Seminar

SU(5) and SO(10) Unification

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24.02.2011

Abstract

This is short review of main ideas of Grand Unified Theories, that has been prepared for a seminar, as a part of Symmetries in Physics subject studying, in academic year 2010/11. First I write about basic motivations to use the idea of symmetry in particle physics, and give short review of Standard Model, then I introduce idea of Grand Unified Theory to give short descriptions of $SU(5)$ and $SO(10)$ models.

Contents

1 Introduction

It has been said that idea of a field is one of the most important concepts in of physics so far. Michael Faraday introduced it, while studying electrical and magnetic phenomena. He thought of a field as points or places "characterized only by certain strength of action"[1], or as space itself, capable of being messenger of forces.

At some point, people started to use Newton's classical mechanics to describe material continuum. They thought of it as huge number of point masses, and instead of thinking about position of any of particles, it was enough to take small portions of it (large enough to contain big number of particles), and it was possible to calculate number of particles and their average velocity in such portions. Idea is that these averages contain all relevant information about behavior of continuum system. These small pieces of space then might be thought of a infinitesimal, and average velocities now become field, function of each space point.

This way many areas of physics became easily available for study: hydrodynamics, aerodynamics, or elasticity. Inventor of the first field theory was Scottish physicist and mathematician James Clerk Maxwell. Not only did he write down correct theory of all electric and magnetic phenomena, this theory had an incredible characteristics: it has shown to us that electrical and magnetic phenomena are united, and this was the beginning of unification program in physics. It was known even before (Faraday, Amperre and others) that these two sorts of phenomena are connected, and Maxwell gave quite concise interpretation of these connections in terms of electric and magnetic fields. His main line of idea is that electrical field is produced by changing magnetic field, and that if there is varying electrical field, the magnetic one is present, even if these fields happen to be in space without charged particles around. Maxwell knew that electric and magnetic fields are two faces of same sort of matter, which is called electromagnetic field, and which is part of reality.

His equations are well known as Maxwell equations, and out of them one could see existence of electromagnetic waves that propagate with speed of light. These were soon experimentally found by H.R Hertz, and it was shown that visible light is also part of electromagnetic spectrum.

Maxwell's theory was thought to be well studied and understood in nineteenth century, but it has contained some hidden symmetries, and one symmetry missing. We know that lines of magnetic field are always closed; we only observe electric charges, but no magnetic monopoles. Maxwell's equations should be invariant under electromagnetic duality, $(E, B) \to (B, -E)$, together with exchange of electric and magnetic currents, which would make the theory more symmetric and give us connection between weakly coupled theory (with electric charges) and strongly coupled theory (the one with magnetic charges), and this connections might be something we would explore and generalize further.

But, more important, at the beginning of twentieth century it has been discovered that Maxwell theory contains two very important symmetries: Lorentz invariance and gauge invariance. First one helped Einstein in discovering unity of space and time, and to conclude that, not only electromagnetic, but any physical theory should be Lorentz invariant. Taking quantum mechanics into account, these symmetries led to further discoveries and some of the best theories physicists presently have.

If we allow that symmetry of the theory is local, meaning that it depends on spacetime points coordinates, we get natural way of introducing interactions into the theory; new fields, that are messengers of interactions (gauge bosons). Such theory is Quantum Electrodynamics, which met great success in comparison with experiment. This is Abelian gauge theory, which means that gauge symmetry group is abelian. Generalization of this concept to non-abelian theories leads to other successful theory, like quantum chromodynamics. With help of idea of spontaneous symmetry breaking of such (non-abelian) gauge symmetries, it became possible to explain origin of mass of gauge bosons that transmit weak nuclear force, which led to Standard model of electroweak force.

Standard Model is very satisfying quantum field theory of all elementary interactions, excluding gravity, and has superbly passed all experimental tests up to this date, but still has some unsolved questions. Group of symmetries of Standard Model is $SU(3) \times SU(2) \times U(1)$ and it contains three gauge couplings for three forces, so it really is not unification of three forces. That is why it is believed that this might be low energy limit of some more fundamental model. There are 12 force messengers in this models, and after spontaneous symmetry braking three of these particles get mass (transmitters of weak force, Z and two W bosons), 8 massless gluons are transmitters of strong force that binds quarks in mesons and nucleons, and photon, the messenger of electromagnetic force, remains massless.

