

Hierarchical sequencing of online social graphs

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Abstract

In online communications, patterns of conduct of individual actors and use of emotions in the process can lead to a complex social graph exhibiting multilayered structure and mesoscopic communities. Using simplicial complexes representation of graphs, we investigate in-depth topology of online social network which is based on MySpace dialogs. The network exhibits original community structure. In addition, we simulate emotion spreading in this network that enables to identify two emotion-propagating layers. The analysis resulting in three structure vectors quantifies the graph's architecture at different topology levels. Notably, structures emerging through shared links, triangles and tetrahedral faces, frequently occur and range from tree-like to maximal 5-cliques and their respective complexes. On the other hand, the structures which spread only negative or only positive emotion messages appear to have much simpler topology consisting of links and triangles. Furthermore, we introduce the node's structure vector which represents the number of simplices at each topology level in which the node resides. The total number of such simplices determines what we define as the node's topological dimension. The presented results suggest that the node's topological dimension provides a suitable measure of the social capital which measures the agent's ability to act as a broker in compact communities, the so called Simmelian brokerage. We also generalize the results to a wider class of computer-generated networks. Investigating components of the node's vector over network layers reveals that same nodes develop different socio-emotional relations and that the influential nodes build social capital by combining their connections in different layers.

Keywords: Multiplex networks; Simplicial complexes; Online social networks; Simmelian brokerage

1 Introduction

Structure of online social networks emerges via self-organizing processes of social dynamics, where the links are being established and used for communications between individuals. The contents (information, emotion) communicated between pairs and groups of participants affects their activity patterns and thus shape the network's evolution. Two prototypical classes of online social networks can be distinguished (1): a hierarchically organized multi-layered structure that reflects the level of knowledge of the involved individuals in the chats-based systems, on one hand, and a wider class of networks with community structure, on the other. Recently, online social networks of both types have been studied based on high-resolution empirical data from a variety of Web portals (2; 3; 4; 5; 6; 7; 8; 9; 10; 11; 12).

In social sciences, the *value of social networking* can be quantified. Specifically, the *social capital*, as an essential element of functioning social networks, measures how social relationships create a competitive advantage of certain individuals at the expense of others (13). For instance, the Simmelian ties and the related brokerage roles measure how an individual can act as a broker (transmitting actor) in a dense social community (14). The availability of massive empirical data of online social interactions and their dynamics that completes the information for the study of social graphs offers the possibility to investigate the mechanisms of social capital build-up by different actors. In view of ever-evolving social graphs, intricate communication processes result in the social-value relationships within communities where the social layers may play a role. Analysis of such social structures on multilayered networks requires new mathematical approach, capable of taking into account the inner (in depth) structure of each society and the role of different nodes in it.

In recent research, efforts have been made on determining the network complexity metric that permits to successfully distinguish between critical and redundant nodes (15), discover the active cores of the network as compared to the network's periphery (16; 10) and quantify the network's multiplexity (17; 18; 19; 20; 7; 11; 21) and the role of higher-order structures in the network dynamics (22; 23; 24). In this respect, several approaches have focused on introducing suitable graph-theoretic vectors that can be defined on local graphlets (25), the network's feature vector (26) or the graph's eigenvalue spectrum (27). These approaches proved very useful in the study of biological systems, for example, in alignment of protein networks (28) and uncovering network function in cancer-related processes (29). Similarly, characterizing topology of molecular graphs (30) as well as determining modules in socio-technological networks by eigenvectors localization (31) have been successful.

In this work, we exploit the topological concept of a *simplex*, structure that extends beyond the nodes and links, i.e., a polyhedron of possibly high dimension, and their aggregates or *simplicial complexes*. We investigate in-depth topology of online social networks and explore the role of nodes in layers and communities. The concept of simplicial complexes of graphs (32; 33) allows precise defi-

dition, using topological, algebraic and combinatorial tools, of the node's natural surroundings in the network. Consequently, it allows to determine the node's social capital such as the Simmelian brokerage in social environments.

The considered network in this paper is constructed from the original data collected from MySpace social network as described in Ref. (5). Typically, in online social networks, such as MySpace and Facebook, certain kind of social graphs exist a priori. However, the use of connections over time as well as the dominance of positive emotions in the texts of messages (5) reveal the dynamical structure that is different from the conventional social networks.

In general, network layers appear due to different types of relationships among nodes (34; 10; 11). In MySpace and Facebook social networks, where text messages of mixed information contents are communicated, the emotion contained in these words can be inferred (35; 5). This fact offers the possibility to identify the network's layered structure in a unique manner. Specifically, one can define layers that propagate emotions with positive or negative valence (attractiveness and aversiveness). Quantitative study of emotions, based on Russell's model (36), and the social dimension of emotional interactions are the subject of an intensive research in recent years (see the summary of the Cyberemotions project (37)). In this context, the dynamics of emotion spreading on networks has been investigated by an agent-based model (38; 39). In this paper, we employ this model to generate network layers. The agents are interacting along the social links of MySpace network influencing each-other's emotional state by exchanging messages. The identity of each message in the simulations is known, its time of inception, source and recipient nodes as well as its emotional content that matches the current emotional state of the agent who created it. Consequently, the links carrying negative/positive emotion messages can be distinguished; together with the involved nodes these links define the corresponding network layer.

The paper is organized as follows. The standard structure of the network, as well as the dynamics leading to the emotion-propagating layers, is briefly described in Section 2.1. The basic definitions and the method are described in 2.2. The analysis of simplices and their complexes in the communities, section 2.3, and in the emotion-propagation layers of the network, section 3, is presented. Section 4 deals with the functional relationship between Simmelian brokerage role of a node and its topological dimension. Section 5 contains a brief summary of the results and conclusions.

