# Towards unification: SU(5) and SO(10)

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June 7, 2012

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# 1 Introduction: Group theory

Particle physics theories, as we know them today, are build on few core principles. The most important one is the principle of symmetry, which was already famous concept in physics, but further developed by particle theories. Symmetry of the physical system is an invariance of the system under transformations. Symmetries were known to lead to conservation laws. Conserved quantities, called charges, are the most important quantities in theoretical physics.

The major breakthrough in particle physics was a discovery that a symmetry can actually describe dynamics! This is what we call gauge principle. The present theory of particle physics, Standard Model is an $SU(3) \times SU(2) \times U(1)$ gauge theory, and is experimentally very well established theory at presently probed scales. Grand unification is a dream of a gauge theory described by one group, which would in the language of particle physics imply unification of forces!

The language of symmetry is a group theory, and hanse it will be our starting point.

## 1.1 Continuous groups

Particle physics is all about group theory. An important part of group theory are continuous groups. A continuous group is a group in which every element can be labeled with a set of continuous parameters $g(\vec{\theta})$, where $\vec{\theta} = (\theta_1, ..., \theta_N)$ are $N$ real independent parameters. We have notion of "closeness" between two group elements if for arbitrarily close set of parameters $\vec{\alpha}$ and $\vec{\beta}$, two corresponding elements $g(\vec{\alpha})$ and $g(\vec{\beta})$ are also close. Such continuous group is called Lie group. Unit element can be chosen to correspond $e = g(\vec{0})$. Group is compact if parameter space forms a closed interval, and connected if every element can be reached from unit element "smoothly" through parameter space (with continuous line in group manifold). Properties of connected Lie groups are completely determined with is behavior in neighborhood of unit element. Let $D_R(g(\delta\vec{\theta}))$ be representation of group element $g(\delta\vec{\theta})$, if parameter is arbitrarily small, we can expand around unity

$$D_R(\delta\theta) = 1 + i\delta\theta^a T_R^a$$

where

$$T_R^a = -i \frac{\partial D_R(\theta)}{\partial \theta^a}\bigg|_{\theta=0}$$

are called group generators. Number of group generators is equal to number of independent parameters of the group and is called dimension of the group. If representation of the group is unitary, group generators are hermitian. Since any group element can be reached with successive application of infinitesimal elements then

$$D_R(\vec{\theta}) = \lim_{N\to\infty} \left(1 + \frac{i\theta^a T_R^a}{N}\right)^N = e^{i\theta^a T_R^a}$$

## 1.2 Lie Algebra

An important property of group generators, which follows from group axiom of closure, is that they form Lie algebra

$$[T_a, T_b] = i f_{abc} T_c$$

where $f_{abc}$ are called structural constants. This relation is not dependent on specific representation of the group, and it contains all important information for the group! For compact Lie algebras it is possible to define scalar product of generators as

$$\mathrm{Tr}[T_R^a T_R^b] = T(R)\delta^{ab}$$

where $T(R)$ is called *index* of representation. Using this definition and algebra relation, we can show following

$$f^{abc} = \frac{1}{iT(R)} \mathrm{Tr}[[T_R^a, T_R^b] T_R^c]$$

Structural constants are completely antisymmetric due the cyclic property of trace.

Special combination of generators $T_R^a T_R^a$, where sum over repeated indices is assumed, commutes with all other generators, and for irreducible representation is proportional to unit matrix $(T_R^a T_R^a)_{ij} = C(R)\delta_{ij}$, where $C(R)$ is *quadratic Casimir*. From the scalar product of generators, we get

$$T(R)D(G) = C(R)D(R)$$

where $D(G)$ is dimension of group and $D(R)$ dimension of representation.

We will introduce some more concepts which will be relevant. Invariant subalgebra is set of generators $X$, which is subset of all generators $Y$, such that commutator $[X, Y]$ gives again a generator of invariant subalgebra. Invariant subalgebra generates invariant subgroup. The whole algebra and 0 are trivial invariant subalgebra. An algebra which has no nontrivial invariant subalgebras is called *simple*. Algebras without Abelian invariant subalgebra are called *semisimple*. They are built out of simple algebras.

## 1.3 Representations

In particle physics, matter fields live in irreducible representations of Lie groups. If $D_R(g)$ is a representation of the group, then $D_{\overline{R}}(g) \equiv (D_R(g))^*$ is also a representation, called complex conjugate representation. Antiparticles live in complex conjugate representation. It is easy to show that generators of complex conjugate representation are given by

$$T_{\overline{R}}^a = -T_R^{a*}$$

Representation is real if it is equivalent to its complex conjugate representation. It means there exists nonsingular matrix $V$ such that $VT_{\overline{R}}^a V^{-1} = T_R^a$ for every $a$. Contrary, representation is called complex.

Important representation of Lie group is *adjoint* representation defined by structural constants

$$[T_A^a]^{bc} = -if^{abc}$$

Using Jacobi identity, it can be shown that this representation satisfy group algebra. Taking complex conjugate of previous relation, we can conclude that adjoint representation is real. Dimension of adjoint representation is equal to number of generators. Adjoint representation of simple lie algebras is irreducible.

An important concept is direct product representation of a group. Suppose we have two representations of a group $(D_{R_1}(g))_{i,j}$ and $(D_{R_2}(g))_{x,y}$, direct product representation $R_1 \otimes R_2$ lives in space formed by tensor product of spaces of two representations, and

$$(D_{R_1 \otimes R_2}(g))_{ix,jy} = (D_{R_1}(g))_{i,j} \, (D_{R_2}(g))_{x,y}$$

Equivalently, at the level of generators we have

$$\left(T^a_{R_1 \otimes R_2}\right)_{ix,jy} = \left(T^a_{R_1}\right)_{i,j} \delta_{x,y} + \delta_{i,j} \left(T^a_{R_2}\right)_{x,y}$$

and obviously $D(R_1 \otimes R_2) = D(R_1)D(R_2)$.

## 1.4 $SU(N)$

$SU(N)$ is a group of $N \times N$ unitary matrices with determinant equal 1,

$$UU^\dagger = U^\dagger U = 1 \ \text{ and } \ det(U) = 1$$

Defining or *fundamental* representation is formed by matrices $U$ themselves. $N$-dimensional multiplet, which lives in vector space of this representation transforms under the action of group elements $\Phi \to U\Phi$. Number of independent parameters of $N \times N$ unitary matrix with determinant 1 is $N^2 - 1$, which is dimension of the group.

Generators of SU(N) are traceless due to property $det(e^A) = e^{Tr(A)}$. Maximal number of generators which commute with each other, and can be simultaneously brought to diagonal form, is called *rank* of the group, and the generators are called Cartan generators. Rank of $SU(N)$ is $N - 1$. Fundamental representation of $SU(N)$ has the index $T(N) = \frac{1}{2}$, and the index of adjoint is $T(A) = N$. Peculiar property of $SU(2)$ group is that its fundamental representation is real, all higher groups have complex fundamental representation.

From definition of adjoint representation, multiplet $A^a$ which lives in this representation transforms as

$$A^a \to A^a + i\theta^c(-if^{cab})A^b + \cdots$$

It is convenient to define matrix $A = A^a T^a$, then

$$A \to A + i\theta^c[T^c, T^b]A^b + \cdots = UAU^\dagger$$

where $U = e^{i\theta^c T^c}$. This form of representing adjoint fields will be useful in discussion of Higgs sector of $SU(5)$.

## 1.5 $SO(N)$

$SO(N)$ is the group of $N \times N$ real orthogonal matrices $OO^T = O^T O = 1$ with $\det(O) = 1$. It can be generated by $N(N-1)/2$ hermitian, antisymmetric matrices

$$O = e^{i\frac{1}{2}\theta_{ij}L_{ij}}$$

where the generators of fundamental representation are given by

$$(L_{ij})_{kl} = -i\left(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}\right)$$

By definition, all its representations are real. Generators obey following commutation relations

$$[L_{ij}, L_{kl}] = i(\delta_{ik}L_{jl} + \delta_{jl}L_{ik} - \delta_{jl}L_{ik} - \delta_{il}L_{jk})$$

Fundamental (vector) representation transforms as $\phi_i \to O_{ij}\phi_j$. Fundamental representation of $SO(N)$ has the index $T(N) = 2$, and the index of adjoint is $T(A) = 2N - 4$.