Elementary particles (quarks and leptons) are classified under the way they transform under the gauge symmetry group of Standard Model, and one of basic properties of theory is its chiral nature: left and right chiralities of fermions transform differently under the symmetry group.

In year 1974, H.Georgi and S.Glashow proposed theory of unification based on $SU(5)$ symmetry group, that contains Standard Model group as its subgroup. Theories of such kind, based on simple symmetry group that enlarges Standard Model group were given name of *Grand Unified Theories*, or GUTs. It is said that $SU(5)$ theory unifies all interactions, because at energy above scale of around $10^{16} GeV$ it has one gauge coupling, but at this scale, it gets spontaneously broken down to symmetry group of Standard model. This model induces 24 gauge bosons, which are all massless, but after spontaneous symmetry breaking, 12 bosons become massive, leaving 12 gauge bosons of Standard Model massless. Standard Model further gets spontaneously broken up to electromagnetic U(1) symmetry, at around $10^2 GeV$, leaving only photon massless.[1]

How one determines scale of unification in this theory is that gauge couplings g_3, g_2, g_1 for $SU(3), SU(2)$ and $U(1)$ slowly change as we move up in energy scale, in such way that they meet at some energy scale. Infinite predictions of such quantum field theory can be removed, as it is proven that Standard Model is renormalizable theory.

In $SU(5)$ quarks and leptons happen to be in same representation, and there are interactions mediated between them that transform quarks into leptons and vice versa. So, this theory predicts decay of proton. Several observatories were designed to search for signals of this process. Other important prediction of $SU(5)$ theory is magnetic monopole, that was also not yet observed. Another suitable model of GUT theory is grand unification based on $SO(10)$ symme-

[1] gluons are also massless, but of short range, because of phenomena known as confinement of QCD

try group, as an extension of $SU(5)$ model. This symmetry now brakes down through some possible cascade of spontaneous symmetry breaking to symmetry group of Standard model.

Now, let us explain what is aesthetically attractive side of idea about unification into larger group. Suppose that F is gauge (internal symmetry) group of a theory, and that V is representation of this group on Hilbert space, where particles "live". Let V also be a representation of some larger symmetry group F, to which G is subgroup. Then V decomposes into fewer irreducible representations as a representation of group F, than as representation of G. We may give a name GUT to unifying theory if group is simple, and not direct product of other groups.

Ideas of grand unification rose in mid seventies and are still very influential, as it has been noticed that enlarging spacetime symmetries by considering supersymmetry, and adding it to GUTs, makes these models even more satisfactory and a possible key to understanding physical phenomena beyond Standard model explanations.

2 Symmetry

Symmetry is very simple and important concept in physics. It can be described as change of our point of view that does not change outcome of any experiments.[2] By the famous theorem of Emmy Noether, symmetries of lagrangians of theories imply conservations of quantities, like energy, momentum, angular momentum, electric charge. Following Hamilton, we want action of the theory to be invariant with respect to some transformation of dynamical variables. In quantum theory, it is only possible to predict probabilities of some process, so it is Feynman's path integral that must be left invariant under the symmetry. There are several reasons why symmetries are important in particle physics.

First, symmetries enable us to give labels to particles, and to classify them. These labels include spin, mass, charge, color, flavor, so that symmetry defines particle's properties, which are conserved by Noether's theorem. Particle is determined by the way corresponding field transforms under corresponding symmetries. This ability to classify particles into multiplets under symmetry groups was leading idea in building of Standard Model.