2 Simplices: Beyond standard structure of social graphs

2.1 Dynamic architecture of online social networks

In this work we use the online social network from the empirical data of Ref. (5), consisting of the links in MySpace along which the messages were exchanged within 2-months time window. The original data and the network mapping are described in (5) together with the study of the dynamics of emotions detected in

the data. The structure of these networks has been defined in terms of several measures. In particular, the degree- and strength- distributions, link correlations—disassortativity, reciprocity, path lengths, clustering, community structure, and testing the weak-tie hypothesis have been determined in Ref. (5) and for the purpose of this work, this study will be termed *standard analysis of a social graph*. These results revealed that the MySpace dialogs graph exhibits the characteristic social community structure. A similar conclusion was derived in Ref. (40) considering the static structure of the Facebook graph. However, the analysis in (5) uncovers that several other topology measures, notably, disassortativity, non-reciprocal in- and out-linking, the role of weak-ties, are different from the features often found in social networks. Moreover, they share a high similarity with the networks of online games studied recently in (6). Together with some other findings (41; 42), the observations in (5) suggest that the dynamical organization of online social networks can be strikingly different from that of the conventional social graphs. The origin of such structure can be traced in the dynamics of exchanged messages with their emotional contents (35; 5; 43; 38). Hence, the patterns of conduct and the role of individuals in these processes (44; 7; 45) are crucial for the emergence of macro-social temporal structures. Therefore, an in-depth topology beyond the standard analysis of the emergent social networks can reveal some of the working mechanisms at the local level. In the remaining parts of this work, we extend the analysis of these networks by means of simplicial complexes. For this purpose, the original network which consists of 32000 nodes is reduced to 3321 nodes by removing nodes with a low connectivity (5). The dynamics of emotional messages is simulated using the agent-based model represented in (39).

2.2 Simplicies and simplicial complexes of graphs

In contrast to the standard graph theoretical approach which reveals connectivities, a topological framework (32) provides information about the structure and patterns of connectivities and enables the multifaceted approach including topological, algebraic and combinatorial techniques. A basic element of the topological approach is a simplex whose spatial representation is a polyhedron. The polyhedra of various dimensions (i.e. the number of vertices and faces) may be connected to each other forming a polyhedral complex, the spatial embodiment of a simplicial complex. In a more formal approach, any subset of a set of vertices $V = \{v_{\alpha_0}, v_{\alpha_1}, \dots, v_{\alpha_n}\}$ determines an n -simplex denoted by $\langle v_{\alpha_0}, v_{\alpha_1}, \dots, v_{\alpha_n} \rangle$ where n is the dimension of the simplex. A q -simplex σ_q is a q -face of an n -simplex σ_n , denoted by $\sigma_q \lesssim \sigma_n$, if every vertex of σ_q is also a vertex of σ_n . The union of faces of a simplex σ_n is a boundary of σ_n . A simplicial complex represents a collection of simplices and its dimension is the largest dimension of its simplices. More formally, a simplicial complex K on a finite set $V = \{v_1, \dots, v_n\}$ of vertices is a nonempty subset of the power set of V , so that the simplicial complex K is closed under the formation of subsets. Hence, if $\sigma \in K$ and $\rho \in \sigma$, then $\rho \in K$. An example of a simplex of dimension zero is a point. A simplex of dimension one is a line; two-dimensional

simplex is a triangle, three-dimensional simplex is tetrahedron and so on.

Simplicial complexes may be constructed directly from the available data or they may be formed from the undirected or directed graphs (digraphs) in several different ways. Here we only consider two of them: the neighborhood complex (46) and the clique complex (47). The neighborhood complex $N(G)$ is constructed from the graph G , with vertices $\{v_1, \dots, v_n\}$ in such a way that for each vertex v of G there is a simplex containing the vertex v , along with all vertices w connected to it and the corresponding faces. The neighborhood complex is obtained by including all faces of those simplices and in terms of matrix representation, the incidence matrix is obtained from the adjacency matrix of G by increasing all diagonal entries by 1.

The vertices of the clique complex $C(G)$ are the same as the vertices of G with the maximal complete subgraphs (cliques) as simplices so that it is essentially the complete subgraph complex. An example of the clique complex construction is presented in Fig. 1.

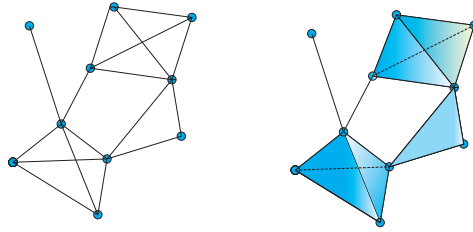


Figure 1: Construction of the clique complex (right) from the graph (left)

These two methods are not the only ones which may be used for constructing simplicial complexes from graphs. Actually, any property of the graph G that is preserved under deletion of vertices or edges may be used for construction purposes. A detailed account of the methods for obtaining simplicial complexes from graphs, among many other issues related to the relationship between graphs and simplicial complexes, may be found in (32).

Initially all maximal cliques (MC) were found using the Bron-Kerbosch algorithm (48). The resulting MC matrix contains full information about cliques, the clique's ID, size and IDs of nodes that are involved in it; from this the connection between different simplices can be deduced (48), (33).

As mentioned earlier, the power of the topological approach is based on the fact that simplicial complexes may be considered from three different aspect: (1) a combinatorial model of a topological space; 2) a combinatorial object; 3) an algebraic model. Hence, different measures and invariants may be associated to the simplicial complex based on an appropriate perspective, and each one of them provides new information about the graph or network from which the simplicial complex was constructed. The first is the dimension of the simplicial complex corresponding to the maximal simplex dimension, K . From the combinatorial aspect three structure vectors of the simplicial complex are defined (33):

- the *first structure vector* Q : The q^{th} entry of the so called Q – vector of length $K + 1$ (or *first structure vector* (22)), denoted by Q_q is equal to the number of q -connectivity classes. This vector provides information on the number of connected components at each level of connectivity with initial level being equal to the dimension of the complex:

$$\mathbf{Q} = \{Q_{q=K}Q_{q=K-1} \dots Q_{q=1}Q_{q=0}\}; \quad (1)$$

- the *second structure vector* N_s is an integer vector with $\dim(K) + 1$ components

$$\mathbf{N}_s = \{n_{q=K}n_{q=K-1} \dots n_{q=1}n_{q=0}\}; \quad (2)$$

where the q – *th* entry, n_q , is equal to the number of simplices with dimension larger or equal to q , that is, it is equal to the number of simplices at the q -level.

- the *third structure vector* \hat{Q} is the global characteristic of the simplicial complex which measures the degree of connectedness on a q -level. In other words, it measures the number of q -connected components per number of simplices whose components \hat{Q}_q are defined as (32):

$$\hat{Q}_q = 1 - \frac{Q_q}{n_q}; \quad (3)$$

where Q_q is q -th entry of the first structure vector, and n_q is q -th entry of the second structure vector. In addition to these structure vectors that describe the network as a whole or its large parts, in the following we will introduce the vector associated to each node, the node's Q -vector, in order to characterize neighborhood of the node in the network.

