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Discrete Flavor Symmetries in Models of Neutrino Mixing

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Abstract

New experimental data on neutrino oscillation suggest a definite form of the neutrino mixing matrix. Within the present bounds a number of models has been suggested, usually in the frame of non-Abelian discrete symmetries. In this work, the most popular group for this purpose, A_4, is studied and a physical model is constructed. Also, the possible origin of such a symmetry is investigated in terms of breaking a continuous symmetry.

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1 Introduction

The standard model (SM) of particles and interactions was constructed in such a way as to contain massless neutrinos, which was a reasonable assumption at the time of its creation. For decades, physicists were suspecting otherwise, assuming that these particles have an extraordinary small, but non-vanishing mass. Numerous experiments on neutrino oscillations finally proved them right, which was the first sign of physics beyond the SM. This means that, like in the quark sector, there is a difference between mass and flavor eigenstates. To put it in a different way, the neutrinos entering interactions in actual measurements, being the ones with well defined flavor, are mixtures of mass eigenstates, or as physicists like to refer to them, the physical states. One can directly connect the according mixing angles between the states in the two mentioned bases to the differences between squared masses of the different mass states, which will be shown in chapter 2. These values lead to mixing patterns suggesting an underlying symmetry. In particular, the matrix inducing the change of basis between flavor and mass eigenstates acquires a definite shape within a few σ and can thus be connected to some symmetry group.

This immediately raises the question of which particular group actually is at work. Non-Abelian discrete symmetries have been studied widely in this context, becoming an important tool for model building in flavor physics. To be specific, A_4, the group of even permutations of four objects is the mostly addressed one in this area, and will be explained in some detail in this seminar. On the other hand, one might also want to explain the origin of the symmetries at work. A possible mechanism is the breaking of a continuous (gauge) group down to its discrete subgroup. Chapter 5 gives the simplest choice of such a group. Other possibilities include extra dimensions and superstring theory, which will not be part of this work ([4], [5]).

An interesting feature of neutrino mixing is that the angles are completely different from those in the quark sector. While the Cabbibo-Kobayashi-Maskawa (CKM) matrix is near unity, the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix, corresponding to neutrino mixing, is far from that. There has been a number of attempts to find a theory which naturally implements both patterns, usually in the picture of grand unified theories (GUTs).

Experiments investigating solar and atmospheric neutrinos measured two distinct mixing angles. This demands the existence of at least three different mass eigenstates of which two, in principle, must be massive. We know that two mixing angles, θ_{23} and θ_{12}, are large, the former being near maximal, while θ_{13}, having a very small upper limit, could as well vanish. ([7], table [1]). Given the notation $\Delta m^2{}_{sun} \equiv |m_2|^2 - |m_1|^2 > 0$ and $\Delta m^2{}_{atm} \equiv |m_3|^2 - |m_2|^2$, one can differ between three possible scenarios:

- Degenerate: $|m_1| \sim |m_2| \sim |m_3| \gg |m_i - m_j|$
- Inverted hierarchy: $|m_1| \sim |m_2| \gg |m_3|$
- Normal hierarchy: $|m_3| \gg |m_{1,2}|$

It is obvious at this point that oscillation experiments do not provide any information on the

Table 1: Fits to neutrino oscillation data

Quantity	Fogli et al., 2008 [1] [2]	Maltoni and Schwetz, 2008 [3]
$\Delta m^2{}_{sun}(10^{-5}eV^2)$	$7.67^{+0.16}_{-0.19}$	$7.65^{+0.23}_{-0.20}$
$\Delta m^2{}_{atm}(10^{-3}eV^2)$	$2.39^{+0.11}_{-0.08}$	$2.40^{+0.12}_{-0.11}$
$\sin^2\theta_{12}$	$0.312^{+0.019}_{-0.018}$	$0.304^{+0.022}_{-0.016}$
$\sin^2\theta_{23}$	$0.466^{+0.073}_{-0.058}$	$0.50^{+0.07}_{-0.06}$
$\sin^2\theta_{13}$	0.016 ± 0.010	$0.010^{+0.016}_{-0.011}$

absolute neutrino mass scale. Upper limits are obtained from the end point of the tritium beta decay spectrum, measurements of the cosmic microwave background (CMB) as well as from neutrinoless beta decay ($0\nu\beta\beta$). The discovery of the latter would not only give us information about the absolute mass scale of neutrinos but would also establish lepton number (L) violation as well as the Majorana nature of these particles. In close connection to that fact, one could explain the neutrinos' extremely small masses by the so called see-saw mechanism, where the neutrino mass is inversely proportional to the large scale at which L is violated.