Symmetries determine interactions among particles, by use of *gauge principle.* We will soon see how symmetry structure of theory can completely determine some interactions (between W and Z bosons, i.e). Consider a field theory of complex scalar field with Lagrangian:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^* - V(\phi\phi^*) \tag{1}$$

Quantum theories of fields are constructed in such way to be consistent with quantum mechanics and invariant under Lorentz transformations, that is why it is said that quantum field theory is joining of quantum mechanics and special relativity. Lorentz invariance might not be needed in quantum field theory for condensed matter, but in high energy physics, it is crucial. Above lagrangian contains scalar field which transforms trivially under Lorentz group,

[2] This is Weinberg's definition, found in his book, The Quantum Theory of Quantized Fields, vol 1, [18]

and potential V is function of product of field with its complex conjugate. It is invariant under transformation:

$$\phi \to exp(i\alpha)\phi. \tag{2}$$

This is $U(1)$, or phase transformation that, by Noether's theorem, leads to conservation of quantity that can be interpreted as electrical charge. After quantization of this theory, field ϕ can be interpreted as the field corresponding to spin zero charged particles, and ϕ^* belongs to antiparticles. As long as α is independent of spacetime, above transformation is called global. First term in lagrangian is called kinetic term, and it is no more invariant if transformation (2) is local, if α is function of spacetime coordinates, $\alpha = \alpha(x)$, because partial derivative changes as:

$$\partial_\mu \to exp(i\alpha(x))(\partial_\mu\phi + i(\partial_\mu\alpha)\phi), \tag{3}$$

but, covariant derivative:

$$D_\mu\phi := \partial_\mu\phi - iA_\mu\phi \tag{4}$$

together with rule for transformation of field A_μ:

$$A_\mu \to A_\mu - \partial_\mu\alpha(x) \tag{5}$$

transforms as:

$$D_\mu\phi \to exp(-i\alpha(x))D_\mu(x), \tag{6}$$

ensuring invariance of lagrangian. So instead of ordinary partial derivative, we use covariant derivative, and lagrangian is symmetric under local transformation, also called gauge transformation. Now rewrite lagrangian as:

$$\mathcal{L} = D_\mu\phi D^\mu\phi^* - V(\phi\phi^*). \tag{7}$$

This lagrangian contains term $eA_\mu\phi A^\mu\phi$ that has fields ϕ and A_μ coupled. So this is how gauge symmetry principle gives interactions of scalar charged field with field A_μ, which is potential of electromagnetic field. It was introduced to restore gauge invariance. In this way we can get Lagrangian of electrons interacting with photons, lagrangian of Quantum Electrodynamics:

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{8}$$

Interaction term here is $eA_\mu\bar{\psi}\gamma^\mu\psi$, and gauge invariant electromagnetic fields are also introduced: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Field A_μ describes photons, and it is easy to see that these particles must be massless, mass term $\frac{1}{2}m^2A_\mu A^\mu$ is not gauge invariant.

In mathematics, unitary group of degree N, named as U(N) is group of $N \times N$ unitary matrices. So, transformations (2) represent 1×1 unitary matrix, which belongs to $U(1)$ group. Since (2) is phase change, we say that Quantum Electrodynamics (QED) possess local phase invariance. Such invariance allows construction of renormalizable field theory that agrees with experiments to very high precision. This symmetry was generalized to non-abelian gauge groups by

Yang and Mills (1954) and by Shaw (1955).

Consider N-component scalar field $\phi(x) = \phi_1(x), \phi_2(x), ..., \phi_N(x)$ that transforms as $\phi(x) \to U\phi(x)$ is symmetry of Lagrangian $\mathcal{L} = \partial\phi^\dagger\partial\phi - V(\phi^\dagger\phi)$, because $\phi^\dagger\phi \to \phi^\dagger\phi$ and $\partial\phi^\dagger\partial\phi \to \partial\phi^\dagger\partial\phi$. Yang and Mills generalized theory to the case when U varies from place to place in spacetime, with following form:

$$U(x) = exp(-ig\alpha^a(x)T^a), \tag{9}$$

which is element of $SU(N)$ and T^a are generators of the group, and $\alpha^a(x)$ are parameters of transformation. Just like in case of abelian local transformation, $\partial\phi^\dagger\partial\phi$ is no longer invariant under such transformation, and to preserve local invariance they introduced covariant derivative of the field as:

$$D_\mu = \partial_\mu - igA_\mu, \tag{10}$$

with $N \times N$ identity matrix that multiplies ∂_μ. Covariant derivative then transforms as:

$$D_\mu \to U(x)D_\mu U^\dagger(x). \tag{11}$$

The field A_μ is $N \times N$ matrix, and is called gauge field (potential), in analogy with electromagnetism. We want $D_\mu\phi(x) \to U(x)D_\mu\phi(x)U^\dagger(x)$, so that covariant derivative transforms in the same way ordinary derivative would transform when U does not depend on x. In order to assure that, we need rule of transformation of $A_\mu(x)$ as follows:

$$A_\mu \to U(x)A_\mu U(x)^\dagger + \frac{i}{g}U(x)\partial_\mu U^\dagger(x). \tag{12}$$

This rule leads to (5) in case of abelian symmetry. Matrices of generators obey commutation relations of the form:

$$[T^a, T^b] = if^{abc}T^c, \tag{13}$$

where f^{abc} structure constants of the group, and can be chosen with following normalization:

$$Tr(T^aT^b) = \frac{1}{2}\delta^{ab}. \tag{14}$$

These two relations can be used to show that f^{abc} are completely antisymmetric in its indices. For $SU(2)$ group, $T^a = \frac{1}{2}\sigma^a$, where σ^a are Pauli matrices and $f^{abc} = \epsilon^{abc}$, completely antisymmetric Levi-Civita symbol.

Now let us construct kinetic term for $A_\mu(x)$. Field strength is given by:

$$F_{\mu\nu}(x) = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]. \tag{15}$$

This is generalization of electromagnetic field strength for non-abelian symmetries. For electromagnetic field, last term in above equation vanishes, so we are left with good definition of electromagnetic field strength. Field $F_{\mu\nu}$ transforms as:

$$F_{\mu\nu} \to U(x)F_{\mu\nu}(x)U^\dagger(x) \tag{16}$$

and kinetic term for this field is constructed as:

$$\mathcal{L}_{kinetic} = -\frac{1}{2} Tr(F^{\mu\nu} F_{\mu\nu}), \tag{17}$$

with thanks to invariance of trace under cyclic permutations, $Tr(ABC) = Tr(CAB)$.

The consistent fact is that A_μ is hermitian matrix, and can be chosen as traceless. Then it can be expanded in terms of matrices of generators:

$$A_\mu = A^a_\mu(x) T^a. \tag{18}$$

With this in mind, field strength can be written as:

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu + g f^{abc} A^a_\mu A^b_\nu, \tag{19}$$

and also kinetic term:

$$\mathcal{L}_{kinetic} = \frac{-1}{4} F^{c\mu\nu} F^c_{\mu\nu}, \tag{20}$$

where summation over repeated indices is understood.
Finally, the lagrangian of Yang-Mills theory is:

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) - \frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu}) + m\phi^\dagger \phi - V(\phi^\dagger \phi). \tag{21}$$

First term contains interaction between scalar and gauge fields, second term is known as pure Yang-Mills term, it contains quadratic term $(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)^2$, cubic term $f^{abc} A^{b\mu} A^{c\nu} (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu)$ and quartic one, $(f^{abc} A^b_\mu A^c_\nu)^2$. Quadratic term describes propagation of massless spin one boson, called gauge boson, and cubic and quartic terms introduce interactions between gauge bosons. Pure Yang-Mills theory (only the second term) is lot more complicated than pure Maxwell theory, and constants f^{abc} are completely determined by group theory, which means that interactions are completely determined by symmetry of theory.

There are also other classes of Yang-Mills theories that include $SO(N)$, or $Sp(2N)$, and there are also exceptional compact groups that can be used: $G(2), F(4), E(6), E(7)$, and $E(8)$. It is important to notice that Yang-Mills theory must be based on group whose generators have property that $Tr(T^a T^b)$ is positive definite matrix, otherwise $\mathcal{L}_{kinetic}$ might be negative and hamiltonian would be unbounded from below.
Following the same procedure, Yang-Mills theories involving spinor fields instead of scalars are constructed. One such theory, based on group $SU(3)$ is *Quantum Chromodynamics*, or QCD, which is so far correct theory of strong force.
One could take scalar or spinor fields in different representations of group, and write covariant derivative as $D_\mu = \partial_\mu - igA^a_\mu T^a_R$, and transformation as $\phi(x) \to U_R(x)\phi(x)$. Transformation of gauge potential must be independent of representation used.
At first, Yang-Mills theories did not look like candidates for same real world theories, as they possessed then unobserved massless particles. We now know why are they unobserved: gluons of QCD are prevented from being observed by mechanism called infrared slavery of Quantum Chromodynamics, and in late