2.3 Simplicial complex analysis of the social network with communities

In this section, we analyze a reduced system containing $N_U = 3321$ nodes and no leaves. Focusing on the large community in this network, we perform the analysis on simplices and determine the network's MC matrix. It appears that the maximal clique simplices (in further text for brevity we use term clique) in the network are 5-cliques, i.e. in the present notation, $q_{max} = 4$. The analyzed structures at different q -levels are shown in Fig. 3.

As q increases, the number of nodes that participate in the higher topological structures is decreasing. Considering the largest community in the network shown in Fig. 3a, we compute its structures at higher topological layers; they are shown in Fig. 3b,c,d corresponding to the level of triangles, $q = 2$, 4-cliques, $q = 3$, and at the level of 5-cliques, $q = 4$. Summary of the network's structure vectors will be given in Section 4 in connection with the discussion of brokerage roles of nodes.

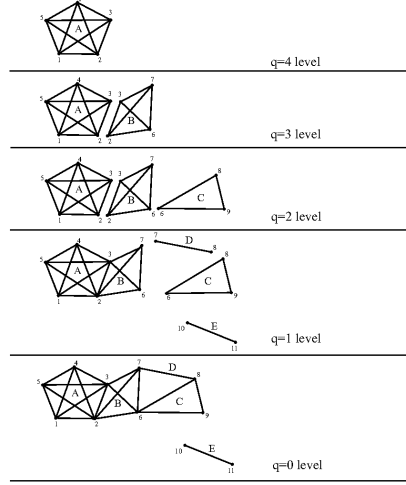


Figure 2: Q-vector disclosing connectivity of the complex at various q -levels. The underlying graph is at the $q = 0$ level.

Here, focus is on the role of individual nodes. Therefore, we introduce the "node's Q -vector", Q^i , that describes the node's i environment in the network.

Definition of the Node's Q -vector. With $q_{max} + 1$ as the dimension of the maximal clique in the network, the *node's Q -vector* is a $q_{max} + 1$ -dimensional vector associated with each node

$$Q^i = \{Q_{q_{max}}^i, Q_{q_{max}-1}^i, \dots, Q_1^i, Q_0^i\}, \quad (4)$$

whose components $\{Q_q^i\}$, $q = q_{max}, q_{max} - 1, \dots, 1, 0$ describe the number of q -dimensional simplices in which the node i participates. We define the *topological dimension* of the node i as the number of all simplices in which the node i participates, i.e., $dimQ^i = \sum_{q=0}^{q_{max}} Q_q^i$. The term "dimension" is motivated in view of the conjugate simplicial complex constructed from the network, where nodes become simplices and simplices become nodes. Therefore, $dimQ^i$ corresponds to the dimension of the conjugate simplex (33). To illustrate the meaning of these components consider the vector $Q^i = \{2, 0, 3, 2, 0\}$ in the network whose maximal clique dimension is 5. In this case, the value of the component $Q_0 = 0$ suggests that the node is not isolated. Further, $Q_1 = 2$ means that among all the nodes connected to the node i , only two are not a part of any higher-order clique. $Q_2 = 3$ indicates that there are three triangles attached to i that are not faces of higher-order cliques; while $Q_4 = 1$ and $Q_3 = 0$ indicate that the node i belongs to one 5-clique, but there are no 4-cliques attached to it.

The components $\{Q_q^i\}$ for each node in the analyzed network can be computed from the MC matrix. Here we determine the components of each node of the network in Fig. 3a. Sorting the nodes according to their topological dimensions $dimQ^i$, one can identify the "influential" nodes in the network's community or a layer. Fig. 4, shows the distribution of the node's dimension for all nodes in

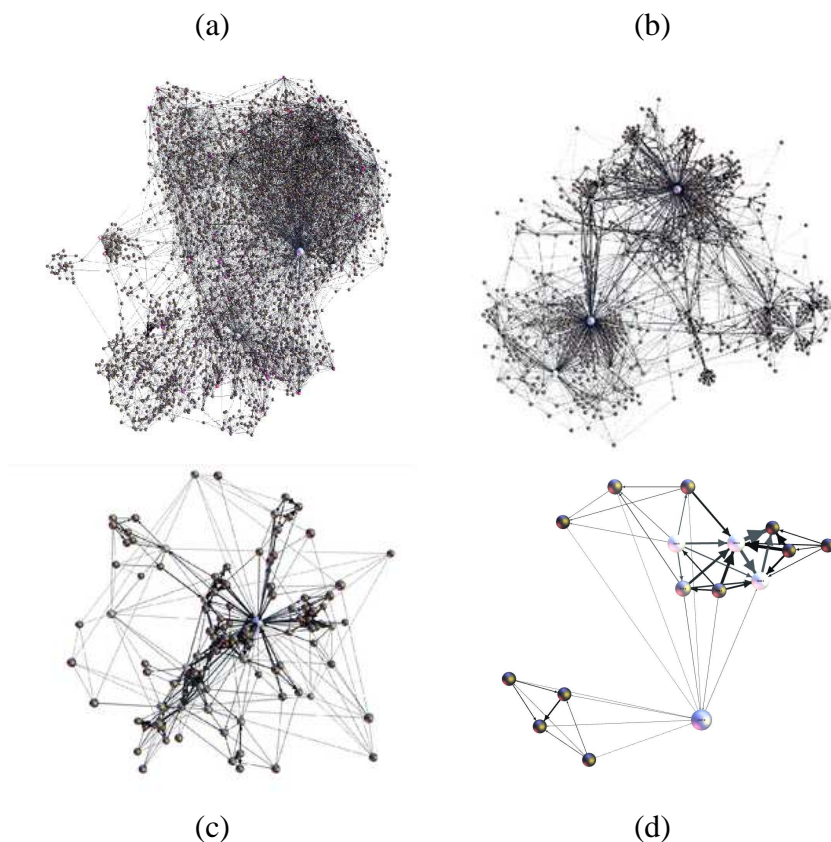


Figure 3: Online social network, constructed from MySpace dialogs, after removal of leaves (a); Structure of the largest social community in that network at different topology layers corresponding to dimensions $q = 2, 3$ and 4 (b,c, and d, respectively).

the network as a function of the node's rank. Notice that the topological dimensions exhibit a broad distribution (Zipf's law) with two rather than a single slope. Such situation often appears in the evolving complex systems (49; 50). Recently, the origin of two slopes in the Zipf's law has been discussed (51) in connection with the scaling and innovation in the use of words in the written text of an increasing length. In the present case, the appearance of new topological forms in the dialogs-based network is related with the activity patterns of users (nodes in the network). According to the analysis in (5), three distinct groups of users can be distinguished considering the number of their actions in relation with the interactivity times. Consequently, very active nodes may build a larger environment resulting in a higher topological dimension. In Fig. 4, the nodes topological dimensions obey a power-law distribution with the slope $\gamma = 0.72 \pm 0.05$ up to the rank 70 (corresponding to the density function power-law with the exponent $\tau = 1/\gamma + 1 \approx 2.37$). In contrast, the nodes of lower dimensions (higher ranking orders) show the slope

close to one, $\gamma = 0.92 \pm 0.06$). Temporal appearance of new topology forms will not be considered in this paper. In the following, we study correlations among topological dimensions of the connected pairs of nodes.