## 1.6 Tensors

We will develop useful tool to deal with complex representation. Let $\phi^i$ transform under complex representation $R$ in following way

$$\phi^i \to (D_R(\theta))^i_j\, \phi^j = (1 + i\theta^a T^a_R)^i_j\, \phi^j$$

Then $(\phi^i)^\dagger \equiv \phi^\dagger_i$ transforms under $\bar{R}$. Since $T^a_{\bar{R}} = -T^{a*}_R$

$$\phi^\dagger_i \to (1 - i\theta^a T^a_R)^j_i\, \phi^\dagger_j$$

Note that contraction of upper and lower indices gives invariant quantity $\phi^\dagger_i\phi^i$. Let us check transformation property of $\delta^i_j$. Since it carries two indices, it transforms under direct product representation $R \otimes \bar{R}$

$$\delta^i_j \to (1 + i\theta^a T^a_R)^i_k\, (1 - i\theta^a T^a_R)^l_j\, \delta^k_l = \delta^i_j + i\theta^a (T^a_R - T^a_R)^i_j + \mathcal{O}(\theta^2)$$

Since it transforms into itself, it forms a singlet representation of the group. Furthermore, it should always be presented in decomposition of

$$R \otimes \bar{R} = 1 \oplus \cdots$$

We can also show that $(T^a_R)^i_j$ is invariant object under $R \otimes \overline{R} \otimes A$

$$(T^a_R)^i_j \to (1 + i\theta^c T^c_R)^i_k\, (1 - i\theta^c T^c_R)^l_j\, \left(\delta^{ab} + i\theta^c(-if^{cab})\right)(T^b_R)^k_l$$

$$= (T^a_R)^i_j + i\theta^c\, (T^c_R)^i_k\, (T^a_R)^k_j - i\theta^c\, (T^c_R)^l_j\, (T^a_R)^i_l + i\theta^c(-if^{cab})(T^b_R)^i_j + \mathcal{O}(\theta^2)$$

$$= (T^a_R)^i_j + i\theta^c \left( [T^c_R, T^a_R]^i_j - i f^{cab} \left(T^b\right)^i_j \right) + \mathcal{O}(\theta^2)$$

$$(T^a_R)^i_j + \mathcal{O}(\theta^2)$$

The fact that $(T^a_R)^i_j$ is an invariant symbol implies that

$$R \otimes \overline{R} \otimes A = 1 \oplus \cdots$$

If we multiply both sides by $A$ and use $A \otimes A = 1 \oplus \cdots$, because $A$ is real, we conclude that

$$R \otimes \bar{R} = A \oplus \cdots$$

Our final conclusion is that direct product of representation with its complex conjugate representation always contains singlet and adjoint representation. In case of $SU(N)$ this equality holds

$$N \otimes \overline{N} = 1 \oplus A$$

since dimensions on left and right side should match, $N \times N = 1 + (N^2 - 1)$. For example in $SU(5)$ $5 \otimes \overline{5} = 1 \oplus 24$.

We should mention another invariant symbol in $SU(N)$

$$\epsilon^{i_1,\ldots,i_N} \to U^{i_1}_{j_1} \cdots U^{i_N}_{j_N} \epsilon^{j_1,\ldots,j_N} = (\det U)\epsilon^{i_1,\ldots,i_N}$$

# 2 Standard theory of particle physics

The present theory of particle physics is so good, that people call it "Standard". It is a theory of strong and electroweak interactions of fundamental building blocks of nature, quarks and leptons. The theory is build on very few guiding principals and very early in the development of the field of particles physics, in sixties. Remarkably, it passed all precision tests conducted in 50 years of collider physics. Today, there is a big effort to search for a new physics (Beyond the Standard Model) from both, theoretical and experimental side, and one of the possible candidates are Grand Unified Theories (GUT). This chapter is organized in following way, first we introduce guiding principals and building blocks of the particle theories, then we describe Standard Model, and at the end we spot possible weak points, and main motivation for GUT.

## 2.1 Gauge theories

Particle physics is a study of smallest scales of nature, mainly study of highly energetic particles and their interactions. Since we deal with small scales, our description is quantum, on the other side, we deal with highly energetic particles, so our description should be relativistic as well. Marriage of Quantum Mechanics and Special Theory of Relativity is known as Quantum Field Theory (QFT), and it is the framework of particle physics. This framework allows to describe creation and annihilation of particles, scattering processes with variable number of particles and so on. Framework was established in forties and highly developed through next decades and is still developing. QFT itself does not predict the form of interactions or particle content, it is just a framework, which once you have a theory, allows to make a predictions.

An answer to what dictates the dynamics and interactions of particles was off course found in symmetries! A prototype interaction from which guiding principle was deduced is Quantum Electrodynamics, quantum version of classical theory of electromagnetism. It is a theory which explains interaction between electron and a photon. Electron is a spin 1/2 particle, and theory of free electron field is given by Dirac Lagrangian

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$$

It is worth nothing that the theory is invariant under global $U(1)$ transformation

$$\psi \to \psi' = e^{i\alpha Q}\psi$$

$$\bar{\psi} \to \bar{\psi}' = e^{-i\alpha Q}\bar{\psi}$$

that is, phase of the field is arbitrary and physics should not depend on it. This also leads to conservation of charge $Q$, which is the generator of this symmetry. However, the free Lagrangian is no longer invariant if one allows the phase transformation to depend on the space-time coordinate, i.e., under local phase redefinitions $\alpha \to \alpha(x)$

$$\partial_\mu \psi' = e^{i\alpha(x)Q}\partial_\mu\psi + e^{i\alpha(x)Q} iQ\partial_\mu\alpha\psi$$

$$\mathcal{L}' = \mathcal{L} - \bar{\psi}\gamma^\mu\psi Q\partial_\mu\alpha$$

Thus, once a given phase convention has been adopted at the reference point $x_0$, the same convention must be taken at all space-time points. This looks very unnatural. The 'gauge principle' is the requirement that the $U(1)$ phase invariance should hold locally. This is only possible if one adds an extra piece to the Lagrangian, transforming in such a way as to cancel the $\partial_\mu\alpha$ term. The only way is to introduce a new field, which should carry Lorentz index $\mu$, that is a vector field $A_\mu(x)$ in following way

$$D_\mu = \partial_\mu + ieQA_\mu$$

If vector field transforms as

$$A_\mu \to A'_\mu = A_\mu - \frac{1}{e}\partial_\mu\alpha$$

it is easy to show

$$D'_\mu\psi' \to e^{i\alpha Q}D_\mu\psi$$

and our new Lagrangian is $U(1)$ gauge invariant

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi - eQ\bar{\psi}\gamma^\mu\psi A_\mu$$

Adding kinetic term for gauge boson field $-\frac{1}{4}F^{\mu\nu}F^{\mu\nu}$, we get full quantum description of electron-photon interaction, which is one of the most precise theories we have! It agrees with experiment incredibly well. The key point was to recognize internal symmetry of matter field, and promote it to LOCAL. This then requires an introduction of a gauge boson field, mediator of the force, and not just that, it precisely gives the structure of interacting term, that is, it dictates dynamics!

It is straight forward job to generalize idea to higher groups. Let us take $N$ fields in fundamental representation of $SU(N)$, $\psi^T \equiv (\psi_1, ..., \psi_N)$. Free theory is given by

$$\mathcal{L} = \bar{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi^i$$

Lagrangian is invariant under global transformation

$$\psi \to U\psi \quad \bar{\psi} \to \bar{\psi}U^\dagger$$

where

$$U = e^{i\theta^a T^a}$$

If we promote symmetry to be local, i.e. $\theta^a \to \theta^a(x)$, we are forced to introduce new gauge boson fields for every generator of the symmetry in following way

$$D_\mu = \partial_\mu + igT^a A^a_\mu$$

with transformation property

$$\mathbf{A}_\mu \to U\mathbf{A}_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$$

where $\mathbf{A}_\mu \equiv A^a_\mu T^a$ and $g$ is the coupling constant of the group. Gauge fields transform under adjoint representation of the gauge group.