In chapter 2 basic concepts of neutrino mass and oscillations are explained, chapter 3 gives an overview of the groups S_4 and A_4 together with their geometrical interpretation, in chapter 4 a concrete model based on the group A_4 is given, while in chapter 5 the idea of $SO(3) \rightarrow A_4$ breaking is discussed.

2 Neutrino masses

If one wants to work in the frame of the SM gauge group of electroweak interactions, $SU(2)_L \times U(1)_Y$, it is possible to give the neutrinos a mass by simply adding new degrees of freedom. The minimal choice is to introduce a very heavy right handed neutrino. Not only does that lead to the familiar Yukawa term, but also to a Majorana mass term in the Lagrangian, which violates L

$$-\mathcal{L}_{m\nu} = y_\nu \ \overline{\nu_R} \ H^c \ l + M \ \overline{\nu_R} \ \nu_R^c + h.c., \tag{1}$$

where y_ν is the Yukawa coupling, l the left handed lepton dublet and M the mass of the right handed neutrino ν_R. Diagonalization of the mass matrix then leads to the famous see-saw formula

$$m_\nu = -{m_D}^T M^{-1} \ m_D, \tag{2}$$

making the mass of neutrinos inversely proportional to the large right handed neutrino mass. For $m_\nu \sim \sqrt{\Delta {m^2}_{atm}} \sim 0.5$ eV and the Dirac mass $m_D \sim v \sim 200$ GeV, M turns out to be near the GUT scale, $M \sim 10^{15}$ GeV, which one could interprete as a link between neutrino masses and grand unified theories.

2.1 Neutrino oscillations

The neutrino beam produced by charged current interactions in an experiment is a superposition of different mass states and the probability of finding a certain particle state in that beam evolves with time. Thus neutrino oscillation experiments usually include great distances between the production point and the detector. It was mentioned before that mixing angles can be expressed through mass squared differences. This can be done as follows [8]. Assume a neutrino with flavor l is created at time $t = 0$. In general, it is a superposition of the physical states ν_α, linked to them by a matrix U. After a time t, the state has evolved into

$$|\nu_l(t)\rangle = \sum_\alpha e^{-iE_\alpha t} U_{l\alpha} |\nu_\alpha\rangle. \tag{3}$$

Now, the probability of finding a different flavor state ν_l' in the original ν_l will be proportional to multiples of the matrix elements of U times $\cos(E_\alpha - E_\beta)$, which directly leads to the mass dependence through the relation $m^2 = E^2 - p^2$. In this way, experimental data on the observed probabilities restrict Δm^2 as a function of the mixing angles.

2.2 Mixing and mass matrices

The matrix U can be parametrized in terms of three mixing angles ($0 \le \theta_{ij} \le \pi/2$) and one phase ϕ ($0 \le \phi \le 2\pi$) like

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{i\phi} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\phi} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{4}$$

There are different approaches, for experimental results still contain large uncertainties (table 1). One can work in the frame of so called "normal" models, where θ_{23} is not too close to maximal, and θ_{13} does not vanish. In this work, the case of "exceptional" will be studied, where one assumes a maximal atmospheric angle and $\theta_{13} = 0$. For $c_{23} \sim s_{23} \sim 1/\sqrt{2}$ and if we keep only linear terms in $u \equiv s_{13}e^{i\phi}$, the mixing matrix becomes

$$U = \begin{pmatrix} c_{12} & s_{12} & u \\ -(s_{12} + c_{12}u^*)/\sqrt{2} & (c_{12} - s_{12}u^*)/\sqrt{2} & -1/\sqrt{2} \\ (s_{12} - c_{12}u^*)/\sqrt{2} & -(c_{12} + s_{12}u^*)/\sqrt{2} & 1/\sqrt{2} \end{pmatrix}. \tag{5}$$

