u} \mathrm{Re}^{3/4}, \tag{29}$$

where T is the total duration of the simulation. The total computing time T_C is roughly proportional to $N_x N_t$, so that we finally arive to the relation [7]

$$T_C \propto \left(\frac{T}{l_0/u} \right) \left(\frac{L_{\mathrm{BOX}}}{l_0}^3 \right) \mathrm{Re}^3. \tag{30}$$

The computational time in a direct numerical simulation grows roughly with the 3rd power of Re. Estimates for T_C at different values of L_{BOX}/l_0 and Re on a petaflop computer are given in table 1.

L_{BOX} $[\times l_0]$	Re = 1000	Re = 5000	Re = 10,000
10	10 s	15 min	2 h
20	1 min	2 h	17 h
50	15 min	1 day	9 days
100	2 h	9 days	2 months

Table 1: Estimates for the computational time on a petaflop (10^{15} operations per second) computer. The numbers were evaluated from estimates for a 1 teraflop computer in [7]. The total simulation run assumed in [7] was five eddy turn-over times, $T = 5(l_0/u)$ and the resolution was $\Delta x = 2\eta$ [7]. According to the 'top 500' list (`http://www.top500.org/`) of supercomputers, there are currently about 20 computers around the world with computational power above 1 petaflop.

5 Conslusions

We have seen that turbulence is a very exciting and difficult subject, and still almost 200 years since Navier (1823) [5] wrote the equations for fluid motion, we are lacking a general theory on turbulence. Many questions therefore still remain open, while others were answered with the help of various approaches ranging from the dimensional arguments by Kolmogorov to experimental findings and more recentely also extensive numerical simulations.

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