## 2.2 Spontaneous Symmetry Breaking

Gauge theories are very powerful in explaining interactions of matter fields and interaction mediators, but they have big shortcoming, they predict massless gauge bosons, since term $\frac{1}{2}m^2A^\mu A_\mu$ explicitly breaks gauge invariance. Also, we know that nature has some massive gauge bosons, for instance mediators of the weak force. Second key ingredient of Standard Model is spontaneous symmetry breaking (SSB), which maintains not just masses, but renormalizability of the theory.

Spontaneous symmetry breaking is a subtle way to break a symmetry by still requiring that the Lagrangian remains invariant under the symmetry transformation. However, the ground state of the symmetry is not invariant, i.e. not a singlet under a symmetry transformation. Thus, the theory is spontaneously broken if there exists at least one generator that does not annihilate the vacuum

$$T^a |0\rangle \neq |0\rangle \quad \text{for some } a$$

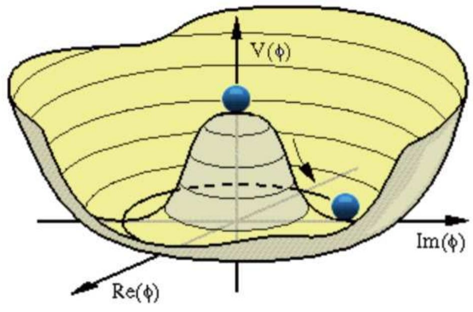


Figure 1: Theory with $U(1)$ invariant potential with "wrong" sign for the mass term. Since it is energetically favorable, theory will land into one of the minima, i.e. to one point on the circle. Going from one point to another needs "no energy", and this mode corresponds to masless Goldstone field.

That is, vacuum is not singlet under the group transformations. In general, this is the case if theory predicts more then one possible vacuum state, state of lowest energy. For illustration lets take real scalar field with a renormalisable potential and $Z_2$ symmetry ($\phi \to -\phi$)

$$V(\phi) = -\mu^2\phi^2 + \lambda\phi^4$$

Two possible configuration can have minimal energy $\langle\phi\rangle_\pm = \pm\sqrt{\frac{\mu^2}{2\lambda}} = \pm\frac{v}{\sqrt{2}}$. We settle down "spontaneously" to one of the possible vacuum states, and rewrite our field as a fluctuation around chosen vacuum state. For instance

$$\phi = \langle\phi\rangle_+ + \eta$$

where $\eta$ is now physical field, and $\langle\phi\rangle$ is called vacuum expectation value (VEV). Rewriting potential in terms of $\eta$, we get term $\propto \eta^3$, so apparently we lost initial symmetry. It is proper to say that symmetry is hidden by asymmetric vacuum state.

Anyway, lets extend idea to continuous groups. Let us take multiplet of scalar fields in some irreducible representation of some Lie group $\phi_i$, and suppose that potential $V(\phi)$ is minimized by some VEV

$$\langle\phi_i\rangle = v_i$$

Generator $T^a$ is called broken if $(T^a)_{ij}v_j \neq 0$, otherwise is called unbroken. An important theorem states that in a theory with spontaneously broken global symmetry, for each broken generator, we have a masless particle called Goldstone boson. Theory has a symmetry, so

$$V\left((1 + i\theta^a T^a)\phi\right) = V(\phi)$$

Expanding to linear order in the infinitesimal parameter $\theta$, we find

$$\frac{\partial V}{\partial\phi_j}(T^a)_{jk}\phi_k = 0$$

Differentiating with respect to $\phi_i$

$$\frac{\partial^2 V}{\partial\phi_j \partial\phi_i}(T^a)_{jk}\phi_k + \frac{\partial V}{\partial\phi_j}(T^a)_{ji} = 0$$

At $\phi = v$, potential has a minimum $\left.\frac{\partial V}{\partial\phi_j}\right|_{\phi=v} = 0$ and $m^2_{ij} \equiv \left.\frac{\partial^2 V}{\partial\phi_i\partial\phi_j}\right|_{\phi=v}$ is positive definite mass-squared matrix. Finally

$$m^2_{ij}(T^a v)_j = 0$$

We see that if $T^a v \neq 0$, then $T^a v$ is an eigenvector of the mass-squared matrix with eigenvalue zero.

If we gauge a theory, that is promote symmetry from global to local, interesting thing happens; Goldstone boson degrees of freedom are eaten up by the gauge fields to form physical, longitudinal polarization which means mass for the gauge boson! This is so called Higgs mechanism. A simple realization is to take complex scalar field

$$\mathcal{L} = (\partial_\mu\phi)(\partial^\mu\phi)^* - V(|\phi|)$$

which has global $U(1)$ symmetry. After gauging ($\partial^\mu \to D^\mu$) and spontaneous symmetry breaking, term $(D_\mu\phi)(D^\mu\phi)^*$ contains mass term for the gauge boson $\frac{v^2 g^2}{2}A^\mu A_\mu$.

## 2.3 Standard Model

Standard Model is an $SU(3)\times SU(2)\times U(1)$ gauge theory. $SU(3)$ part is referred to as QCD and describes strong interactions. It has 8 massless gauge fields called gluons. $SU(2)\times U(1)$ part describes electroweak interactions and is spontaneously broken down to $U(1)$ of QED, so three gauge bosons are massive, $W^\pm$ and $Z$, and the remaining one is masless photon. Matter content is made out of quarks and leptons, totally 15 Weyl fermions in 5 representation of the gauge group

$$Q \equiv \left(3, 2, +\frac{1}{6}\right)$$

$$L \equiv \left(1, 2, -\frac{1}{2}\right)$$

$$u^c \equiv \left(\bar{3}, 1, -\frac{2}{3}\right)$$

$$d^c \equiv \left(\bar{3}, 1, \frac{1}{3}\right)$$

$$e^c \equiv (1, 1, +1)$$

First column is representation under $SU(3)$ where 3, $\bar{3}$ and 1 stand for fundamental, anti-fundamental and singlet respectively. Second column is representation under $SU(2)$, where

2 stands for fundamental and 1 for singlet.[1] Third column is hypercharge $Y$, generator of $U(1)$, which is connected to QED charge

$$Q = T_3 + Y$$

where $T_3$ is Cartan generator of $SU(2)$. We work with left-handed fermions only. Common notation for the matter content is $Q$ and $L$ being left-handed Weyl fermions and $u \equiv \left(3, 1, \frac{2}{3}\right)$, $d \equiv \left(3, 1, -\frac{1}{3}\right)$ and $e \equiv (1, 1, -1)$ being right-handed Weyl fermions. In GUT framework it is convenient to work with left-handed fields only. Since

$$(\psi^c)_L = C\bar{\psi}_R^T$$

and charge conjugation changes all internal quantum numbers, that is, right-handed in representation $R$ is equivalent to left-handed in complex conjugate representation $\bar{R}$. This matter content repeats itself three times, in three generations. There is also a complex scalar field in representation

$$H \equiv \left(1, 2, \frac{1}{2}\right)$$

which is responsible for SSB, Higgs field. The Lagrangian of the theory is composed of kinetic part for fermions and gauge bosons, Higgs part and Yukawa part. Yukawa part of the Lagrangian is an interaction of Higgs field with fermions, through which fermions obtain mass

$$\mathcal{L}_Y = Y_u Q^T C i\sigma_2 H u^c + Y_d Q^T C H^* d^c + Y_e L^T C H^* e^c + h.c$$

To complete the specification of the Standard Model, we need twenty real numbers: the three gauge couplings; the three diagonal entries of each of the three diagonalized Yukawa coupling matrices (for the up quarks, the down quarks, and the charged leptons); the four angles in the CKM mixing matrix for the quarks; the vacuum angles for the SU(3) and SU(2) gauge groups; the scalar quartic coupling; and the scalar mass-squared.

## 2.4 Motivation for GUT

Phenomenological success of the Standard Model leads to a conclusion that Standard Model is a good theory of nature up to energies of 100 GeV, and if we are looking for theory beyond, in low energy limit the theory should correspond to Standard Model.

Why should we look for a theory beyond, especially for the GUT? Some of the motivation is simply search for more predictive and compact theory. Standard Model has 20 parameters, and GUT should relate some of these. For example, GUT theory instead of having three couplings, would have only one. Also, it would typically have less different representations for matter fields then 5 representations we have in SM. This implies some connections among different SM Yukawa couplings.