Putting θ_{13} exactly to zero and using the relation

$$m_\nu = U^* \mathrm{diag}(m_1, m_2, m_3) U^\dagger, \tag{6}$$

one gets the most general neutrino mass matrix that is symmetric under 2-3 (or $\mu - \tau$) exchange [10]

$$m_\nu = \begin{pmatrix} x & y & y \\ y & z & w \\ y & w & z \end{pmatrix}, \tag{7}$$

with complex coefficients x, y, z and w.

Finally, one can also use the ansatz $\sin^2\theta_{12} = 8/9$, which leads to tri-bimaximal (TB) mixing and the Harrison-Perkins-Scott (HPS) matrix

$$U_{HPS} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{8}$$

This matrix can be reproduced in certain symmetry groups, which will be explained in the following chapters. The name tri-bimaximal originates from merging the terms trimaximal and bimaximal, the former reflecting the idea of uniform mixing of the second mass state, ν_2 with the other two, while the third, ν_3, mixes only (that is 'bimaximally' for the given θ_{23} and θ_{13}) with ν_2.

3 S_N and its subgroups

As already mentioned before, A_4 is widely used in the context of neutrino mixing. Here a short overview of the permutation group S_N in general will be given and afterwards S_4 and A_4 will be discussed.

S_N is also called the symmetric group and it consists of $N!$ elements, describing all possible permutations among N objects. A common and very intuitive notation is

$$P = \begin{pmatrix} 1 & 2 & 3 & ... & n \\ p_1 & p_2 & p_3 & ... & p_n \end{pmatrix}, \tag{9}$$

where the object i is replaced by the object p_i. An arbitrary permutation may be broken up into a product of cycles of lengths $l_1, l_2, ...$ where e.g. a cycle of length $l = 1$ leaves a number unmoved, while a cycle of length $l = 2$ symply transposes two numbers. All elements of S_N associated with the same structure in terms of cycle length $[l_1 l_2 ...]$ are in the same class [12]. More commonly, for two permutations P and P' to be in the same class, there must be a permutation Q such that $QPQ^{-1} = P'$. Let us now turn to representations in certain vector spaces, for this approach is more suitable for describing physical phenomena.

3.1 S_4

As an illustration, let us have a look at S_4, the group of permutations among four objects. It consists of $4! = 24$ elements and geometrically it has the symmetry of a cube. One can see this by looking at a face of a cube and assigning its vertices numbers from one to four. Then the remaining vertices will be labelled such that the same numbers correspond to two vertices seperated by a spatial diagonal (fig. 1).

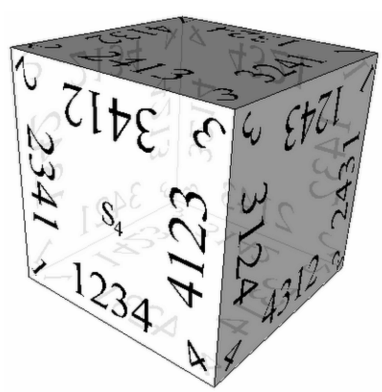


Figure 1: S_4 as the symmetry of a cube

The cube has 13 symmetry axes, three of order $h = 4$, six of order $h = 2$ and four of order $h = 3$ (fig 2). The corresponding rotations form five conjugacy classes:

- C_1: identity
- C_2: 3 rotations by 180° about the 4-fold axes
- C_3: 6 rotations by 180° about the 2-fold axes
- C_4: 4 rotations by 120° and 4 rotations by 240° about the 3-fold axes
- C_5: 3 rotations by 90° and 3 rotaions by 270° about the 4-fold axes

Thanks to the geometrical aspect of S_4 it is possible to imagine the logic behind this classification. For rotation groups, two rotations R and R' are in the same class if the group contains a rotation which carries the axis of R into the axis of R'. Lookig at the above classes one can easily verify the result.