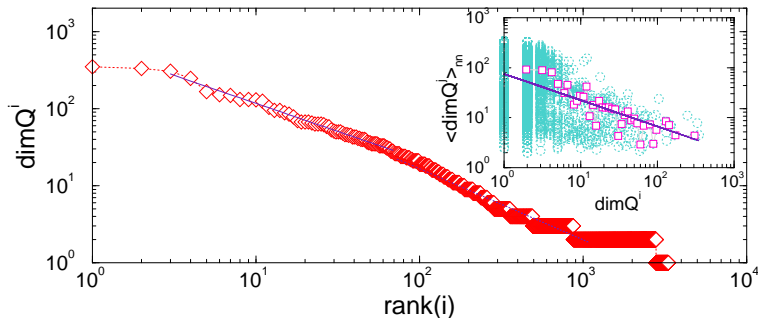


Figure 4: Ranking plot (Zipf's law) of topological dimensions $dimQ^i$ of all nodes in the network, main panel. Inset: Average topological dimension of the node's i nearest neighbors plotted against the node's i dimension exhibiting disassortative behavior (slope of the fit line is $\mu = -0.527 \pm 0.026$).

In an analogy with standard assortativity measure in social networks (52), we plot the node's topological dimension against the average topological dimension of its neighbors. The results, shown in the inset of Fig. 4, indicate that at the level of triangles and cliques of higher dimension, the graph exhibits disassortativity. The general trend of all points can be approximated with the function $\langle dimQ^i \rangle_{nn} \sim (dimQ^i)^{-0.52}$. This means that gradually fewer number of nodes with high dimension are connecting between structures of a lesser complexity. These findings complement the disassortativity results for the same network found in (5) at the level of links (i.e., including leaves) for varied combinations of the directed link orientations.

Fine structure of a node Q -vector allows for further differentiation between nodes and their roles in the network. A 3-dimensional plot in Fig. 5a shows the components Q_q^i of the first 100 nodes as a function of the node's dimension $dimQ^i$ and the dimension of the corresponding simplices q . It also shows the number of triangles ($q = 2$), tetrahedra ($q = 3$) and 5-cliques to which the leading nodes belong. In Fig. 5b a scattered plot of the node's dimension computed from the topology layer $q \geq 2$ is shown against $q \geq 3$. Different colors reflect the number of nodes corresponding to a given combination in the plot. For each node a nonzero value in this plot ($dimQ_{q \geq 3}^i, dimQ_{q \geq 2}^i$) indicates the number of triangles $n_{\Delta}^i = dimQ_{q \geq 2}^i - dimQ_{q \geq 3}^i$ surrounding that node that are not shared faces of higher-order structure. Obviously, the number of nodes which participate in higher-order structures (of higher dimension) is steadily decreasing. Thus, compared with a standard clustering coefficient, the in-depth-topology analysis with simplicial complexes differentiates between types of structures in which the node

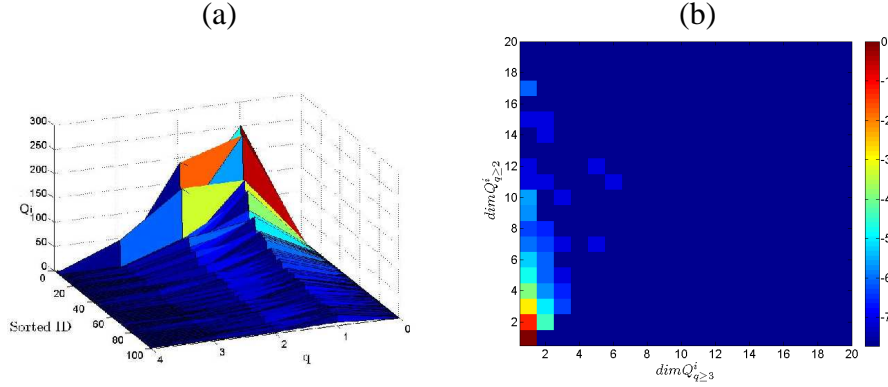


Figure 5: (a) Components of the topology vector Q_q^i of the first 100 nodes plotted against node's index i and simplex dimension q . Ranking order of nodes according to the topological dimension $dim Q^i$ applies. (b) Scatterplot of the node's i topological dimension $dim Q_{q \geq 2}^i$ against its dimension at higher level of clicques, $dim Q_{q \geq 3}^i$.

resides.

The nodes that build up these higher-order structures can be clearly distinguished owing to the unique ID assigned to every node. As it will be presented in Sec. 4, these nodes often have a considerable social capital. An example is the node with ID=1372, connecting three 5-cliques in Fig. 3d. The composition of its components in the entire network is given in Table 2. Apart from three 5-cliques, this node joins 53 different 4-cliques at lower q -levels, 144 triangles and 46 single-link simplices. Another type of the node with a large social capital is the central node in a predominantly star-like structure; an example is the node with ID=2238, also given in Table 2. The changed roles of these and other nodes on emotion-propagating layers of the network are studied in the subsequent sections.