A serious motivation is to explain charge quantization. Since SM contains $U(1)$, hypercharge generators are fixed by experiment, and it is a puzzle why we get rational numbers.

[1] Remember that 2 and $\bar{2}$ are equivalent for $SU(2)$.

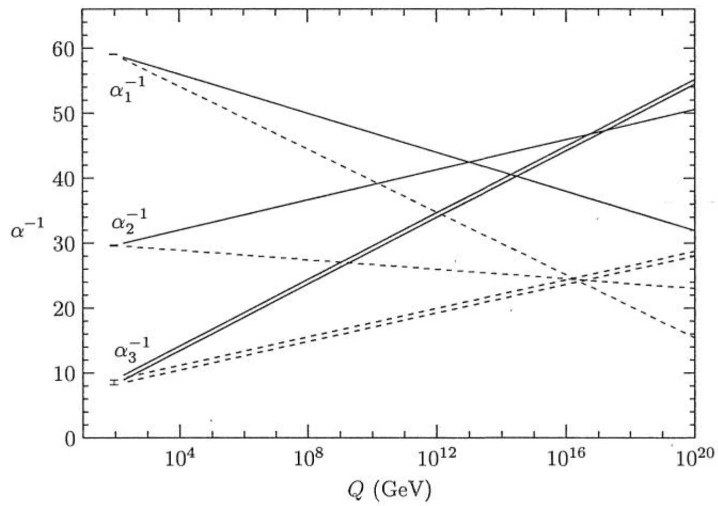


Figure 2: The running of $SU(3) \times SU(2) \times U(1)$ coupling constants. The solid line represents purely Standard Model calculation, while the dashed line represents calculation in Minimal Supersymmetric Standard Model.

Although, anomaly cancellation constraints do predict the electric charge quantization in the SM, this does not have any further experimental consequences. A theory based on one simple gauge group would solve this easily. In GUT, the electric charge operator is made out of a linear combination of non-abelian gauge algebra generators, and its eigenvalues are obviously quantized.

Also, there is an issue of neutrino mass. Minimal Standard Model predicts masless neutrinos which is in contradiction to experiment. Some GUT models can naturally give masses to neutrinos.

An important question to be asked is how does couplings of the SM group unify. Here it is crucial the notion of running coupling constants. It is well known feature of quantum field theories that coupling depends on the scale of interaction. This is called running of the coupling. Unification means that all three couplings of the Standard Model should at some scale meet into one coupling.

At one loop level, renormalization group equations (RGE) are given by

$$\frac{dg_i}{d\log\mu} = -\frac{b_i}{(4\pi)^2} g_i^3$$

where generic formula for the running coefficient

$$b_i = \frac{11}{3} T_G - \frac{2}{3} T_F - \frac{1}{3} T_B$$

and G,F,B stay for gauge bosons, fermions and bosons. $T_R$ is an index of representation ($\mathrm{Tr}[T_R^a T_R^b] = T(R)\delta^{ab}$). This gives for $SU(3)$, $SU(2)$ and $U(1)$, respectively

$$b_3 = \frac{33}{3} - \frac{4}{3} N_g$$

$$b_2 = \frac{22}{3} - \frac{4}{3}N_g - \frac{1}{6}$$

$$b_1 = \frac{3}{5}b_Y = -\frac{4}{3}N_g - \frac{1}{10}$$

where $N_g$ is the number of generations. Factor $\frac{3}{5}$ needs an explanation. In $SU(N)$ gauge theories, index for the fundamental is fixed to $\frac{1}{2}$. We have to normalize $U(1)$ hypercharge generator to correspond to one of the generators of GUT, that is

$$g'Y = g_1 T$$

We will see later that one of the $SU(5)$ representations is an antifundamental made out of $d^c$ and $L$. We thus have

$$g_1^2 Tr[T^2] = g_1^2 \left(\frac{1}{2}\right)^2 = g'^2 Tr[Y^2] = g'^2 \left(3\left(\frac{1}{3}\right)^2 + 2\left(-\frac{1}{2}\right)^2\right)$$

$$g' = \sqrt{\frac{3}{5}} g_1$$

From RGE we can conclude

$$b_1 = \frac{3}{5}b_Y$$

Also, there is a prediction of Weinberg angle at GUT scale $\sin^2\theta_W = \frac{3}{8}$. So, running coefficients in SM are $(7, \frac{19}{6}, \frac{-41}{10})$. By solving RGE, it is easy to prove that the couplings do not meet in one point, but if we add supersymmetry, we will get unification at the scale of $10^{16}$ GeV.

Although supersymmetry is not found yet, we can consider this calculation as one more motivation for GUT. Also, it is always possible to add new representations of fields, which would eventually cure SM calculation, and give a unification, without introducing supersymmetry.

# 3 $SU(5)$

## 3.1 Minimal setup (Georgi-Glashow Model)

First attempt to unify all forces in one group was done by Georgi and Glashow [6]. Starting point is SM gauge group $SU(3) \times SU(2) \times U(1)$, and 15 Weyl left-handed fermions in 5 different representations. Our goal is to find simple group which contains SM group as a subgroup, and an irreducible representations for fermions such that they have right property under SM group transformations. SM group has $8 + 3 + 1 = 12$ generators total, and 4 of them are Cartan generators. It means that our unifying group should be of rank grater or equal to 4 and have more then 12 generators. The minimal group with this properties is $SU(5)$, which is rank 4. Fundamental representation of $SU(5)$ is 5, and we can put for instance

$$5 \to \left(3, 1, -\frac{1}{3}\right) \oplus \left(1, 2, \frac{1}{2}\right)$$

equivalently

$$\overline{5} \to \left(\overline{3}, 1, \frac{1}{3}\right) \oplus \left(1, 2, -\frac{1}{2}\right)$$

or represented as a column matrix

$$\overline{5} \equiv \begin{bmatrix} d_1^c \\ d_2^c \\ d_3^c \\ e \\ -\nu \end{bmatrix}$$

so we put $d^c$ and $L$ into $\bar{5}$. Note for instance that we could not put $u^c$ with $L$, since hypercharge generator must be traceless because it is now one of the generators of $SU(5)$. [2] Our choice of 5 is consistent since $3 \times \frac{1}{3} + 2 \times \frac{-1}{2} = 0$. Following this, let us try to embed $SU(3) \times SU(2) \times U(1)$ into $SU(5)$. $SU(3)$ generators $\frac{\lambda^a}{2}$, $a = 1, ..., 8$ are traceless matrices acting on first three indices of 5

$$\begin{bmatrix} \frac{\lambda^a}{2} & 0 \\ 0 & 0 \end{bmatrix}$$

$SU(2)$ generators $\frac{\sigma^i}{2}$, $i = 1, 2, 3$ are also traceless and act on two last indices

$$\begin{bmatrix} 0 & 0 \\ 0 & \frac{\sigma^i}{2} \end{bmatrix}$$

and generator of hypercharge commutes with both groups

$$Y = \begin{bmatrix} -\frac{1}{3} & & & & \\ & -\frac{1}{3} & & & \\ & & -\frac{1}{3} & & \\ & & & \frac{1}{2} & \\ & & & & \frac{1}{2} \end{bmatrix}$$

[2] It is possible if we introduce additional $U(1)$ group, this leads to so called *flipped* $SU(5)$ model.

from here it is obvious why corresponding generator in $SU(5)$ is $T = \sqrt{\frac{3}{5}}Y$, since $Tr[T^2] = \frac{3}{5}Tr[Y^2] = \frac{1}{2}$. SM group algebra is satisfied by construction and is embed into $SU(5)$.

