Origin of Scaling on Networks, Structural Inhomogeneity and Preference in Dynamical Behaviour

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ABSTRACT

We examine the fluctuation properties of packet traffic on scale-free networks and random graphs using two different dynamical rules for moving packets; random diffusion and a locally navigated diffusive motion with preferred edges. We find that preferential behaviour in either the topology or in the dynamics leads to the scaling of fluctuations of the number of packets passing nodes and the number of packets flowing along edges, respectively. We show that the absence of any preference results in the absence of scaling, and when scaling occurs it is non-universal with the scaling exponents depending on the acquisition time window, the network structure and the diffusion rule.

1. INTRODUCTION

Most real systems exhibit complex dynamical behaviour. An interesting property of complex systems is that the scaling of fluctuations found in rivers, stock markets, computer networks, WWW and electric circuits.[1,2] This occurs when the average activity $< X_i >$ of the component i of a system is related to the dispersion σ_i of its time series by power law behaviour $\sigma_i \sim< X_i >^{\mu}$, where the value of the exponent μ is between 0.5 and 1.0. Initial studies of the systems[1] found the exponent μ takes only two values: 0.5 or 1.0, which were related to the internal and the external dynamics, respectively. However, the simulations on networks found the value of μ to be dependent on the traffic parameters like input rate R and packet's life time S or the process of data acquisition by the value of time window T_{WIN}.[3,4] Similar dependencies of the acquisition time window are observed in the analysis of the empirical time-series of the stock markets[5] and in the gene expression data, where the natural time window is determined by the cell-cycle dynamics.[6] The occurrence of scaling and reasons of its nonuniversality have been subject of debate,[1–5] with conclusions sometimes obscured by the nature of the empirical data or limitations in the applied models. Recently we have introduced a fairly realistic model for traffic of information packets (see review and references in[7]), with dynamic creation of packets, their navigation towards given address, packet queuing at nodes and delivery upon arrival. By simulating traffic of packets on an uncorrelated scale-free network with the edge-preferred navigation rules, we have shown[4] that certain preference in either topology (i.e., for nodes on a scale-free network), or in the dynamics (i.e., for dynamically preferred edges), is necessary for the occurrence of the scaling. Furthermore, we have shown that the nonuniversal dependence of the exponent μ on the time window appears to be different for nodes and for edges, and determined the related return-time distributions.[4]

In this work we extend the study of Ref.[4] in order to further investigate the role of network topology and the role of navigation rules in the occurrence of scaling. For this purpose we study in parallel:

- traffic on scale-free network and on the random graph;
- random diffusion and a probabilistic edge-preferred local navigation rule on both networks;
- fluctuating time series recorded at nodes, $\{h_i(t)\}$, and at edges, $\{f_{ij}(t)\}$, of the network within a specified time window T_{WIN}.

Therefore we define two types of the dispersion relations, for the actrivity of nodes and for flow along the edges:

$$\sigma_i \sim < h_i >^{\mu} \; ; \quad \sigma_{ij} \sim < f_{ij} >^{\mu} \; . \tag{1}$$

Our findings in this larger class of network structures and dynamic rules confirm the necessity of a preference for the scaling of the fluctuations in Eq. (1). Furthermore, we demonstrate that the variations of the values of the scaling exponents with the time window, both for nodes and links, are strictly related to the network topology and to the navigation rules.

We introduce the network structures and navigation rules and briefly describe the basic traffic properties in Section 2. In Section 3 we focus on the fluctuations of the time series recorded at nodes and at edges from the simulations with the random diffusion on both network types and similarly, in Section 4, the results for the edge-preferred navigation on both network types. In Section 5 we give a short summary of the results.

2. NETWORK STRUCTURES AND TRANSPORT RULES

Networks. To stress the role of network structure on the fluctuating time series we study network with a scale-free connectivity distribution and a random graph. In both cases the network consists of $N = 1003$ nodes and $E = 2N$ edges (links). We grow the *scale-free network* with the preferential attachment described in detail in the book.[11] The network with the connectivity $k_i \sim (i/N)^{-1/(1+\alpha)}$ at ith added node emerges, leading to the degree distribution $P(k) \sim k^{-(2+\alpha)}$, with $\alpha = 0.5$ used. The *random graph* of the same size and number of edges we make starting from N nodes from each of which $m = 2$ links are randomly connected to two other nodes. Multiple links are not allowed and we also take care to produce the graph with all nodes connected to the giant cluster. Although the avegare connectivity per node is $< k >= 2$, our random graph has nodes with the connectivity varying in the range from 2 to 9. It should be stressed that both networks are uncorrelated and have low clustering coefficient. Once the networks are generated, we consider their structure fixed, and given by the adjacency matrix C_{ij}.