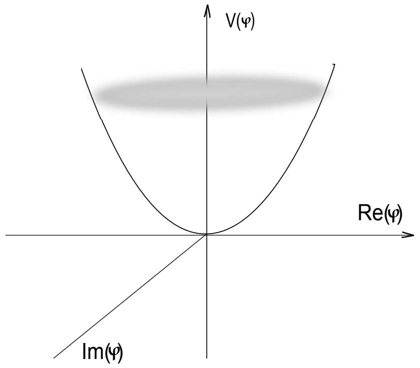

Figure 1:

seventies it was understood how gauge bosons can become massive, by mechanism of spontaneous symmetry breaking. Soon it has been proved by Veltman and t'Hooft that these theories are renormalizable, in other words quantum Yang-Mills theories make sense, and so idea of Yang and Mills became one of the greatest discoveries in physics in 20th century. This idea led to Standard Model, which is our current standard theory of matter and interactions.
Basic building blocks of matter are spinor fields, that interact with each other through messenger particles, spin 1 bosons that correspond to Yang-Mills gauge potentials (fields). There are some reviews of geometry of gauge fields in literature, in example [1], [5], or [6], and quantization of gauge theories is explained in [1], [5] or [7], among others.

3 Hidden symmetry

One of the most curious facts about symmetries is that they can be hidden (spontaneously broken).
Let us turn to our model of complex scalar field theory, now with specific form of potential, forced by requirement of being renormalizable:

$$\mathcal{L} = \partial_\mu \phi \partial^\mu \phi^\dagger - m\phi^\dagger \phi - \frac{\lambda}{2}(\phi^\dagger \phi)^2. \tag{22}$$

This lagrangian has global phase symmetry,$\phi \to exp(i\alpha)\phi$. We should identify the ground state of the theory, i.e minimum of energy (Hamiltonian). If we would explicitly write hamiltonian, we would see that its minimum is found by taking constant value of the field and by minimizing potential term. Hamiltonian corresponding to this lagrangian will have unique ground state, as shown in a figure 1.

Upon quantization, we would use Feynman's integral, for our theory:

$$\mathcal{Z} = A \int \mathcal{D}\phi \mathcal{D}\phi^\dagger e^{i\int d^4x(\partial_\mu\phi\partial^\mu\phi^\dagger - m\phi^\dagger\phi - \frac{\lambda}{2}(\phi^\dagger\phi)^2)}. \tag{23}$$

In perturbation theory we would study small changes of field around minimum of the action. One of the central relations in QFT, Lehman-Symanzik-Zimmermann (LSZ) reduction formula, that relates amplitude of a process to

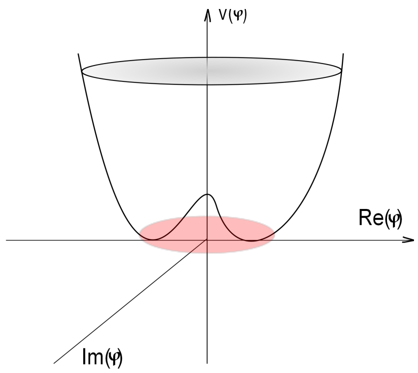

Figure 2:

Green function calculated from Feynamn's integral, requires existence of unique vacuum, and also that vacuum expectation value of fields vanish (see [4], chapter 5). For parameter m being negative, the potential has minimum for:

$$\phi(x) = \frac{1}{\sqrt{2}} v e^{-i\theta}, \tag{24}$$

where θ is arbitrary, which means that we now have continuous family of minima for potential. In quantum theory there is continuous family of vacua (one for each θ, and with property of:

$$\langle\theta|\phi(x)|\theta\rangle = \frac{1}{\sqrt{2}} v e^{-i\theta}. \tag{25}$$