We are still left with 10 fields in 3 different representations of SM gauge group. To find irreducible representation of $SU(5)$ which contains these fields lets decompose direct product of

$$5 \otimes 5 = 15_S \otimes 10_A$$

Representations on right hand side carry two indices, since fundamental carries one. 15 is symmetric and 10 is antisymmetric under exchange of the indices. On the other hand,

$$5 \otimes 5 \to \left(\left(3, 1, -\frac{1}{3}\right) \oplus \left(1, 2, \frac{1}{2}\right)\right) \otimes \left(\left(3, 1, -\frac{1}{3}\right) \oplus \left(1, 2, \frac{1}{2}\right)\right)$$

$$= \left(6, 1, -\frac{2}{3}\right) \oplus \left(3, 2, \frac{1}{6}\right) \oplus (1, 3, 1) \oplus \left(3, 2, \frac{1}{6}\right) \oplus \left(\bar{3}, 1, -\frac{2}{3}\right) \oplus (1, 1, 1)$$

since in $SU(3)$ group $3 \otimes 3 = 6_S \oplus \bar{3}_A$ and in $SU(2)$ group $2 \otimes 2 = 3_S \oplus 1_A$, obviously antisymmetric 10 is

$$10 \to \left(3, 2, \frac{1}{6}\right) \oplus \left(\bar{3}, 1, -\frac{2}{3}\right) \oplus (1, 1, 1)$$

and this is exactly three remaining SM representations; $Q$, $u^c$ and $e^c$! Remarkably, one generation of quark and lepton fields fits exactly into the $\overline{5} \oplus 10$ representation of $SU(5)$. In matrix form, antisymmetric 10 is given by

$$10 \equiv \begin{bmatrix} 0 & u_3^c & -u_2^c & u_1 & d_1 \\ -u_3^c & 0 & u_1^c & u_2 & d_2 \\ u_2^c & -u_1^c & 0 & u_3 & d_3 \\ -u_1 & -u_2 & -u_3 & 0 & e^c \\ -d_1 & -d_2 & -d_3 & -e^c & 0 \end{bmatrix}$$

## 3.2 Higgs sector

We have to properly break $SU(5)$ to $SU(3) \times SU(2) \times U(1)$ at GUT scale by some Higgs field. Since $SU(5)$ has 24 gauge bosons, and SM has 12, the rest of the gauge bosons should get mass after SSB. So, we need to get at least 12 Goldstone bosons. Minimal representation of the Higgs which can do the job is 24, adjoint Higgs.

Let us discuss spontaneous symmetry breaking of $SU(N)$ group by Higgs field in its adjoint representation. We take real Higgs fields $\Phi^a$, $a = 1, ..., N^2 - 1$, and as already discussed, define a matrix

$$\Phi = \Phi^a T^a$$

Transformation rule is

$$\Phi \to U \Phi U^\dagger$$

The most general, renormalizable potential consistent with $SU(N)$ symmetry and a additional $Z_2$ symmetry $\Phi \to -\Phi$ is

$$V(\Phi) = -\frac{\mu^2}{2} Tr\Phi^2 + \frac{\lambda_1}{4} Tr\Phi^4 + \frac{\lambda_2}{4}\left(Tr\Phi^2\right)^2$$

Let $\Sigma$ be the VEV of the Higgs field. It is constant, traceless, hermitian $N \times N$ matrix. By group transformation $\Sigma \to U\Sigma U^\dagger$, we go from one vacuum to another and obviously, potential does not change. We have freedom to chose vacuum, such that $\Sigma$ is in the block diagonal form. Since, $U\Sigma U^\dagger = \Sigma + i\theta^c[T^c, \Sigma] + \cdots$, broken generators do not commute with the VEV, but generators of unbroken symmetry do. Suppose the diagonal entries consist of $N_1$ $v_1$'s, followed by $N_2$ $v_2$'s, etc., where $\sum N_i v_i = 0$. Then all generators whose nonzero entries lie entirely within the $i$-th block commute with $\Sigma$, and hence form an unbroken $SU(N_i)$ subgroup. Furthermore, the linear combination of diagonal generators that is proportional to $\Sigma$ also commutes with $\Sigma$ , and forms $U(1)$ subgroup. Thus the unbroken gauge group is $SU(N_1) \times SU(N_2) \times \cdots \times U(1)$.

As already discussed, we can work in a basis

$$\Phi = v\text{diag}\left(\alpha_1, ..., \alpha_N\right)$$

Since matrix is traceless, we have constraint $\sum \alpha_i = 0$. Also, we impose $\sum \alpha_i^2 = 1$, which is simply definition of $v$. We have $N-1$ independent variables, say $v, \alpha_1, ..., \alpha_{N-2}$. Plugging into potential, we get

$$V(\Phi) = -\frac{\mu^2}{2} v^2 + v^4 \left(\frac{\lambda_1}{4}\sum \alpha_i^4 + \frac{\lambda_2}{4}\right)$$

Since $1 > \sum \alpha_i^4 > 0$, which comes from $\sum \alpha_i^2 = 1$, for potential to be bounded from below, following condition must be satisfied $\lambda_1 \sum \alpha_i^4 + \lambda_2 > 0$. To find minimum of the potential, we solve

$$\frac{\partial V}{\partial v} = 0$$

which gives

$$v^2 = \frac{\mu^2}{\lambda_1 \sum \alpha_i^4 + \lambda_2}$$

Plugging into potential, we get

$$V = \frac{-\frac{\mu^4}{4}}{\lambda_1 \sum \alpha_i^4 + \lambda_2}$$

Global minimum is achieved when $\lambda_1 \sum \alpha_i^4 + \lambda_2$ is minimal. Our next task is to find minimum of the function $f(\alpha_1, ..., \alpha_{N-2}) = \lambda_1 \sum \alpha_i^4 + \lambda_2$, with constraints $\sum \alpha_i = 0$ and $\sum \alpha_i^2 = 1$. We can use Lagrange multipliers to handle constraints

$$g(\alpha_1, ..., \alpha_N, a, b) = \lambda_1 \sum \alpha_i^4 + \lambda_2 - a\left(\sum \alpha_i^2 - 1\right) - b\sum \alpha_i$$

and arrive to set of equations for extremum

$$\frac{\partial g}{\partial \alpha_j} = 4\lambda_1 \alpha_j^3 - 2a\alpha_j - b = 0$$

$$\sum \alpha_i = 0$$
$$\sum \alpha_i^2 = 1$$

Cubic equation can have at most three different solutions, and sum of them is 0, since there is no quadratic term! Diagonal entries of VEV can have either one, two or three different values. First possibility is that all $\alpha_j$ are the same, but from second equation they must be 0. In general case, it can be shown that global minima has two different values on diagonal, which come $N_+$ and $N_-$ times. With given constraints, it is straightforward to show the function we want to minimize, $\sum \alpha_i^4$ is proportional to $\frac{N_+}{N_-} + \frac{N_-}{N_+}$. So, global minimum for even $N$ is $N_+ = N_- = \frac{N}{2}$, and for odd $N$ is $N_\pm = \frac{N\pm 1}{2}$. To conclude, $SU(N)$ is spontaneously broken down to $SU(N_+) \times SU(N_-) \times U(1)$ by adjoint Higgs representation, with assumption we want to live in global minima.

Obviously, this result is perfect for $SU(5)$! We break it down to $SU(3) \times SU(2) \times U(1)$ with $\Sigma = \frac{v}{\sqrt{30}} \mathrm{diag}\,(2,2,2,-3,-3)$ and global minima comes as free lunch! From the form of VEV, we can explicitly see it is preserved by SM generators listed in previous section. Also, hypercharge generator is proportional to VEV, which is consistent with general discussion from the beginning of this section.[3]

Let us work out spectrum of Higgs fields under unbroken $SU(3) \times SU(2) \times U(1)$. Since $5 \to \left(3,1,-\frac{1}{3}\right) \oplus \left(1,2,\frac{1}{2}\right)$, and as already shown for $SU(5)$

$$5 \otimes \bar{5} = 1 \otimes 24$$

On the other hand

$$5 \otimes \bar{5} \to \left(\left(3,1,-\frac{1}{3}\right) \oplus \left(1,2,\frac{1}{2}\right)\right) \otimes \left(\left(\bar{3},1,\frac{1}{3}\right) \oplus \left(1,2,-\frac{1}{2}\right)\right)$$
$$= (8,1,0) \oplus (1,1,0) \oplus \left(3,2,-\frac{5}{6}\right) \oplus \left(\bar{3},2,\frac{5}{6}\right) \oplus (1,3,0) \oplus (1,1,0)$$

So the spectrum of adjoint Higgs under SM gauge group is

$$24 \to O\,(8,1,0) \oplus S\,(1,1,0) \oplus X\left(3,2,-\frac{5}{6}\right) \oplus \bar{X}\left(\bar{3},2,\frac{5}{6}\right) \oplus T\,(1,3,0)$$

With obvious notation

$$\Phi = \begin{bmatrix} O & X \\ \bar{X} & T \end{bmatrix} + \left(1 + \frac{S}{v}\right)\Sigma$$

To find the masses, we plug $\Phi$ into potential. For simplicity, we can take one field from each SM representation, since the remaining symmetry guaranties the same mass within a representation. After some algebra, we get a mass spectrum

- color octet $O$: $m_O^2 = \frac{\lambda_1 v^2}{6}$

[3] It means, the choice for $d^c$ representation is consistent with the choice of Higgs.