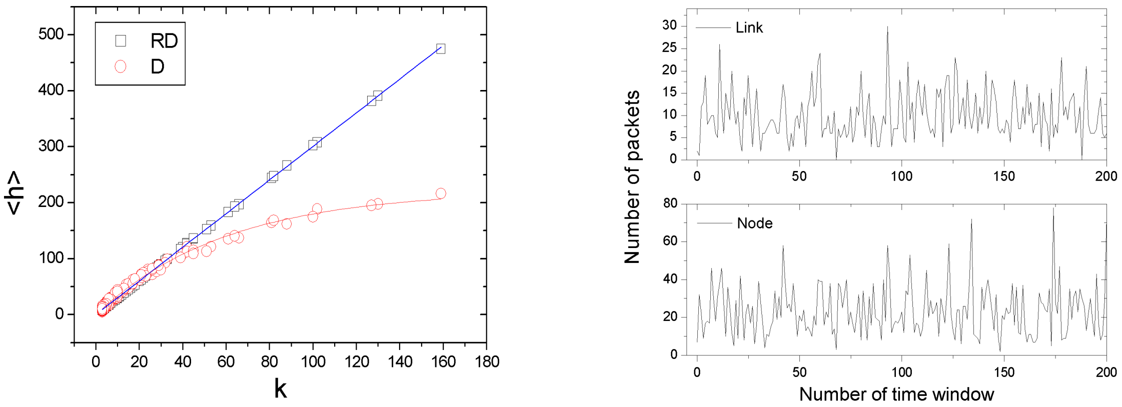
Navigation rules. We simulate the transport of packets on these networks within the traffic model.[7,8] The packets are created with a rate R and each packet is given a destination address where it us eventually delivered and removed from the traffic. The packets are moved through the network in parallel using a local navigation rule.[7,12] Packets are queuing at nodes with the FIFO preference queue. Here we use two strictly local rules to navigate packets towards their in-advance specified destinations. These rules are defined with the probability p_{ij} that a node i forwards the packet towards one of its neighbour nodes j:

$$p_{ij}^{D} = 1 - \frac{k_j}{\sum_{j=1}^{N} C_{ij} k_j} \; ; \quad p_{ij}^{RD} = \frac{1}{\sum_{j=1}^{N} C_{ij} k_j} \; ; \tag{2}$$

where C_{ij} is the adjacency matrix and k_j is the degree of node j and subscripts RD and D denote *random diffusion* and *degree-dependent* D-navigation rule, respectively. Note in contrast to Refs. ,[7,12] the D-rule in Eq. (2) does not imply any in-depth search, and thus it is similar to the random diffusion. However, according to the D- rule, the *edge pointing towards the less-connected neighbour node is dynamically preferred*. In effect, the central node loses its topological preference. This is in contrast to the random diffusion, where ithe number of visits of a random walker to a node are proportional to node's connectivity,[?] i.e., average number of packets processed by a node i, $< h_i >$ is given by node's degree k_i

$$< h_i > \sim k_i . \tag{3}$$

Consequently, the average number of packets processed by a node, as shown in Fig. 1 a, is different in the two dynamic rules. With the egde-preferred local rule D, described above, we observe the dynamic homogeneity of the network, i.e., $< h_i > \sim$ *const* for large k_i. In the following we study in detail the fluctuating time series which are representing the time fluctuations of the number of packets processed by nodes, $\{h_i(t)\}$, for all $i = 1, 2 \cdots N$ nodes in the network and time fluctuations of the number of packets processed along a link (packet flow), $\{f_{ij}(t)\}$, for all E links (connected pairs ij) in the network. Examples of such time series for the scale-free network and two diffusion rules are shown in Fig. 1b.