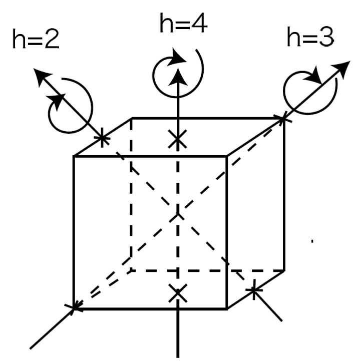


Figure 2: The symmetry axes of a cube

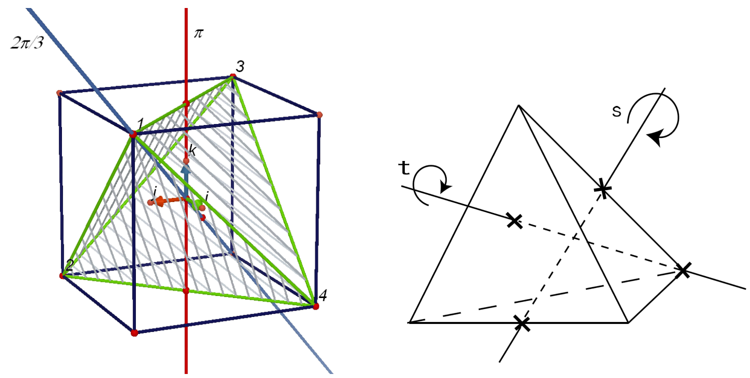
Thus S_4 has five irreducible representations. If one denotes by m_n the multiplicity of the n-dimensional representation, using $\sum_n m_n = 5$ and the relation [11]

$$\sum_\alpha [\chi_\alpha(C_1)]^2 = \sum_n m_n n^2 = m_1 + 4m_2 + 9m_3 + ... = 24, \tag{10}$$

one gets that the irreducible representations of S_4 contain two singlets $\mathbf{1}$ and $\mathbf{1}'$, one doublet $\mathbf{2}$ and two triplets $\mathbf{3}$ and $\mathbf{3}'$.

3.2 A_4

A subgroup of S_N is the so called alternating group, A_N, which consists of only the even permutations and thus has order $N!/2$. It turns out that A_3 is nothing but Z_3, the cyclic group of order 3, and the smallest non-Abelian group is A_4. It has 12 elements and can be viewed as the symmetry group of a tetrahedron, which is why it is often denoted by T.



(a) (b)

Figure 3: The symmetry axes of a tetrahedron

The tetrahedron has three symmetry axes of order $h = 2$ and four of order $h = 3$ (fig. 3). There are four conjugacy classes:

- C_1: identity
- C_2: 4 rotations by 120° about the 3-fold axes
- C_3: 4 rotations by 240° about the 3-fold axes
- C_4: 3 rotations by 180° about the 2-fold axes

It is possible to generate all elements of A_4 by two basic permutations, $S = (4321)$ and $T = (2314)$ [9], whith the property

$$S^2 = T^3 = (ST)^3 = 1. \tag{11}$$

In this presentation, the classes of A_4 consist of [7]

- $C_1 : I = (1234)$
- $C_2 : T = (2314), ST = (4132), TS = (3241), STS = (1423)$
- $C_3 : T^2 = (3124), ST^2 = (4213), T^2S = (2431), TST = (1342)$
- $C_4 : S = (4321), T^2ST = (3412), TST^2 = (2143),$

Table 2: Characters of A_4

Class	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	3
C_2	1	ω	ω^2	0
C_3	1	ω^2	ω	0
C_4	1	1	1	-1

where it can be seen that the classes are according to powers of T (using $T^3 = 1$). Going on in the same way as with S_4, one obtains that A_4 has four irreducible representations, three singlets, $\mathbf{1}$, $\mathbf{1'}$ and $\mathbf{1''}$, and a triplet, $\mathbf{3}$. Now it is possible to study characters of A_4, having in mind that they are identical for elements in the same equivalence class. Let us first look at the singlet representations. From $S^2 = I$, C_4 can have character $+1$ or -1, but as T and ST belong to the same class, it follows $\chi_\alpha(C_4) = 1$. Similarly, $T^3 = 1$ leads to $\chi_\alpha(T) = 1, \omega, \omega^2$ (where $\omega^3 = 1$), a different value for each singlet representation. On the other hand, $C_3 = (C_2)^2$, and in order for their elements to have different characters, one simply interchanges ω and ω^2.