3 Structure Vectors of Emotion-Propagating Network Layers

As discussed in section 2.1, the emotion-propagation dynamics involves different types of contacts among individuals in online social network. The diversity of the emotional content of communicated messages, described by two variables—emotional arousal and valence, enables to identify different network layers corresponding to a particular type of emotional content. Consequently, two network layers are recognized. The positive layer consists of the links along which messages with a positive emotion valence were communicated up to a given instance of time, and the negative layer with links carrying messages with negative valence. In order to enhance the difference between these layers, using the agent-based model

of Ref. (38), we numerically simulated two situations. In one, the external input noise has prevailing positive emotion "astonished" and, in the other, the negative emotion "ashamed". As it was shown in (38; 53; 54), in such situations the temporal correlations of message streams, that marks the emotion propagation dynamics, lead to collective emotion states in networks. Here, the positive/negative valence of the input emotion eventually prevails. In this way, in each case one can distinguish a dominant layer that diffuses the "winning" emotion from the counter-emotion layer. These are named PP (positive-positive) and NN (negative-negative) as dominant layers in positive and negative input, respectively; counter-emotion layers are NP and PN. Here NP designates a layer with links propagating positive emotion in the case when the majority of messages in the network are negative (following negative emotion input). While PN denotes a layer with the negative emotion links when the prevailing emotion is positive. Note that, by definition, the same nodes (and sometimes overlapping links between them) can belong to both layers. The focus is on in-depth-topology analysis of these layers and in quantifying the roles of relevant nodes residing in each one of them. The structure of connections in counter-emotion layers for the two cases of emotion-propagating dynamics is shown in Fig. 6. The results of topology analysis of all emotion-propagating layers are summarized in Table 1.

Table 1: Layers in the OSN propagating negative and positive emotion messages when the majority of messages are of negative emotional content (NN and NP) and when the majority of messages in the network are of positive emotional content (PN and PP). Compared with other structure vectors for the whole network, given in Table 3, in the emotion-propagating layers 5-cliques ($q = 4$ components of the structure vectors) are absent; in counter-emotion layers 4-cliques also do not appear.

	$NN_{w>30}$			NP		
q	Q	N	\hat{Q}	Q	N	\hat{Q}
4	-	-	-	-	-	-
3	4	4	0	-	-	-
2	198	198	0	12	12	0
1	6242	6353	0.017	1134	1138	0.003
0	1	6353	0.999	2310	3404	0.321
	PN			$PP_{w>30}$		
q	Q	N	\hat{Q}	Q	N	\hat{Q}
4	-	-	-	-	-	-
3	-	-	-	5	5	0
2	11	11	0	204	205	0.005
1	1069	1074	0.005	6228	6287	0.009
0	2360	3390	0.304	4	6352	0.999

As may be noticed in Fig. 6 and Table 1, the counter-emotion layers consist of

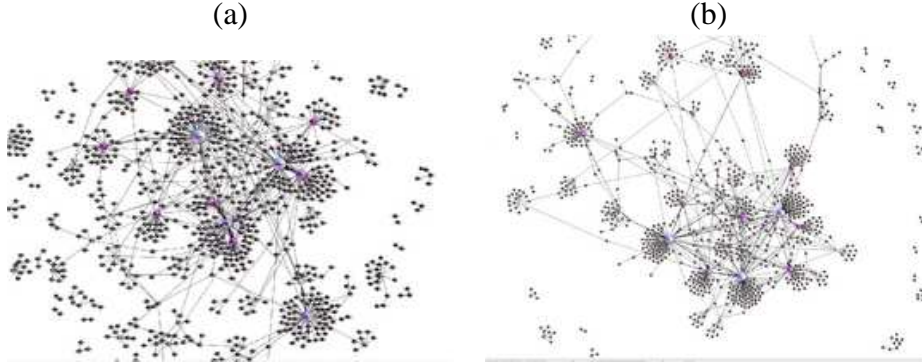


Figure 6: Counter-emotion layers NP (left) and PN (right) of the online social network from Fig. 3a.

a reduced number of connections as compared to the dominant-emotion layer and the whole system. Furthermore, the structure of connections is different, depending on the dominant emotion, and much simpler than in the dominant-emotion layer. The cliques of dimension 4 and 5 existing in the simplicial complex of the network are absent in the counter-emotion layers. The highest cliques are triangles ($q = 2$ components of the structure vectors). Comparison of the first and the second structure vectors of NP and PN layers shows that only few triangles, 4 in the case of NP and 5 in PN layer, join the other simplices at the $q = 1$ level. In both cases, a large number of disconnected components remains at the $q = 0$ level, in contrast to the dominant layers. The dominant-emotion layers (PP and NN) are richer, although the number of 4-cliques is much smaller than in the entire network, and the 5-cliques are still absent. For comparison, the structure vectors of the entire network are given in Table 3 when all links are counted irrespectively of the type of emotion diffusing along them.

The structure of emotion-propagating layers reveals altered position of individual nodes and their social capital. In Fig. 7 we show the scatterplot of the topological dimension of each node in two counter-emotion layers. The color code indicates the number of nodes that correspond to a given combination of dimensions ($dimQ_{PN}^i, dimQ_{NP}^i$). For the majority of nodes with at least one dimension larger than one, an off-diagonal pattern occurs. These findings indicate that the nodes have a different topological dimension, i.e. their neighborhoods differ in these two layers. A node that builds a compact neighborhood (large topological dimension) in the PN layer often has a small dimension (poor neighborhood) in the NP layer and vice versa. In other words, a large number of simplices, mostly $q = 1$ simplices cf. Table 1, do not change their emotion polarity under the influence of the collective emotion in the network.

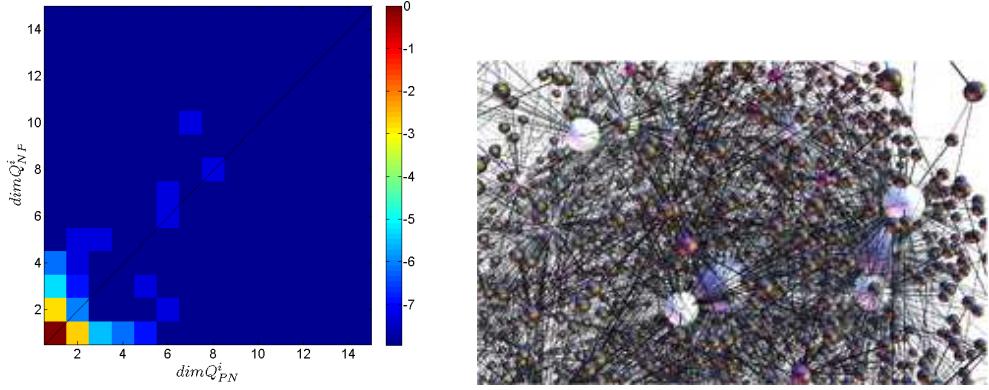


Figure 7: Left: Scatterplot of topological dimensions $dimQ^i$ of the leading nodes in the layers where the propagated emotion opposes the majority of emotional messages: the nodes are sorted (along x-axis) according the $dimQ^i_{NP}$ in the positive layer when the majority of messages are negative. Position of the node along y-axis corresponds to its $dimQ^i_{PN}$ found in the negative layer when the majority of messages in the network are positive. Right: Close-up of the part of the network in the vicinity of big nodes (for the case of majority of positive messages).