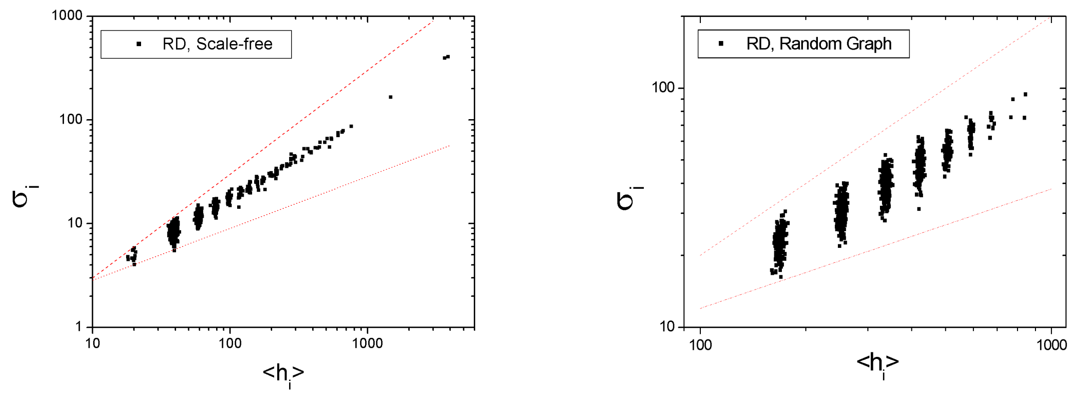


(a) (b)

Figure 1. (a) The average number of packets processed by a node against node degree for random diffusion (RD) and navigated diffusion (D) (Fits: $f(x) = ax^b, b = 1.0$ and $f(x) = a\left(1 - e^{-\frac{x}{x_0}}\right), x_0 \approx 59$). (b) Example of time series recorded at a preferred node with random diffusion rules (bottom) and time series recorded at a preferred edge with the D navigation algorithm for time-window $T_{WIN} = 1000$ steps.

3. SCALING OF FLUCTUATIONS FOR RANDOM DIFFUSION ON NETWORKS

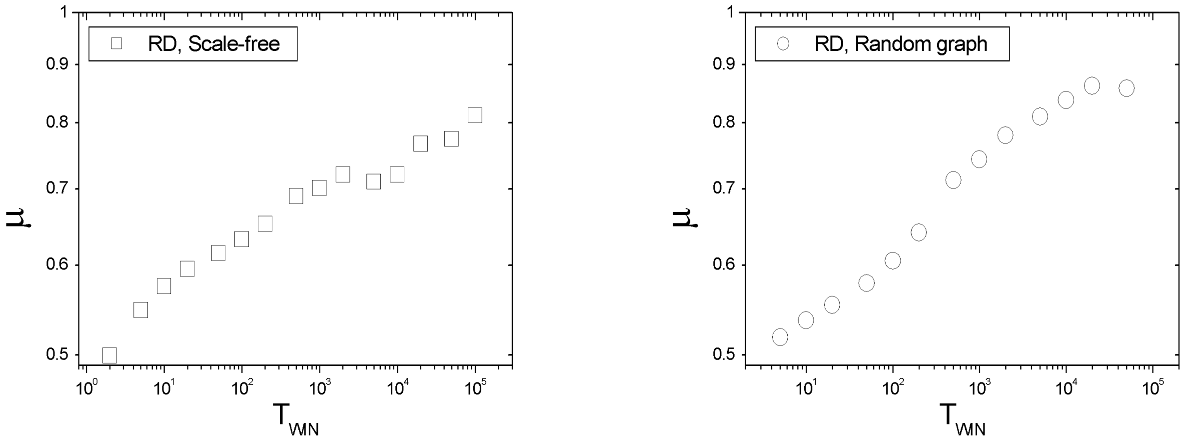
In this Section we investigate the scaling of noise fluctuations $\{h_i(t)\}$ for random diffusion process on two types of underlying structures, the scale-free network and the random graph, described in the previous section. Fig. 2 shows the relation between the dispersion σ_i and the averaged noise $< h_i >$, where each node i is represented as a point. The plots for the scale-free network and the random graph follow the general scaling relation in Eq.1. In both cases the scaling exponents μ are between the border values 0.5 and 1.0 (indicated by the thin lines). According to Eq. 3 the average noise $< h_i >$ is related to node's degree k_i. This property results in clearly



(a) (b)