In the three-dimensional unitary representation where S is diagonal, S and T are of the form

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \qquad T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \tag{12}$$

Considering their traces, the character table [2] is finished.

One immeditately sees that the multiplication rules for the singlets are: $1' \times 1' = 1''$, $1' \times 1'' = 1$, $1'' \times 1'' = 1'$, as in the one-dimensional case characters equal representations. The reduction of the 3×3 product can be obtained using the relation [11]

$$m_\alpha = \frac{1}{N} \sum_p c_p \chi_p^{(\alpha)*} \chi_p, \tag{13}$$

where the subsctipt α refers to the according irreducible representation, c_p is the number of elements in a class, and $\chi_p = \chi_p^R \chi_p^S$ with $R = S = 3$ in this case. The result is $3 \times 3 = 1 + 1' + 1'' + 3 + 3$ and taking the product of two such triplets, (a_1, a_2, a_3) and (b_1, b_2, b_3), gives [6]

$$
\begin{aligned}
1 &= a_1b_1 + a_2b_2 + a_3b_3 \\
1' &= a_1b_1 + \omega^2 a_2b_2 + \omega a_3b_3 \\
1'' &= a_1b_1 + \omega a_2b_2 + \omega^2 a_3b_3 \\
3 &\sim (a_2b_3, a_3b_1, a_1b_2) \\
3 &\sim (a_3b_2, a_1b_3, a_2b_1).
\end{aligned} \tag{14}
$$

Consider for example the $1''$, which is invariant under S, equation (12) and $T1'' = \omega^2 1''$. From table [2] we see that ω^2 corresponds to $\chi_{1''}(C_2) = \chi_{1''}(T)$.
Furthermore, it is useful to have a look at the matrices in a basis where T is diagonal

$$
T' = VTV^\dagger = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \quad S = VSV^\dagger = \frac{1}{3}\begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix}, \tag{15}
$$

where

$$
V = \frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}. \tag{16}
$$

This will correspond to a diagonalization of the charged lepton mass matrix in physical models.

4 A flavor model with A_4

Now we are in a position to apply A_4 symmetries to lepton masses and mixing. We assign leptons to the four irreducible representations of A_4 such that the left-handed doublets l transform as the triplet **3**, and the right handed charged leptons e_R, μ_R and τ_R as the singlets $\mathbf{1}$, $\mathbf{1}'$ and $\mathbf{1}''$, respectively. We also include right handed neutrinos in an A_4 triplet representation. In addition, two real triplets, φ and φ', both of them gauge singlets, are introduced to break the family symmetry of leptons, and are hence called flavons. The SM Higgs doublet H is invariant under A_4. A Yukawa Langrangian is constructed as follows:

$$\begin{aligned} -\mathcal{L}_Y &= \frac{1}{\Lambda} y_e\, \overline{e_R} H^c\,(\varphi l) + \frac{1}{\Lambda} y_\mu\, \overline{\mu_R} H^c (\varphi l)'' + \frac{1}{\Lambda} y_\tau\, \overline{\tau_R} H^c (\varphi l)' \\ &+ M\, \overline{\nu_R} \nu_R^c + y_\nu\, \overline{\nu_R} H l + x_\nu\, \overline{\nu_R} \nu_R^c\, \varphi' + h.c. + \ldots \end{aligned} \tag{17}$$