All these vacua are physically equivalent. If we rewrite our complex scalar field as: $\phi = \phi_1 + i\phi_2$, the choice of minimum can be $\phi_1 = v = \sqrt{\frac{m^2}{\lambda}}, \theta = 0$, and $\phi = 0$. Vacuum expectation value of $\phi_1(x)$ is not 0, but some v. In this case we can redefine our field as:

$$\phi(x) = v + \phi_1'(x) \tag{26}$$

Expectation value of $\phi_1'(x)$ is zero, so it is suitable for further work. This is just change of the name of operator, and there is no effect on physics. Now we can rewrite the lagrangian in terms of $\phi_1'(x)$ and get:

$$\mathcal{L} = \frac{m^4}{4\lambda} + 1/2[(\partial\phi_1')^2 + (\partial\phi_2)^2] - m^2\phi^2 \tag{27}$$

Now, field ϕ_1' appears with mass $\sqrt{2}m$ [3], but field ϕ_2 appears as massless. That field is complex part of ϕ, so oscillations in this field correspond to radial directions of points corresponding to vacua in figure 2.

When continuous symmetry gets "spontaneously broken" there emerges massless field known as Nambu-Goldstone bosons. To every continuous symmetry there is conserved charge operator, which is generator of symmetry. Being conserved means that it dos not change in time:

$$[H, Q] = 0, \tag{28}$$

[3]Recall that mass term in lagrangian looks like $1/(2m)\phi^2$

where H is hamiltonian that leads the time evolution of physical system. If vacuum is invariant under symmetry, $e^{i\theta}|0\rangle = |0\rangle$, or:

$$Q|0\rangle = 0. \tag{29}$$

If there is a vacuum chosen as preferred one, $Q|0\rangle \neq 0$, energy of state $Q|0\rangle$ is

$$HQ|0\rangle = [H, Q]|0\rangle = 0, \tag{30}$$

so that $Q|0\rangle$ has same energy as $|0\rangle$. In field theory:

$$Q = \int d^3x J^0(x). \tag{31}$$

Consider a state:

$$|s\rangle = \int d^3x e^{-i\overrightarrow{k}\overrightarrow{x}} J^0(x)|0\rangle, \tag{32}$$

and as $\overrightarrow{k}$ goes to zero, $|s\rangle$ goes to $Q|0\rangle$ which has zero energy. So that means that $|s\rangle$ corresponds to massless particle. This is formulation of Goldstone theorem that applies to any hidden symmetry. We can see that number of Nambu-Goldstone bosons is equal to number of conserved charges that do not leave vacuum state invariant. What would happen if Lagrangian (22) is gauge invariant?

$$\mathcal{L} = (D_\mu)^\dagger D\phi + m\phi^\dagger\phi - \frac{\lambda}{2}(\phi^\dagger\phi)^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}. \tag{33}$$

Now, for reason of clarity, redefine field into polar coordinates, $\phi = \rho e^{i\theta}$, and $D_\mu\phi = [\partial_\mu\rho + i\rho(\partial_\mu\theta - A_\mu)]e^{i\theta}$ so Lagrangian is:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \rho^2(\partial_\mu\theta - A_\mu)^2 + (\partial\rho)^2 + m^2\rho^2 - \frac{\lambda}{2}\rho^4. \tag{34}$$

Under gauge transformation $\phi \to e^{i\lambda}\phi$,$\theta \to \theta + \alpha$, $A_\mu \to A_\mu - \partial_\mu\alpha(x)$, but combination:

$$B_\mu := A_\mu - \partial_\mu\theta \tag{35}$$

is gauge invariant. First two terms in previous lagrangian are now: $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + e^2\rho^2B_\mu^2$. Kinetic term $F_{\mu\nu}$ stays the same for B_μ: $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \partial_\mu B_\nu - \partial_\nu B_\mu$. Upon choosing vacuum as $\rho = \frac{1}{\sqrt{2}}(v+\eta)$, the lagrangian becomes:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2B_\mu^2 + e^2v\eta B_\mu^2 + \frac{1}{2}(\partial\eta)^2 - m^2\eta^2 - \frac{\lambda}{8}\eta^4 - \sqrt{\frac{\lambda}{2}}m\eta^3 + \frac{m^4}{2} \tag{36}$$