- isospin triplet $T$: $m_T^2 = \frac{2}{3}\lambda_1 v^2$
- Goldstone bosons $X$, $\bar{X}$: $m_{X\bar{X}}^2 = 0$
- SM singlet $S$: $m_s^2 = 2\mu^2$

As expected, after SSB we get 12 Goldstone bosons, which will eventually be eaten up in the next section.

## 3.3 The gauge boson mass

Kinetic term for the Higgs field in adjoint of global $SU(N)$ group is

$$\mathrm{Tr}\left((\partial_\mu \Phi)^\dagger (\partial^\mu \Phi)\right) = \partial_\mu \Phi^a \partial^\mu \Phi^b \mathrm{Tr}\left(T^a T^b\right) = \frac{1}{2}(\partial^\mu \Phi^a)^2$$

We promote global symmetry to local by replacing ordinary derivative with covariant derivative

$$D_\mu \Phi = \partial_\mu - igA_\mu^a \left[T^a, \Phi\right]$$

where $A_\mu^a$ are gauge bosons, and $A_\mu \equiv A_\mu^a T^a$. To check that covariant derivative transforms properly, we employ $\Phi \to U\Phi U^\dagger$ and $A_\mu \to UA_\mu U^\dagger - \frac{i}{g}(\partial_\mu U)U^\dagger$. After some algebra, we get $D_\mu \Phi \to U\left(D_\mu \Phi\right)U^\dagger$, which obviously gives gauge invariance.

After Higgs gets a VEV, gauge bosons will get a mass from kinetic term

$$\mathrm{Tr}\left((igA_\mu^a \left[T^a, \Sigma\right])^\dagger (igA^{b\mu}\left[T^b, \Sigma\right])\right) = g^2 \mathrm{Tr}\left(([A_\mu, \Sigma])^\dagger ([A^\mu, \Sigma])\right)$$

Gauge bosons corresponding to the generators of unbroken group (SM gauge bosons), will remain massless. The gauge fields corresponding to twelve broken generators can be grouped into a complex vector field in the representation $\left(3, 2, -\frac{5}{6}\right)$ and will all get the same mass. For instance

$$A_x^\mu \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + A_y^\mu \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & 0 & 0 & X^\mu & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ X^{\mu\dagger} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

where $X^\mu = \frac{A_x^\mu + iA_y^\mu}{\sqrt{2}}$. Writing

$$A^\mu = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 & X \\ X^\dagger & 0 \end{bmatrix}$$

$$[A_\mu, \Sigma] = \frac{-5v}{\sqrt{30}\sqrt{2}}\begin{bmatrix} 0 & X \\ X^\dagger & 0 \end{bmatrix}$$

we get a mass term

$$\frac{5}{6} g^2 v^2 X^\mu X_\mu^\dagger$$

So, the mass of gauge bosons corresponding to broken generators is $M_X^2 = \frac{5}{6}g^2v^2$. This also defines the GUT scale. Since gauge bosons live in the adjoint representation of $SU(5)$, decomposition to SM representations is similar to one for the Higgs field. For completeness we explicitly write $A^\mu$ matrix with physical fields

$$\begin{bmatrix} G_r^r - \frac{1}{3}\sqrt{\frac{3}{5}}B & G_r^g & G_r^b & \frac{1}{\sqrt{2}}X_r^1 & \frac{1}{\sqrt{2}}X_r^2 \\ G_g^r & G_g^g - \frac{1}{3}\sqrt{\frac{3}{5}}B & G_g^b & \frac{1}{\sqrt{2}}X_g^1 & \frac{1}{\sqrt{2}}X_g^2 \\ G_b^r & G_b^g & G_b^b - \frac{1}{3}\sqrt{\frac{3}{5}}B & \frac{1}{\sqrt{2}}X_b^1 & \frac{1}{\sqrt{2}}X_b^2 \\ \frac{1}{\sqrt{2}}X_r^{\dagger 1} & \frac{1}{\sqrt{2}}X_g^{\dagger 1} & \frac{1}{\sqrt{2}}X_b^{\dagger 1} & \frac{1}{2}W^3 + \frac{1}{2}\sqrt{\frac{3}{5}}B & \frac{1}{\sqrt{2}}W^+ \\ \frac{1}{\sqrt{2}}X_r^{\dagger 2} & \frac{1}{\sqrt{2}}X_g^{\dagger 2} & \frac{1}{\sqrt{2}}X_b^{\dagger 2} & \frac{1}{\sqrt{2}}W^- & -\frac{1}{2}W^3 + \frac{1}{2}\sqrt{\frac{3}{5}}B \end{bmatrix}$$

## 3.4 Yukawa sector

Our matter fields live in two irreducible representations of $SU(5)$

$$10_F = \begin{bmatrix} \epsilon_3 u^c & Q \\ -Q^T & \epsilon_2 e^c \end{bmatrix} \quad \overline{5}_F = \begin{bmatrix} d^c \\ \epsilon_2 L \end{bmatrix}$$

where $\epsilon_3$ and $\epsilon_2$ are Levi-Civita tensor in 3 and 2 dimensions. First representation is antisymmetric and transforms tensor with two indices up $\chi^{ij} = -\chi^{ij}$, and second representation transforms a tensor $\psi_i$. As already mentioned in tensor methods, contraction of upper and lower indices gives an invariant.

We are still missing SM Higgs field, which is responsible for electroweak symmetry breaking of $SU(2) \times U(1) \to U(1)_{em}$ and which gives masses to all massive particles of SM. SM Higgs is an $SU(2)$ doublet, and minimal way to put it into theory, is to add additional fundamental representation of $SU(5)$. It would contain SM Higgs, and an additional colour triplet

$$5_H \equiv \begin{bmatrix} T \\ H \end{bmatrix}$$

This field can give masses to all fermions through renormalizable Yukawa interaction

$$-\mathcal{L}_Y = \overline{5}_F Y_5 10_F 5_H^* + \epsilon_5 10_F Y_{10} 10_F 5_H$$

in the first term, two upper indices of 10 are contracted by lower indices of $\overline{5}_F$ and $5_H^*$. In the second term we use 5 dimensional Levi-Cevita tensor to contract all upper indices. $Y_5$ and $Y_{10}$ are matrices in generation space.

We are interested in Yukawa terms with SM Higgs

$$\overline{5}_F Y_5 10_F 5_H^* = \begin{bmatrix} d^c & -L\epsilon_2 \end{bmatrix} Y_5 \begin{bmatrix} \epsilon_3 u^c & Q \\ -Q^T & \epsilon \end{bmatrix} \begin{bmatrix} T^* \\ H^* \end{bmatrix}$$

$$\to d^c Y_5 Q H^* + L Y_5 e^c H^*$$

As we already noted, GUT will typically relate different Yukawa couplings. In this case, prediction of the minimal theory is $Y_D = Y_E^T$, that is Yukawa mass matrix for down quarks is just the transpose of the Yukawa mass matrix for the charged leptons. Unfortunately, this relation is not satisfied in nature. Going beyond Georgi-Glashow model can be done by introducing $45^{\gamma}_{\alpha\beta}$ dimensional Higgs at renormalizable level, or introducing 5 dimensional operators, for instance $\bar{5}^c Y_5^{(1)} 10_F \left(\frac{\Phi}{\Lambda} 5_H\right)^*$, and since adjoint times fundamental transforms as fundamental, this term is gauge invariant. If we add these additional terms, we have enough freedom to get correct Yukawas. Going beyond minimal theory always reduces predictivity.

Up quark Yukawa interaction with SM Higgs can be obtained from the second term in Yukawa Lagrangian. After some algebra, we get interacting term $4u^c(Y_{10} + Y_{10}^T)QH$ and thus predict Yukawa mass matrix for the up quarks to be symmetric.