This needs some explanation. First of all, in the notation at hand (33) transforms as 1, $(33)'$ as $1'$ and $(33)''$ as $1''$. The heavy neutrino states due to the see-saw mechanism can be integrated out, and the dots stand for higher dimensional operators. All of the terms in $\mathcal{L}_Y$ have to be trivial singlets under the symmetry in question. Applying the reduction rules from section (3.2) one sees that this is true. In the first three terms, the vacuum expectation value (VEV) of H selects the charged leptons from the gauge doublets, which leads to their masses. One requirement is still to be made that is not accounted for naturally by A_4, and this is the symmetry between $\mu - \tau$ exchange in the neutrino sector. A suitable vacuum alignment of the flavons will take care of that problem. One takes

$$\begin{aligned} \langle\varphi\rangle &= (v, v, v) \\ \langle\varphi'\rangle &= (v', 0, 0), \end{aligned} \tag{18}$$

and as a result, $\langle\varphi\rangle$ will break A_4 down to Z_3 in the charged lepton sector and $\langle\varphi'\rangle$ will break it to Z_2 in the neutrino sector. $(\overline{\nu_R^c}\nu_R)$ in equation (17) is a 3×3 and in order to get a singlet in the end, one needs the triplets in the reduction. As they both have a (2-3) mixing in the first component, the vacuum allignment of φ' selects exactly what we need. Thus the neutrino mass matrix m_ν is of the form

$$m_\nu = \begin{pmatrix} a & 0 & 0 \\ 0 & a & d \\ 0 & d & a \end{pmatrix}, \tag{19}$$

where the (2-3) components indeed reflect the Z_2 symmetry. The charged lepton mass matrix is

$$m_l = v\frac{v_d}{\Lambda}\begin{pmatrix} y_e & y_e & y_e \\ y_\mu & y_\mu\omega^2 & y_\mu\omega \\ y_\tau & y_\tau\omega & y_\tau\omega^2 \end{pmatrix}. \tag{20}$$

It turns out that m_l is diagonalized by the matrix V introduced in section (3.2). Actually, the flavor basis is the basis in which S is diagonal, and the mass basis of charged leptons is the one where T is diagonal. To reproduce the hierarchy among charged leptons, one can add an Abelian flavor symmetry $U(1)_{FN}$ [14]. In such models, e_R, μ_R and τ_R are assigned Froggatt-Nielsen (FN) charges, for example 4, 2 and 0, respectively. Also, a flavon θ has to be added, with FN charge -1 and a VEV $\langle\theta\rangle \equiv \lambda < 1$. Then, to make the terms in the Lagrangian invariant under the $U(1)_{FN}$ symmetry, different powers of λ are needed for the different generations of leptons, leading to a mass hierarchy.After changing to the charged lepton mass basis, the neutrino matrix becomes

$$m_\nu = \begin{pmatrix} a+2d/3 & -d/3 & -d/3 \\ -d/3 & 2d/3 & a-d/3 \\ -d/3 & a-d/3 & 2d/3 \end{pmatrix}, \tag{21}$$

which is indeed of the form in equation (7) of section (2.2) and thus will be diagonalized by the HPS matrix (8).

5 The breaking of $SO(3)$ to A_4

If indeed A_4 explains the flavor structure in the lepton sector, the question of its origin remains. One of the numerous existing ideas is the spontaneous breaking of a continuous symmetry. The minimal choice is to consider $SO(3)$ [13], which has A_4 as a subgroup. In the next section, a decomposition of the irreducible representations of $SO(3)$ in terms of irreducible representations of A_4 will be given, which will allow us to implement the particle content of our model into the higher symmetry.

5.1 The connection between A_4 and $SO(3)$

In section (3.2), a parallel between A_4 and the symmetry of a tetrahedron was drawn, which indicates that A_4 is a subgroup of rotations in three dimensions. In general, the characters of $SO(3)$ are given by

$$\chi_j(\theta) = \frac{\sin[(2j+1)\theta/2]}{\sin(\theta/2)}, \tag{22}$$

where j labels the irreducible representation and θ is the angle of rotation. This gives us the character table [3] of the first six irreducible representations in the four classes that correspond to A_4.

Table 3: Characters of $SO(3)$

Class	χ_0	χ_1	χ_2	χ_3	χ_4	χ_5
0°	1	3	5	7	9	11
120°	1	0	-1	1	0	-1
240°	1	0	-1	1	0	-1
180°	1	-1	1	-1	1	-1

Using table [3] and from relation (13), section (3.2), one obtains the decomposition of $SO(3)$, table [4].