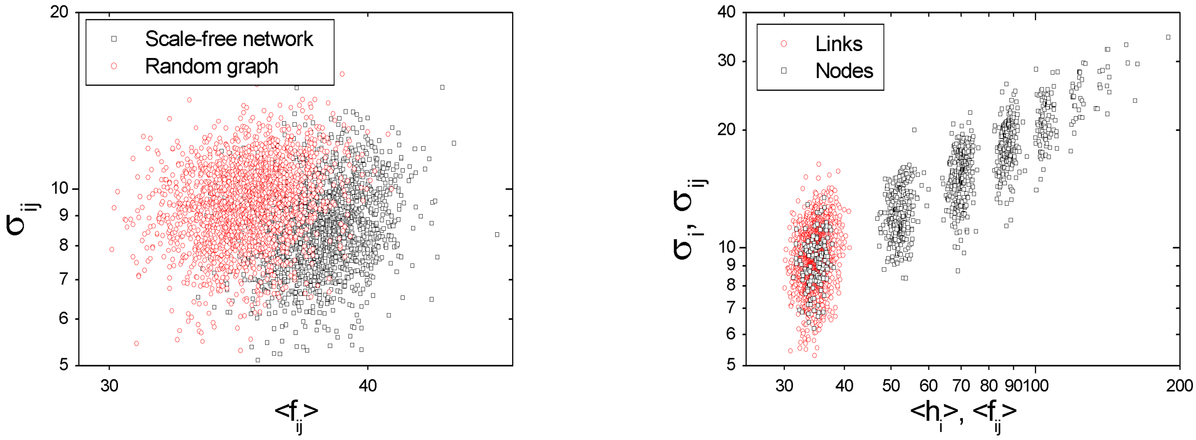
Figure 2. Random diffusion: Dispersion σ_i against average $< h_i >$ of the time series recorded at nodes of the network within a fixed time window $T_{WIN} = 4000$ on (a) scale-free network and (b) random graph.

separated groups of points, where each group contains only nodes with the same degree. The value of the scaling exponent μ may depend on the traffic conditions[3] such as input rate R or the closeness to jamming,[?] and on the acquisition time window T_{WIN}.[2–5] In our simulations we use imput rates R much below the jamming.[?] We further investigate the dependence on the time window $\mu(T_{WIN})$ in the case of both our network structures. The results are presented in Fig. 3.



(a) (b)

Figure 3. Random diffusion: Dependence of the scaling exponent μ of the node activity fluctuations $\sigma_i \sim < h_i >^{\mu}$ on the width of the time window T_{WIN} on scale-free network (a) and on random graph (b).



(a) (b)

Figure 4. Random diffusion: (a) Dispersion σ_{ij} against average flow $< f_{ij} >$ of the time series recorded at links of the network within a fixed time window $T_{WIN} = 4000$ on a scale-free network and a random graph. (b) The comparison of the fluctuations of the node activity and flow along the links on the random graph for the random diffusion process. Several groups of nodes are distinguishable, according to their connectivity, whereas all links fall into a single group.

In a way similar to the multi-channel analysis for nodes, presented above, the same technique can be applied to analyse the fluctuations of flow along the links. In this case we measure the *flow*, f_{ij}, which is defined as the number of packets posted down from $i \to j$ and from $j \to i$ within a given time window T_{WIN}. The results of the relation between the flow dispersion σ_{ij} and average flow $< f_{ij} >$ are plotted in Fig. 4a for the scale-free network and the random graph. In this plots each point represents one edge of the network. In this case, however, no scaling was found for both network structures. This result can be easily understood in view of the Eq. (3) if we consider a link as an element with two inputs/outputs, similar to a node with degree $k = 2$. Indeed, all network links form a single group (cf. Fig. 4a,b). Moreover, the group of all links overlaps with the group of nodes with the degree 2,

$$< f_{ij} \approx < h_i >_{k_i=2}, \tag{4}$$

as shown in Fig. 4b in the case of random graph. In this figure, groups of nodes with increasing connectivity are systematically shifted to the right, in agreement with Eq. (3).

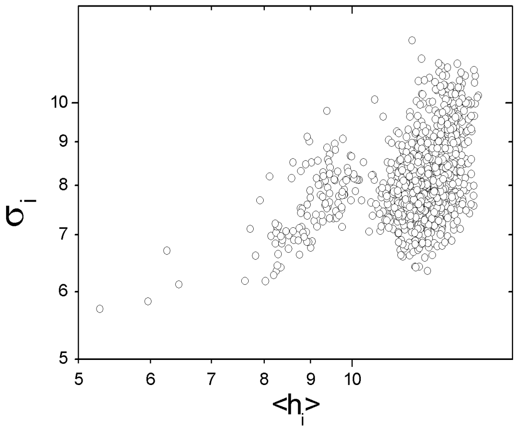


Figure 5. Random diffusion on a regular square lattice with 32×32 nodes: Dispersion σ_i against average node activity $< h_i >$. Four corner nodes (left), the groups of boundary nodes (middle) and the interior nodes (right) are distinguishable.