4 Simmelian brokerage and the node's Q-vector

4.1 Simmelian brokerage of nodes in online social networks

In order to estimate the social capital of nodes (users) in the network, we measure the Simmelian brokerage B_i for each node $i = 1, 2, \dots, N_U$. According to Ref. (14), for a given node i Simmelian brokerage “captures opportunities of brokerage between otherwise disconnected cohesive groups of contacts”. Quantitatively, B_i is determined via the node's efficiency E_i as (14):

$$B_i = n_i - (n_i - 1)E_i, \quad (5)$$

where n_i is the number of neighbors of the node i inside a considered group N_i ; the node's local efficiency E_i is determined by

$$E_i = \frac{1}{n_i(n_i - 1)} \sum_{l \in N_i} \sum_{m \in N_i} \frac{1}{d_{lm}}, \quad (6)$$

where d_{lm} is the distance between all distinct pairs of nodes $l \neq m$ in the set N_i when the node i is removed. As stated earlier in our approach, the node's neighborhood N_i is precisely defined at different topology levels by simplices and simplicial complexes in which the node i resides.

Performing the computation indicated by Eqs. (6-5), we determine Simmelian brokerage B_i for each node in the network shown in Fig. 3a. In Fig. 8a B_i is plotted against the node's topological dimension $dimQ^i$, where each point represents one

node of the network. In addition, plotted are the node's brokerage values that are computed within the emotion-propagating layers. The bottom panel includes similar plots but only for the higher topological levels $q = 2, 3, 4$, matching the social graphs in Fig. 3b,c,d, respectively. Note that, in this notation, the whole network corresponds to the level $q = 0$. It is remarkable that, in these plots, the majority of nodes follow a universal pattern that can be expressed by functional dependence

$$B_i \sim (\dim Q^i)^\mu ; \quad (7)$$

where the exponent $\mu \lesssim 1$ (see Table 3). Dispersion along vertical axis correlates with the number of higher-order cliques in the considered graph.

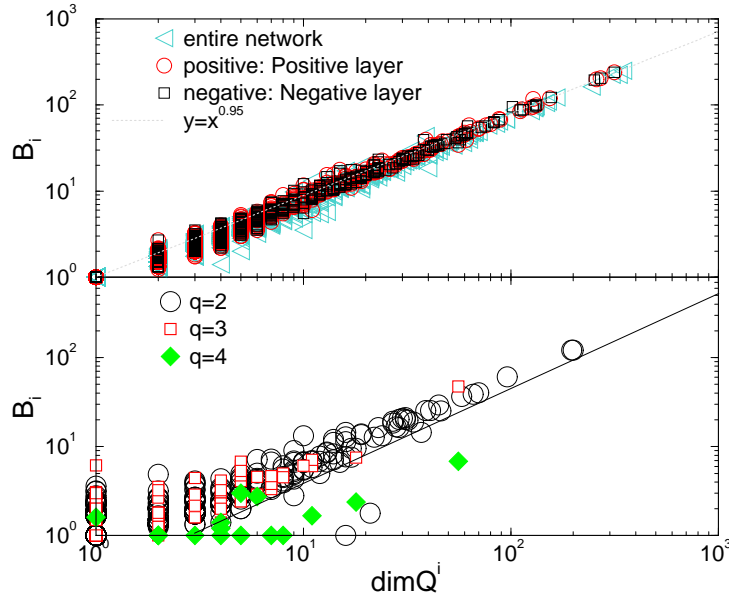


Figure 8: For the networks in Fig. 3, Simmelian brokerage B_i of nodes plotted against the node's topological dimension $\dim Q^i$. The symbols indicate social graphs at different topology levels $q = 2, 3, 4$ (bottom panel), and the entire network $q = 0$, and the two emotion-propagating layers (top panel).

4.2 Relationship between the social capital and the topological dimension of nodes in a wider class of networks

In this section, we confirm the robustness of the functional dependence expressed by Eq. (7). Particularly, we present an approximate analytical expression (which is exact in some limiting geometries) as well as numerical work for a wider class of networks.

Table 2: For two influential nodes within the community in Fig. 3, components of the topology vector $Q_i = \{Q_4^i, Q_3^i, Q_2^i, Q_1^i, Q_0^i\}$ and the related Simmelian brokerage are computed in the whole network and in two emotion-propagating layers. Different roles of the same node in the (counter-)emotion layer when the majority of messages carries positive/negative emotion are demonstrated; compared with the whole network indicates how the nodes build their social capital by combining links at different layers.

<i>NodeID</i>		<i>whole network</i>	<i>NN</i>	<i>NP</i>
1327	Q_q^i	{3 53 144 46 0}	{- 1 18 83 0}	{- - 0 44 0}
	<i>Brokerage</i>	162.79	95.44	43.85
2238	Q_q^i	{0 0 96 253 0}	{- 0 0 316 0}	{- - 0 94 0}
	<i>Brokerage</i>	250.80	239.69	90.49

Combining expressions (5) and (6), the brokerage of the node i is given by

$$B_i = n_i - \frac{\sum_{l \in N_i} \sum_{m \in N_i} \frac{1}{d_{lm}}}{n_i}. \quad (8)$$

where the shortest paths between each pair of nodes $m \neq l$ within the graph are computed after the node i is removed from that graph. This sum can be computed analytically in some limiting cases. For instance, consider the situation where the node i belongs to a $(q+1)$ -clique, which implies that its number of neighbors is $n_i = q$ and all distances within the clique are $d_{lm} = 1$. After removal of the node i , the remaining nodes contribute to the sum as $\sum_{l \in N_i} \sum_{m \in N_i} \frac{1}{d_{lm}} = q(q-1)$. A straightforward extension to the situation where the node i connects k such cliques leads to $\sum_{l \in N_i} \sum_{m \in N_i} \frac{1}{d_{lm}} = kq(q-1)$. In this case, removal of the node leaves the cliques separated from each other, i.e., the distance between pairs of nodes from different cliques is infinite, while the distance inside each clique is one. Note that in the topology analysis described in sec. 2.3, the number of $(q+1)$ cliques related to the node i is given by the node's q -level component, Q_q^i . Hence, the sum at q -level gives $\sum_{l \in N_i} \sum_{m \in N_i} \frac{1}{d_{lm}} = Q_q^i q(q-1)$. The situation is exact for top-level cliques, for example 5-cliques in Fig. 3d.