5.2 A model based on $SO(3) \rightarrow A_4$

To break the symmetry, we need to introduce a scalar, T, which transforms as an irreducible representaion of $SO(3)$ and in order to end up in an A_4 invariant vacuum, this representation has to contain a singlet under A_4 in its decomposition. From table [4] we see that the

Table 4: Multiplicities of A_4 irreducible representaions in $SO(3)$

j	0	1	2	3	4	5
n_1	1	0	0	1	1	0
$n_{1'}$	0	0	1	0	1	1
$n_{1'\prime}$	0	0	1	0	1	1
n_3	0	1	1	2	2	3

smallest non-trivial representation for this porpuse is the **7** of $SO(3)$ and is a rank three tensor. Turning to the fermions, one sees that there is no problem with l, e_R and ν_R as they have the same representations as under A_4. On the other hand, the $\mathbf{1}'$ and the $\mathbf{1}''$, representing the right-handed μ and τ, respectively, do not correspond to irreducible representations of $SO(3)$ but can, in the simplest scenario, be obtained from the **5**. Consequently, some extra right-handed fields (ρ) arise, that transform as a **3** under A_4. By adding also an extra left-handed triplet field η, the new states can be integrated out if we give them a large Dirac mass. The right handed **5** can be written as [13]

$$R = \begin{pmatrix} \mu_R + \tau_R & \rho_3 & \rho_2 \\ \rho_3 & \omega\mu + \omega^2\tau & \rho_1 \\ \rho_2 & \rho_1 & \omega^2\mu + \omega\tau \end{pmatrix}. \tag{23}$$

Having in mind the multiplication rules for representations of $SO(3)$, the most general mass Lagrangian for charged leptons in this model is

$$\begin{aligned} -\mathcal{L} \;&=\; \frac{1}{\Lambda} y_e\, \overline{e_R}\varphi^a H^c\, l^a + \frac{1}{\Lambda} y_R\, \overline{R^{ab}}\varphi^a H^c\, l^b + \frac{1}{\Lambda} y_R^T\, \overline{R^{ab}} T^{abc} H^c\, l^c + y_e' \overline{e_R}\varphi^a \eta^a \\ &+\; y_R'\, \overline{R^{ab}}\varphi^a \eta^b + y_R^{T\prime}\, \overline{R^{ab}} T^{abc}\eta^c + \frac{1}{\Lambda} y_\phi\, \epsilon^{abc}\overline{R^{ad}}\phi^{bd} H^c\, l^c + y_\phi'\, \epsilon^{abc}\overline{R^{ad}}\phi^{bd}\eta^c. \end{aligned} \tag{24}$$

Here, a new scalar ϕ (a **5** in $SO(3)$) was introduced in order to make the μ and τ masses nondegenerate. This is an alternative to the Froggatt-Nielsen mechanism mentioned in section 4. φ and φ' get VEVs as before, $\langle T\rangle \sim v_T$, while

$$\langle\phi\rangle = \begin{pmatrix} 0 & v_\phi & v_\phi \\ v_\phi & 0 & v_\phi \\ v_\phi & v_\phi & 0 \end{pmatrix}. \tag{25}$$

The hierarchy of the constants is assumed to be

$$\Lambda \gg v_T \gg v \sim v' \sim v_\phi \gg v_h. \tag{26}$$

Taking this into account, the charged lepton mass matrix, written in 3×3 blocks, is of the form

$$m_l \sim \begin{pmatrix} v_h v/\Lambda & v \\ v_h v_T/\Lambda & v_T \end{pmatrix}. \tag{27}$$

Basically, the mixings between the three light states (e, μ, τ) and the heavy new states can be neglected. The block corresponding to charged leptons is

$$m_l = \begin{pmatrix} y_e \frac{v_h v}{\Lambda} & y_e \frac{v_h v}{\Lambda} & y_e \frac{v_h v}{\Lambda} \\ q & \omega q & \omega^2 q \\ -q & -\omega^2 q & -\omega q \end{pmatrix}, \tag{28}$$

with

$$q = y_R \frac{v_h v}{\Lambda} + y_\phi (\omega^2 - \omega) \frac{v_h v_\phi}{\Lambda}. \tag{29}$$