## 3.5 Proton decay

Baryon and lepton number conservation are accidental symmetries of the Standard Model. It is simply that if we postulate SM gauge group with SM matter content, it is not possible to write renormalizable, Lorentz invariant term which would violate baryon and lepton number.

Baryon and lepton number are naturally violated in GUT. Since we put leptons and quarks in the same irreducible representation of the gauge group, there must be an interaction between them. Obviously, massive gauge bosons from $\left(3, 2, -\frac{5}{6}\right)$ mediate interactions between quarks and leptons.

Let us estimate lifetime of a proton. We have two scales in the problem; $M_{GUT}$, which is the mass of the gauge bosons, and mass of the proton $m_p$. Since amplitude for the process should have gauge boson mass squared in the propagator, decay width on dimensional grounds is

$$\Gamma(p \to \pi^0 e^+) \approx \frac{m_p^5}{M_{GUT}^4}$$

Presently, experimental lower limit on lifetime of a proton is around $10^{34}$ years, which pushes GUT scale to

$$M_{GUT} \gtrsim 10^{16}\,\text{GeV}$$

This is very close to the predicted unification scale from supersymmetric GUT, which makes searches for the proton decay more exciting.

## 3.6 The doublet-triplet hierarchy problem

The SM Higgs doublet, when embedded in GUT representation typically comes with a triplet partner which can mediate proton decay. Yukawa interactions for triplet in the minimal $SU(5)$ model

$$-\mathcal{L}(T) = T^*(LY_5Q - d^cY_5u^c) - (\frac{1}{2}QY_{10}Q + u^cY_{10}e^c)T^*$$

which obviously allows for quark to lepton transitions. In this case, amplitude for proton decay is roughly proportional to $\frac{y_u y_d}{M_T^2}$. Comparing to previous result, we get constraint on the mass of scalar leptoquark $M_T \gtrsim 10^{12}\,\text{GeV}$, which is very far from electroweak scale and mass of the SM Higgs boson. It is very difficult to naturally achieve heavy color triplet and light weak doublet from the same multiplet, and one need in general huge amount of fine tuning. This is famous doublet-triplet splitting problem.

Let us make this point more quantitative. Interaction between Higgs responsible for $SU(5)$ breaking and Higgs responsible for Yukawa terms is

$$5^*_H(a24_H + b)5_H$$

Once adjoint Higgs gets a VEV, triplet and doublet Higgs get a mass

$$\left(2a\frac{v}{\sqrt{30}} + b\right)|T|^2 + \left(-3a\frac{v}{\sqrt{30}} + b\right)|H|^2 = M_T^2\,|T|^2 + M_H^2\,|H|^2$$

schematically; $x + y \approx 10^{24}$ and $x - y \approx 10^4$, so we need large fine-tuning of parameters!

## 3.7 Current status of $SU(5)$

Minimal Georgi-Glashow model is a history by now, since it predicts wrong Yukawa couplings, massless neutrinos and gauge couplings do not unify at all! Nevertheless, extensions of the minimal model, can solve these drawbacks, and leave $SU(5)$ in the game. We list two realistic non-susy $SU(5)$ models:

- Additional symmetric complex scalar field $15_H$, and higher dimensional operators for charged fermion masses [7]. Generic prediction of this setup is a set of rather light scalar leptoquarks and the model offers interesting phenomenology [8].

- Additional adjoint fermion field $24_F$ [9]. One needs higher dimensional operators both for charged fermions and for realistic neutrino Dirac Yukawa couplings. The main prediction of the theory is the light fermionic $SU(2)$ triplet with mass at the electroweak scale. Its phenomenology is quite interesting for it leads to lepton number violation at colliders in the form of same sign dileptons.

Supersymmetric $SU(5)$ is the famous one. It successfully unified gauge couplings, predicted unification scale, and lead to indication of large top mass! However, it has a big disadvantage since it allows for dimension 5 operators through the exchange of heavy color triplet Higgsino and eventually enhance proton decay. Even if we take very large mass $M_T \simeq 10^{16}\,\text{GeV}$, we get $\tau_p \simeq 10^{30-31}$ years! This model is barely alive today, but some uncertainties in sfermion masses and mixings still keep it possible.

To conclude with, $SU(5)$ GUT was a big theoretical success and it will always remain as an important playground for theorist seeking toward unified models of nature.

# 4 $SO(10)$

$SO(10)$ is larger group then $SU(5)$ and much richer in structure. It can break to several interesting subgroups, such as $SU(5)\times U(1)$, which is the group of flipped $SU(5)$, or to $SO(4)\times SO(6)$ group of Pati and Salam. Unification can easily be achieved with or without supersymmetry. $SO(10)$ is in fact minimal grand unified theory that unifies matter on top of interactions and suggests naturally small neutrino masses through the seesaw mechanism. $SO(10)$ has so called 16 spinoral representation, which decomposes under $SU(5)$ as $10\oplus\bar{5}\oplus 1$, and can unify all SM matter of one family with a prediction of right-handed neutrino! Its rich structure, and physically important subgroups makes $SO(10)$ the main GUT candidate and very interesting group to study.

## 4.1 $SO(2N)$ group theory

As already stated in general part about group theory, orthogonal group $SO(N)$ is the group of orthogonal real $N\times N$ matrices $O^TO=1$ with $\det O=1$. Fundamental representation is formed by matrices $O$ themselves. Vector $v^i$ transforms under fundamental as

$$v^i \to O^{ij}v^j$$

Infinitesimal transformations are

$$O^{ij}\to \delta^{ij}+\omega^{ij}$$

From the orthogonality condition $O^{ij}O^{kj}=\delta^{ik}$ we get $\omega^{ij}=-\omega^{ji}$, so we have $\frac{N(N-1)}{2}$ independent parameters of the group.

Tensor representations are build out by adding more indices, and every index transforms under fundamental representation, for instance

$$T^{ijk}\to O^{il}O^{jm}O^{kn}T^{lmn}$$

We do not distinguish between upper and lower indices since tensor representations are real! Obviously, sum over the repeated index gives an invariant. It is also valid that $\epsilon^{i_1 i_2\cdots i_N}$ is an invariant tensor as already shown in tensor methods.

Consider tensor with two indices $T^{ij}$. We can form symmetric $S^{ij}\equiv\frac{1}{2}(T^{ij}+T^{ji})$ and antisymmetric combinations $A^{ij}\equiv\frac{1}{2}(T^{ij}-T^{ji})$. The symmetric combination transforms into $O^{il}O^{jm}S^{lm}$, which is obviously symmetric. Similarly, $A^{ij}$ stays antisymmetric after transformation. In other words, the set of $N^2$ objects contained in $T^{ij}$ split into two sets: $\frac{1}{2}N(N+1)$ symmetric objects $\frac{1}{2}N(N-1)$ antisymmetric object which form invariant subspace, that is we reduced representation $T^{ij}$.

We can reduce symmetric representation further. Take a combination $T\equiv T^{ij}\delta^{ij}$ which is obviously invariant under group representation. We can subtract the trace from $T^{ij}$ forming the traceless tensor $Q^{ij}\equiv T^{ij}-\frac{1}{N}\delta^{ij}T$ . The $\frac{1}{2}N(N+1)-1$ objects contained in $Q^{ij}$ transform among themselves.

As an example, building block of $SO(10)$ group is fundamental 10 representation. Direct product with itself decomposes to

$$10 \otimes 10 \to 45 \oplus 54 \oplus 1$$

as already discussed. Here, $45^{ij} = -45^{ji}$ is antisymmetric representation, and $54^{ij} = 54^{ji}$ is traceless symmetric representation. Representation with 3 indices completely antisymmetric is 120 $(= 10 \times 9 \times 8/3!)$, and with 5-indices completely antisymmetric with an extra self duality relation is $126_{ijklm} = \pm\frac{i}{5}\epsilon_{ijklmnopqr}126_{nopqr}$ where $126 = \frac{1}{2}10 \times 9 \times 8 \times 7 \times 6/5!$.