A similar reasoning can be extended to $(q-1)$ -level, provided that at this level no shared faces occur between the q -level cliques. Hence, in this case Eq. (8) can be written as

$$B_i = \sum_q Q_q^i q - \frac{\sum_q Q_q^i q(q-1)}{\sum_q Q_q^i q}, \quad (9)$$

where we also note that n_i can be expressed via the components of the topology vector as $n_i = \sum_q Q_q^i q$. It should be stressed that the expression is exact for those topology levels of a graph at which there are no shared faces of the higher level cliques, i.e., where the third structure vector $\hat{Q}_q = 0$ (cf. Table 3). Otherwise, by extending the summation over topology levels, we obtain an approximate expres-

sion that can be written as

$$B_i = \dim Q^i \langle q \rangle - \frac{\langle q^2 \rangle}{\langle q \rangle} + 1, \quad (10)$$

where we used the definition $\dim Q^i = \sum_{q=0}^{q_{\max}} Q_q^i$ and introduced an abbreviation $\langle q \rangle = \sum_{q=0}^{q_{\max}} Q_q^i q / \sum_{q=0}^{q_{\max}} Q_q^i$.

Note that in the case of tree structures, where the highest topology level corresponds to links, i.e., $q = 1$ is the maximal clique, we have that $B_i = \dim Q^i$ with the exact exponent $\mu = 1$. In a more complex network cliques of higher order occur and are weakly connected at lower topology levels, as in the case of studied online social network (OSN in Table 3). Consequently, the contribution of the second term in (10) induces corrections eventually resulting with an exponent $\mu \lesssim 1$. The dispersion in the number of cliques in (10) and the number of their shared faces at lower q -levels can be considerably greater in the case of more compact networks. Nevertheless, a power-law dependence (7) appears, with different values of the exponent μ . The reasons for the occurrence of such power-law dependence in a general network structure are not evident. Here, we provide a numerical proof by considering a wider class of networks.

The analysis in previous sections suggests that the relation between Simmelian brokerage of nodes and their topological dimension depends on the graph architecture. Therefore, by varying the building rules of the network, one can vary the topological dimensions of different nodes and test the robustness of Eq. (7). We consider several types that are shown in Fig. 9. In these networks, the node's neighborhood can be varied by control parameters of the growth, ranging from the tree-like to a highly clustered compact structure. These networks, consisting of approximately 1000 connected nodes, are generated by the algorithm that is initially described in Ref. (31) for growth of scale-free networks with clustering and communities. The basic idea of clustered scale-free networks by preferential attachment and preferential rewiring of Ref. (55) is implemented for the case where different communities (node groups) are allowed to grow. Thus, the attachment of new nodes is preferred within a currently growing community while rewiring can take part both within and outside of that community. Three parameters that control the structure are: p —the probability of a new community, α and β —that appear in the preferential shift-linear rules for attachment and rewiring probabilities (55), respectively, and M —the number of nodes added per growth step. In addition, we consider a dense single-community graph consisting of 100 nodes, shown in Fig. 9d. The results of the topology analysis of these networks is summarized in Table 3. Topological dimension $\dim Q^i$ of each node in these networks is also determined. Then Simmelian brokerage is computed, according to the original formula in Eq. (8), for each node and plotted against the node's topological dimension. The results are shown in Fig. 10. The power-law dependence holds for each network type in the corresponding range of nodes topological dimensions. Values of the corresponding exponent μ are also given in Table 3.

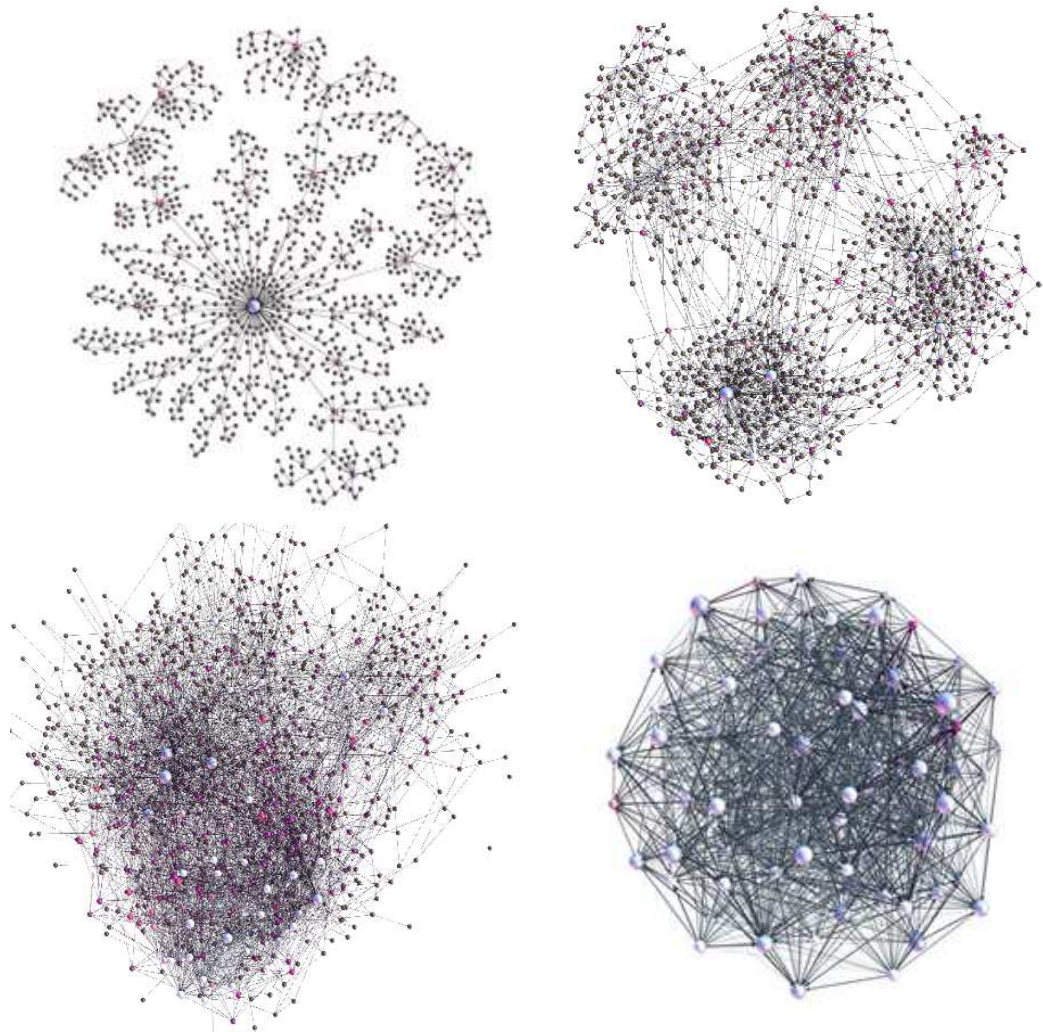


Figure 9: Computer-generated networks with different topology: (a) scale-free tree; (b) network with clustered and weakly interlinked scale-free communities; (c) same parameters as (b) but stronger connections between communities; (d) a strongly clustered single-community network.