This matrix can again be diagonalized by V from section (3.2). Thus one ends up with the masses

$$\begin{aligned} m_e &= |y_e \frac{v_h v}{\Lambda}| \\ m_\mu &= |y_R \frac{v_h v}{\Lambda} - i\sqrt{3} y_\phi \frac{v_h v_\phi}{\Lambda}| \\ m_\tau &= |y_R \frac{v_h v}{\Lambda} + i\sqrt{3} y_\phi \frac{v_h v_\phi}{\Lambda}|. \end{aligned} \tag{30}$$

Now it should be clear why the scalar ϕ was introduced. Without it, the μ and the τ have the same mass. In the neutrino sector the only change in comparison to the case of A_4 is the appearance of the scalar T. The Langrangian is

$$-\mathcal{L}_{Y_\nu} = M\, \overline{\nu_R^a} (\nu_R^c)^a + y_\nu\, \overline{\nu_R^a} H l^a + \frac{x_\nu}{\Lambda}\, \overline{\nu_R^a} (\nu_R^c)^b\, \varphi' T^{abc}, \tag{31}$$

leading to a mass matrix of the form (19). After changing to the basis of charged lepton mass states, it is diagonalized by U_{HPS} and has eigenvalues

$$m_1 = y_\nu^2 v_h^2 \frac{\Lambda}{M\Lambda + x_\nu v' v_T}$$

$$
\begin{aligned}
m_2 &= y_\nu^2 v_h^2 \frac{\Lambda}{M} \\
m_3 &= y_\nu^2 v_h^2 \frac{\Lambda}{M\Lambda - x_\nu v' v_T}.
\end{aligned}
\tag{32}
$$

The measured values of Δm_{12}^2 and Δm_{23}^2 constraint $x_\nu v' v_T$ and $M\Lambda$ to be of the same order, which leads to the requirement $v' \gg M$. This is possible to achieve because both can be much above the weak scale. So we see that the model is constructed in such a way that it is the U_{HPS} matrix that diagonalizes the neutrino mass matrix in the end, thus obeying the present experimental constraints.

6 Conclusion

Recent experiments have provided not only proof for the existence of neutrino mass, but also data on the differences of squared masses. Their connection to mixing angles gives constraints on the final form of the U_{PMNS} matrix, which has inspired a big number of suggestions of underlying symmetries to the mechanism. In this work, A_4, the smallest non-Abelian discrete group, was taken as the underlying symmetry. Under the assumptions of maximal atmospheric mixing, $\sin^2\theta_{13}$ being exactly zero and $\sin^2\theta_{12} = 8/9$, U_{PMNS} takes the form of U_{HPS}. Knowing this, one is lead to the most general form of the neutrino mass matrix, which has to be reproduced in the frame of the assumed symmetry. Considering the three dimensional representations of the basic generators of A_4, it was shown that the two different bases in which one of S, T is diagonal correspond to the flavor and mass basis of charged leptons, respectively. One of the disadvantages of this approach is that it doesn't reproduce the mass hierarchy in the charged lepton sector, which raises the need for additional symmetries or fields. What's more, the experimental constraints on the entries of U_{PMNS} are still too weak to construct a solid model.

Assuming the introduced symmetry gives a good description of lepton flavor patterns, the next step is to explain the origin of the stated symmetry group. In the last section, the spontaneous breaking of an $SO(3)$ gauge group was considered. The particle content of the standard model (plus right handed neutrinos) can be implemented in this model, but to the cost of introducing more new states. Also, due to some constraints on the vacuum expectation values of the scalars, the neglect of higher orders in $1/\Lambda$ ceases to be justified. To be specific, relation (31) suggests that the charged lepton mass scale is decreased in comparison to the electroweak scale by a factor v/Λ, so Λ cannot be more then two orders of magnitude higher than v.

In conclusion, even though it is tempting to base patterns of Yukawa bindings on grounds of a mathematical symmetry group, no elegant description has yet been constructed and it is going to take at least the collection of new experimental data to create a justified and solid model.

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