General representation of $SO(N)$ group can be written in the following form

$$e^{-i\frac{\omega^{ij}}{2}J^{ij}}$$

where $J^{ij}$ are generators of the group and $J^{ij} = -J^{ji}$, $i, j = 1, ..., N$. For fundamental representation

$$\left(J^{ij}\right)^{kl} = i\left(\delta^{ik}\delta^{jl} - \delta^{il}\delta^{jk}\right)$$

which is consistent with a definition of the parameters

$$\omega^{ij} = -i\frac{\omega^{kl}}{2}(J^{kl})^{ij}$$

Fundamental representation is the defining representation of the group, from which we can extract group algebra

$$\left[J^{ij}, J^{kl}\right] = i(\delta^{ik}J^{jl} - \delta^{jk}J^{il} + \delta^{jl}J^{ik} - \delta^{il}J^{jk})$$

$SO(2N)$ groups have very important representation beyond tensor representations.

## 4.2 Spinor representation of $SO(2N)$

$SO(2N)$ orthogonal groups have additional complex representation, which is called spinor representation and is the generalization of Lorentz spinors. Let us construct this representation explicitly. For any integer $n$ it is possible to find $2n$ hermitian matrices $\gamma^i$ which satisfy Clifford algebra

$$\left\{\gamma^i, \gamma^j\right\} = 2\delta^{ij}$$

Proof is by induction. If $n = 1$, obviously we can choose $\gamma^1 = \tau^1$ and $\gamma^2 = \tau^2$, where $\vec{\tau}$ are Pauli matrices. Given $2n$ $\gamma$ matrices, we can construct $(2n + 2)$ $\gamma$ matrices for $SO(2n + 2)$ in following way

$$\gamma_j^{(n+1)} = \gamma_j^{(n)} \otimes \tau^3 = \begin{bmatrix} \gamma_j^{(n)} & 0 \\ 0 & -\gamma_j^{(n)} \end{bmatrix} \qquad j = 1, ..., 2n$$

$$\gamma_{2n+1}^{(n+1)} = 1 \otimes \tau^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\gamma^{(n+1)}_{2n+2} = 1 \otimes \tau^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

It is straightforward to check explicitly that if $\gamma^{(n)}$ satisfy Clifford algebra, than $\gamma^{(n+1)}$ also satisfy. By this, we conclude our construction of $\gamma$ matrices which have $2^n \times 2^n$ dimension for $SO(2N)$. For illustration let us write explicitly $\gamma$ matrices in the case of $SO(4)$ and $SO(6)$. $SO(4)$ has 4 matrices with dimension 4

$$\gamma^1 = \begin{bmatrix} \tau^1 & 0 \\ 0 & -\tau^1 \end{bmatrix} \; \gamma^2 = \begin{bmatrix} \tau^2 & 0 \\ 0 & -\tau^2 \end{bmatrix} \; \gamma^3 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; \gamma^4 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

while $SO(6)$ has 6 matrices with dimension 8

$$\Gamma^i = \begin{bmatrix} \gamma^i & 0 \\ 0 & -\gamma^i \end{bmatrix} \; i = 1, .., 4$$

$$\Gamma^5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \; \Gamma^6 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Now we define $\frac{2n(2n-1)}{2}$ hermitian matrices

$$\sigma^{ij} = \frac{i}{2}\left[\gamma^i, \gamma^j\right]$$

Due to antisymmetry in indices, generators are nonzero just for $i \neq j$, and in that case $\sigma^{ij} = i\gamma^i\gamma^j$. Also, one can explicitly check that $\frac{\sigma^{ij}}{2}$ satisfy Lie algebra of the group and hence they are the generators of the group in so called spinor representation. Spinor is column vector $\psi$, with $2^n$ rows, which transforms under $SO(2N)$ group in following way

$$\psi \to e^{-i\frac{\omega^{ij}}{2}\sigma^{ij}}\psi$$

With spinors and $\gamma$ matrix, we can construct special form which transforms under fundamental representation of $SO(2N)$

$$\psi^\dagger\gamma^k\psi \to \psi^\dagger e^{i\frac{\omega^{ij}}{2}\sigma^{ij}}\gamma^k e^{-i\frac{\omega^{ij}}{2}\sigma^{ij}}\psi = \psi^\dagger\gamma^k\psi + i\frac{\omega^{ij}}{2}\psi^\dagger\left[\sigma^{ij}, \gamma^k\right]\psi + \cdots$$

By using definition of the generators of spinor representation, we can show $\left[\sigma^{ij}, \gamma^k\right] = -2i\left(\delta^{ik}\gamma^j - \delta^{jk}\gamma^i\right)$, plugging into previous expression, we get

$$\psi^\dagger\gamma^k\psi \to \psi^\dagger\gamma^k\psi + \omega^{kj}\psi^\dagger\gamma^j\psi$$

which is transformation rule for fundamental representation.

Following similarity to Lorentz spinors, on can define

$$\gamma^5 = (-i)^n\gamma^1\gamma^2\cdots\gamma^n$$

Also left handed and right handed spinors are defined in following way

$$\psi_L = \frac{1-\gamma^5}{2}\psi \quad \psi_R = \frac{1+\gamma^5}{2}\psi$$

and $\gamma^5\psi_{L/R} = \mp\psi_{L/R}$. Since $[\gamma^5, \sigma^{ij}] = 0$, $SO(2N)$ group transformations keep handness unchanged

$$\psi_L \to e^{-i\frac{\omega^{ij}}{2}\sigma^{ij}}\frac{1-\gamma^5}{2}\psi = \frac{1-\gamma^5}{2}e^{-i\frac{\omega^{ij}}{2}\sigma^{ij}}\psi = \psi_L'$$

The projection into left and right handed spinors cut the number of components into halves and thus we arrive at the important conclusion that the two irreducible spinor representations of $SO(2N)$ have dimension $2^{n-1}$.

For instance, in $SO(10)$

$$16 = \frac{1}{2}(1-\gamma^5)32 \quad \bar{16} = \frac{1}{2}(1+\gamma^5)32$$

## 4.3 Model building in $SO(10)$

As we saw in $SU(5)$, we put all the fermions into two irreducible representations $\bar{5}\oplus 10$. Power of $SO(10)$ is that we can incorporate all 15 Weyl fields into one irreducible representation, 16 spinor representation, with a prediction of a new field, which is desperately needed, right handed neutrino.

$$16_F = (Q, u^c, d^c, L, \nu^c, e^c)$$

Due to property of unifying all matter into one irreducible representation, one can derive various constraints on Standard Model Yukawa couplings. In $SO(10)$ models, only three types of Yukawas are possible, i.e. only 10, 120 and 126 dimensional Higgses can couple to spinorial bilinears, since

$$16 \otimes 16 \to 10 \oplus 120 \oplus 126$$

Generally, people take 10 and 126 only, with the SM Higgs doublets (remember that in MSSM there must be two Higgs doublets) living in both 10 and 126 (i.e. linear combinations of doublets there).

Since $SO(10)$ has rank 5, and we have to break it to rank 4 of SM, this can be done together with giving mases to right-handed neutrino just by giving VEV to SM singlet in 126. Also, breaking of SO(10) into SM needs following representations: 54 and 45 or 210 only. Model building in $SO(10)$ is a rich subject.

# 5 Conclusion

In this paper I gave a short review of Grand Unification Theories, with an emphases on group theory behind. In the first part, I discussed basics of group theory and its relation to particle physics. I explained that fundamental fields transforms under irreducible representations of the symmetry group, and all properties of the particles are defined by this representation, which clearly indicates importance of symmetry and group theory.

I also gave a short review on Standard Model and basic principles and ideas behind the theory, such as gauge theories and spontaneous symmetry breaking. An important concept of running coupling was introduced. This was a preparation for the main part, Grand Unification.

The main part of the work was the discussion of the $SU(5)$ Grand Unification Theory which was the first such theory. I explained the Minimal Model in details, its group structure, representation structure, Higgs sector and Yukawa sector and its important phenomenological predication, proton decay. I also pointed out main drawbacks of the minimal model, and possible solutions by going beyond minimal.

In the $SO(10)$ part, I put emphasis on representation structure more then on a concrete model. I have constructed irreducible representation of $SO(2N)$ groups, with special care for spinor representation.

Grand unified theories are still attractive Beyond Standard Model theories. Unfortunately, beauty of the theory confronts up-to-date experimental capabilities.

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