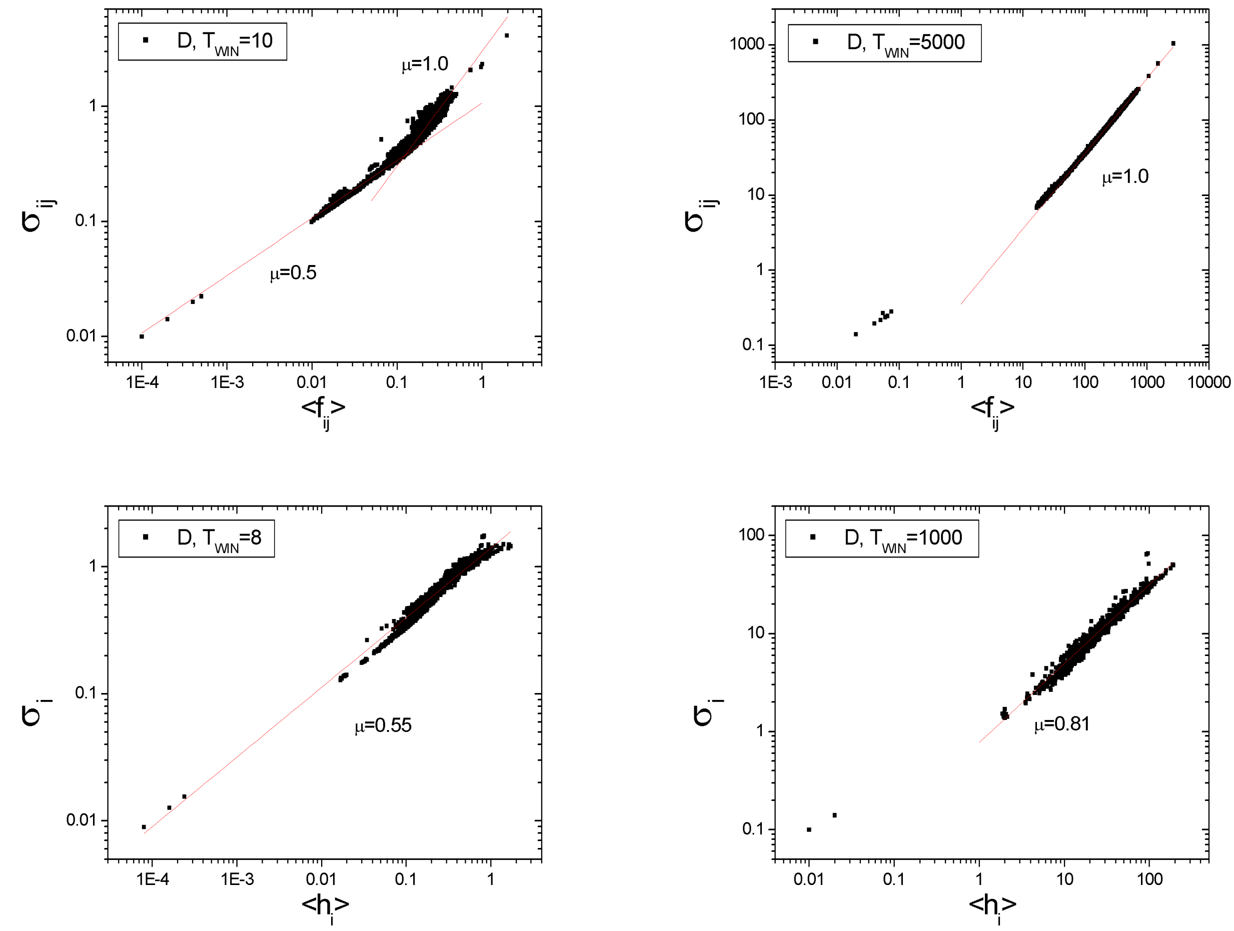
The close relationship between the node's connectivity and the differentiation between node groups in the plots, as in Fig. 3 and 4b, is characteristic for the random diffusion processes. In short, the role that a node plays in teh random diffusion is entirely determined by number of links attached to it. This can be seen even in the simple structures as a regular square lattice with open boundaries, shown in Fig. 5, where groups of boundary nodes are differentiated from the interior nodes.