5 Conclusions

The topological framework based on the simplicial complex representation of graphs offers systematic and in-depth characterization of complex networks beyond the standard methods. In this work, we have used simplicial complexes for addressing both local and global structures of online social networks. The studied graph based on MySpace dialogs data from (5), is a prototypical online social structure: the network organization exhibits communities and layers closely related with the communication patterns between users and the dynamics of emotions. The in-

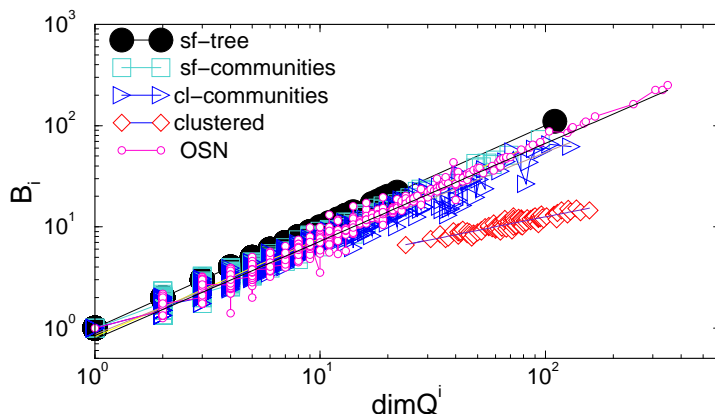


Figure 10: Brokerage versus topological dimension of nodes for computer-generated network types and for OSN MySpace, as indicated. Network characteristics are summarized in Table 3.

depth topology analysis reveals appearance of higher-order cliques and their complexes that are, besides a tree-like local organization, often attached to some relevant nodes. Complementing the standard analysis of the network (5), the three structure vectors introduced here, Q , N_s and \hat{Q} , describe the network architecture at all topology levels from $q = 0$ (the whole network) to q_{max} (the level of maximal $(q_{max} + 1)$ -clique). It demonstrates that social network dynamics leads to unusual local structures, including cliques up to the fifth order and to complexes consisting of the cliques sharing triangle faces and tetrahedra. The structure of emotion-propagating layers is much simpler, containing triangles as the highest order structures. It should be noted that, in another study (45), triadic closure dynamics was recognized as one of the fundamental dynamic principles in social multiplex networks.

The node's Q^i -vector associated with every node of the simplicial complex, defined here for the first time, differentiates in a precise manner different topological and hence graph structures which contain the reference node. Its components $\{Q_q^i\}$ indicate the number of structures at each hierarchical level in which the node participates. The total number of cliques of all sizes in which the node resides defines the node's topological dimension. The concept of the node's structure vector has been useful in sequencing the social network, revealing a new insight into network organization from local to the global level and the role of individual nodes in it. In particular, ranking the nodes according to their topological dimension obeys Zipf's law with two slopes. This feature helps to distinguish a smaller group of innovative nodes that build large surroundings or surroundings with a larger topological complexity, from the rest of the system. Furthermore, we have demonstrated that this vector provides a useful quantitative measure of social capital of nodes in the communities and layers. Specifically, Simmelian brokerage, which measures the

Table 3: Extended topological properties of four computer-generated networks and OSN.

network	$\langle k \rangle$	CC	d	$\langle p \rangle$	modul.	q	Q	N_s	\hat{Q}	μ
sf-tree	1	0	20	6.46	0.916	1	998	998	0	1
						0	1	998	0.99	
sf-comm	1.94	0.201	9	4.42	0.398	3	2	2	0	0.974(5)
						2	328	328	0	
						1	1322	1602	0.175	
						0	10	1611	0.994	
cl-comm	3.83	0.081	7	3.36	0.395	4	16	16	0	0.898(4)
						3	65	80	0.188	
						2	837	900	0.07	
						1	2269	3090	0.269	
clustered	14.52	0.269	2	1.71	0.207	0	12	3101	0.996	0.44(2)
						5	9	9	0	
						4	264	266	0.008	
						3	1302	1522	0.145	
						2	367	1887	0.806	
OSN-	2.302	0.183	8	4.067	0.741	1	1	1887	0.995	0.969(2)
						0	1	1887	0.995	
						4	5	5	0	
						3	90	91	0.011	
						2	1064	1103	0.035	
1	5397	6437	0.161							
0	1	6437	0.999							

ability of a node to act as a broker between groups of other nodes in the recognized community or layer, scales as a power of the node's topological dimension. This functional relationship holds in a class of networks with varied composition. The scaling exponent has an exact value $\mu = 1$ in tree structures and decreases towards lower values in the case of graphs that contain a number of larger complexes (cliques with shared faces).

The influential nodes which possess considerable social capital can be further differentiated by considering the individual components $\{Q_q^i\}$ of their topology vectors. Two types of influential nodes can be observed. First, there are “informer” nodes that act as centers of star-like structures in the network; occurrence of such nodes is in the direct relationship with disassortativity of network's dynamical structure, observed in (5). Second, the nodes that connect a large number of higher-order structures (for example in Fig. 3d) act as star centers in the *conjugate* simplicial complex network. Their appearance may be related with the disassortativity of the topological dimensions, demonstrated in the inset of Fig. 4. The influential nodes build their social capital by combining links in different layers.

This conclusion is reached by inspection and comparison of the simplicial structure containing these nodes in the emotion-propagating layers and the structure of the entire network, Table 2.

The metric based on the topological framework used in this work, can be also applied to discover “key players” in conventional social networks (21; 15). Similarly, it can be used to determine in-depth structure of techno-social networks that grow from scratch co-evolving with the dynamics of chats and blogs (1), where the origin of communities and layers can be entirely different. We expect that this approach can be useful in a variety of other network-based studies of complex systems. Some examples are the brain dynamics and learning (56), and innovation and collaboration systems (49; 57), where different patterns of the actor’s behavior are essential for the global dynamics.

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