We can now conclude that for the random diffusion, the scaling of fluctuations in Eq. (**??**) occurs only in the systems where the diversification of some property (like node degree) is present. In the studied networks the degree distribution for scale-free network and even the random graph provides enough diversification in degree values to observe the scaling in node activity fluctuations. In the case of flow fluctuations, however, we do not find any scaling property because in the random diffusion every link transfers almost the same number of packets. In the next section we will show that, although the links on both scale-free and random graph are topologically equally important, the dynamical preference within the navigated diffusion rule (D) will induce the flow differentiation that is necessary for the scaling to appear.

4. SCALING OF FLUCTUATIONS FOR EDGE-PREFERRED NAVIGATION

We adopt the local navigation rule with the edge-preferred diffusion, defined by in the Eq. (2) with the probability p_{ij}^D. As discussed in Section 2, with this rule, the packets are preferably posted along the adges pointing towards the neighbour node with lesser degree. The rule is more effective at nodes close to the hub in the sacle-free network, precisely, the the packets are awoiding the large-degree nodes, whereas the nodes at the graph boundary and in the random graph, the majority of nodes have similar degree and thus small differences between edges connecting them could only weakly affect the dynamics. Note that we study here the diffusion rule D for the

(a) (b)



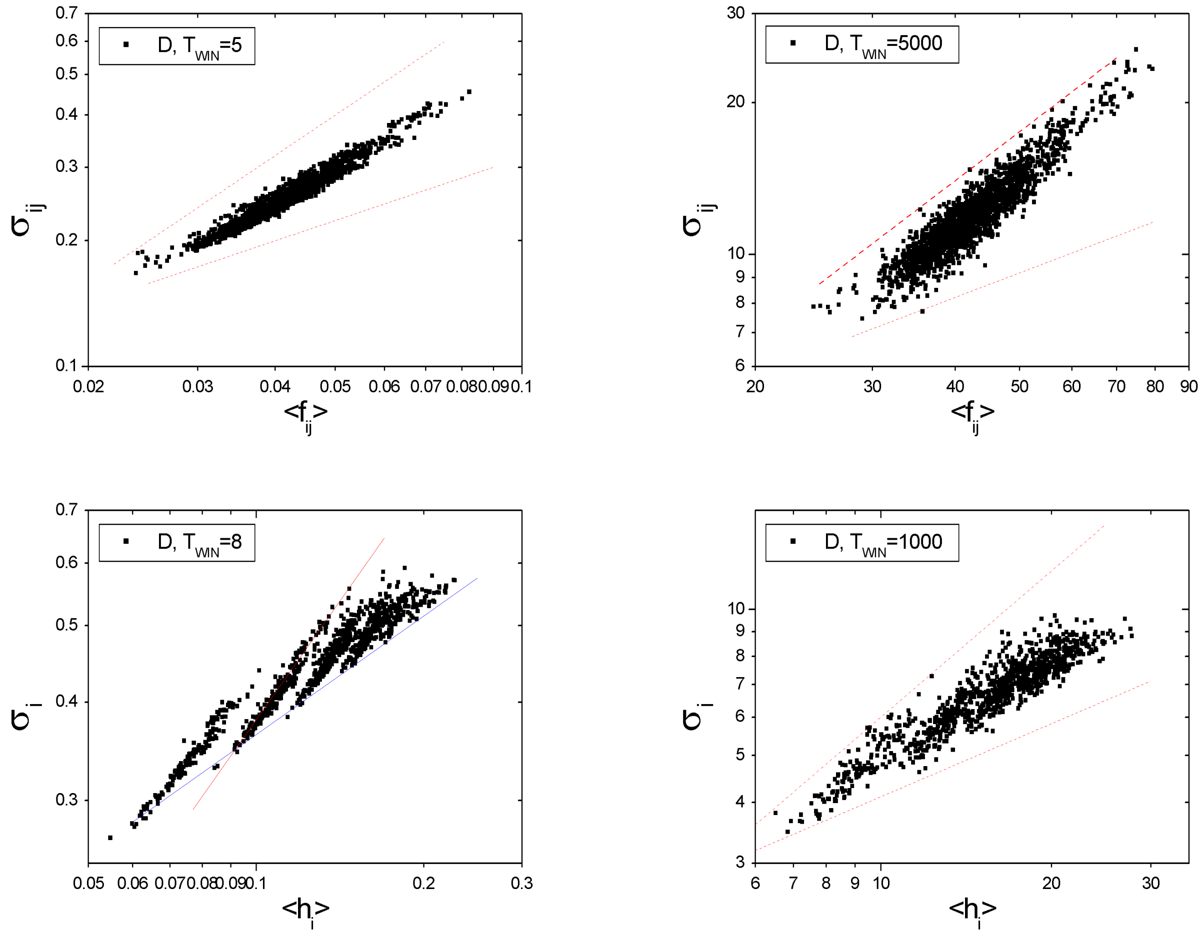
(c) (d)

Figure 6. Edge-preferred D-navigation on scale-free network: Dispersion σ_{ij} of flow along the edges against average flow $< f_{ij} >$ for the time window $T_{WIN} = 10$ (a) and $T_{WIN} = 5000$ (b). Dispersion σ_i of node activity against average activity $< h_i >$ for time window $T_{WIN} = 8$ c) and $T_{WIN} = 1000$ (d).

purpose to demonstrate the origin of scaling at network edges. We do not discuss here the potential effects of the rule on the traffic efficiency (see refs.[7,8,12–14]).

Fig.6a–d and Fig.7a–d show the relation between dispersion σ_{ij} and the averaged flow $< f_{ij} >$ for the scale-free network and the random graph, respectively. In the case of the scale-free network, as a new feature of the navigated diffusion we find the scaling in the flow fluctuations, see Fig. 6a,b. In particular, we find two scaling regimes with exponent μ either 0.5 or 1.0. Further study has shown that the occurrence of two scaling regimes is rather robust to the variations in the time window T_{WIN}. However, we find that the population of points, representing different links of the graph, migrate from upper part of the scatter plot with slope $\mu = 1$ to the lower part, where the slope $\mu = 1/2$ was found, when the time window is *decreased*. For the limiting case $T_{WIN} = 1$ almost all points fit to the curve with exponent μ equal to 0.5. Hence, a continuous variation of $\mu(T_{WIN})$ is absent, in contrast to the scaling of the node activity. For the edge-preferred navigation the plots of node activity on the scale-free graph are also shown in Fig. 6c,d. We find the qualitatively similar behaviour as in the case of random diffusion, namely, the nonuniversal scaling with the exponent continuously varying with the time window. However, the numerical values of the exponents are different for the two diffusion rules. The dependence $\mu(T_{WIN})$ is plotted in Fig. 4a.

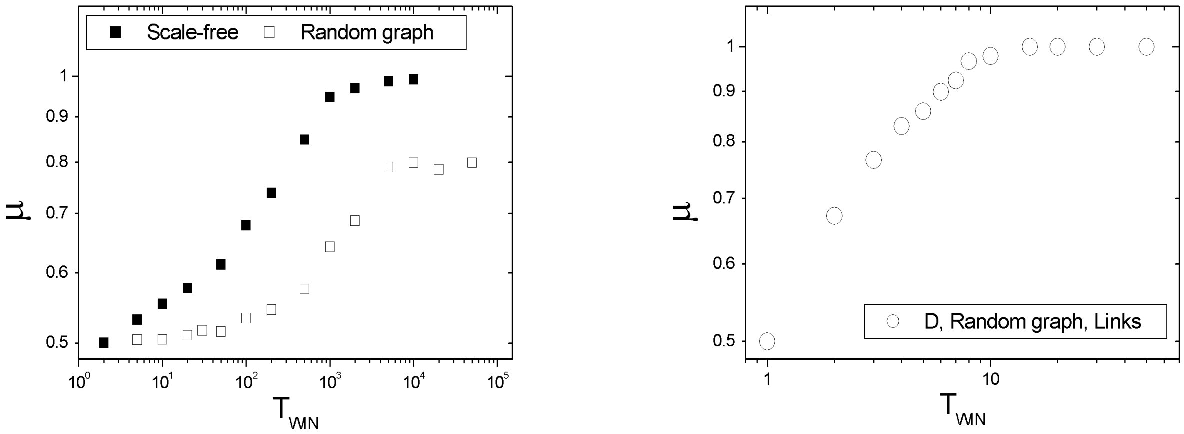
(a) (b)



(c) (d)

Figure 7. Edge-preferred D-navigation on random graph: Dispersion σ_{ij} of flow along edges against average flow $< f_{ij} >$ for time window $T_{WIN} = 5$ (a) and $T_{WIN} = 5000$ (b). Dispersion σ_i of node activity against average activity $< h_i >$ for time window $T_{WIN} = 8$ (c) and $T_{WIN} = 1000$ (d). Thin lines indicate slopes $\mu = 1$ and $\mu = 1/2$.

For the edge-preferred navigation on the random graph the scatter plots are shown in Fig.7a–d. For the fluctuations of flow on edges in this case we find a qualitatively similar behavior as for the fluctuations of node activity. In particular, the scaling of Eg. (1) occurs with a continuously varying exponent when the time window is changed. The numerical values, however, are different. As shown in Fig. 4b for the flow fluctuations, the exponent drops below the value $\mu = 1$ only for very small time windows. Whereas, in the case of node activity for, see Fig. 4a, lower curve, the characteristic crossover region between $\mu \sim 0.5$ and $\mu \sim 0.8$ was found. Note that for the node activity fluctuations the range of T_{WIN} where exponent μ scales is much larger, but the maximum value of μ we found is still lower than 1.0 and in this particular case it is around 0.8. We run our simulations on different random graphs (constant N=1000) and μ_{max} depends on a graph ensemble used in the simulation, however, for all cases μ_{max} were always smaller than 1.0 Note also that different groups of nodes are also distinguishable in the navigated diffusion on the random graph, Fig. 7c,d, whereas such groups can not be identified in the case of edges, Fig. 7a,b. In contrast to the random diffusion, the edge-preferred navigation induces the dynamical difference between edges, that results in the occurrence of scaling in the plots both for scale-free and for random graph, Figs. 6 and 7. The study of the node activity fluctuations for D navigation algorithm shown in Fig.6c,d and Fig.7c,d reveals different dependences of the scaling exponent on the time window, compared to the random diffusion on the same graphs, but also it shows the differences between the traffic on a random graph and on a scale-free network.



(a) (b)

Figure 8. Edge-preferred diffusion: Dependence of the scaling exponent μ on the width of the time window T_{WIN} for: (a) fluctuations of the node activity for the scale-free network (filled squares) and the random graph (empty squares), and (b) fluctuation of flow on edges on the random graph.

5. CONCLUSION

Using the model of traffic on networks with packet creation and delivery, local navigation and queuing at nodes, we analysed the fluctuations of time series of node activities and traffic flow along the links. Two types of networks—uncorrelated scale-free network and a radnom graph and two local diffusion rules—random diffusion and edge-preferred navigation are applied to study the occurrence and universality of scaling defined by Eq. (1). Our findings, summarized in sections 3 and 4, confirm that for the occurrence of the power-law behavior in the scatter plots in Figs. 2,4,6,7, a certain preferential behavior in the diffusion is necessary.[?] Such preference is either induced by the topology, i.e., dispersion of node's connectivity which directly influences the random diffusion process, or by local alteration in the diffusion rules, in our case the edge-preferred diffusion is related with the degree of the node across the link. Moreover, for the fluctuations of node activity, the span of the scaling region in the plots is directly related to the span of the node's connectivity. This results suggest that the scatter plots of the type discussed here can be used for the structure recognition, when the diffusion process is known. We demonstrated systematical dependence of the scaling exponents with the width of the acquisition time window. Generally, the exponents increase with larger time windows, however, the functional dependences $\mu(T_{WIN})$ are related to the network and to the diffusion rule.

We also gave convincing arguments that in structurally inhomogeneous networks, such as the scale-free structures, for the time series measured atnetwork edges (i.e., in the case of edhge-preferred navigation) the scaling features are different from those obtained for the node activity fluctuations. Therefore, it is important to make the distinction for instance in the nalysis of the empirical data of the traffic on the Internet, where usually the flow along an edge is monitored. We would like to stress that for the purposes of this work the network structures that we studied have low clustering (vanishing in the $N \to \infty$ limit). In more realistic packet traffic models both in-depth search algorithms and corerlated network structures may lead to additional features in the scaling of the nose and flow time series.[7,9]

Acknowledgments: We acknowledge support from Programme P1-0044 of the Ministry of Higher Education, Science and Technology of the Republic of Slovenia, the British Council Partnerships in Science Project 22/2006 and the EC Marie Curie Early Stage Training Programme NET-ACE (MEST-CT-2004-6724) and COST P10 action.

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