In theory, there is vector field B_μ with mass $M = ev$. This field interacts with scalar field η. Field θ that would correspond to Goldstone boson in non-gauge theory, disappears. It is said that "gauge field A_μ has eaten Nambu-Goldstone boson and became massive." That is now field B_μ. This phenomenon is known as Higgs mechanism, and has been discovered by P. Anderson, R. Brout, F.Englert, G.Guralnik, C.Hagen, T.Kibble and P. Higgs. Massive vector particle has three degrees of freedom, corresponding to three possible polarizations and third degree was gained from one degree of freedom of Nambu-Goldstone boson.

If we start with theory with symmetry group G, having $n(G)$ generators, and this group gets "spontaneously broken" to subgroup H, with $n(H)$ generators, it means that generators of group H **leave** vacuum invariant. Now,

$n(G) - n(H)$ generators **do not leave** vacuum state invariant, and these correspond to Nambu-Goldstone bosons of non-gauge theory, that are in gauge theory "eaten" by the same number of gauge bosons, which now gain masses. What is left are $n(H)$ massless gauge bosons. This happens in Standard model, where, $SU(3) \times SU(2) \times U(1)$ symmetry group is broken to $U(1)$, which has one generator, that corresponds to one massless gauge boson, that is photon.

4 Short review of Standard Model

It was Fermi's theory of weak interactions, that has been shown as not well behaving one, at higher energies. It described four fermion interactions happening through direct coupling. Physicists clearly saw theory as nonrenormalizable, and the model gave very good agreement with observations, as it is low energy effective model of some more fundamental theory.

It has been experimentally found that only left handed neutrinos are interacting in weak interactions. Idea to start with (to build new theory) was the use of $SU(2)$ gauge theory with triplet of gauge bosons, A^a_μ, $a = 1, 2, 3$. Now, ν_L, e_l are in doublet representation, and right handed electron, e_R is into singlet representation of the group. These particles are called leptons. Similar is with quarks. For now, we will consider only first generation of quarks and leptons. Other two generations are added easily. In one place, we will consider these fermions:

$$\psi_{leptons} := \begin{pmatrix} \nu \\ e \end{pmatrix}_L, e_R \tag{37}$$

$$\psi_{quarks} := \begin{pmatrix} u \\ d \end{pmatrix}_L, u_R, d_R \tag{38}$$

Interactions are:

$$\mathcal{L}_{int} = \bar{\psi}\gamma^\mu A_\mu \psi = \bar{\psi}\gamma^\mu A^a_\mu T^a \psi = \bar{\psi}\gamma^\mu A^a_\mu \frac{\sigma^a}{2} T^a \psi, \tag{39}$$

as we are in $SU(2)$ now and we use Pauli matrices, σ^a as generators. Define currents as:

$$J^a_\mu = \bar{\psi}\gamma^\mu \frac{\sigma^a}{2}\psi, \tag{40}$$

and:

$$\mathcal{L}_{int} = \frac{g}{2}(J^{1\mu} + iJ^{2\mu})(A^1_\mu - iA^2_\mu) + \frac{g}{2}(J^{1\mu} - iJ^{2\mu})(A^1_\mu + iA^2_\mu) + gJ^3\mu A^3_\mu \tag{41}$$

Mixed terms, containing $J^{1\mu}$ and $J^{1\mu}$ cancel in first terms of interaction lagrangian, and define:

$$\frac{A^1_\mu \mp iA^2_\mu}{\sqrt{2}} := W^\pm_\mu \tag{42}$$

being complex field with associated currents $J^\mu_\pm = J^{1\mu} \pm iJ^{2\mu}$, and in terms of W:

$$\mathcal{L}_{int} = \frac{g}{\sqrt{2}}(J^\mu_+ W^+_\mu + J^\mu_- W^-_\mu) + gJ^{3\mu}A^3_\mu. \tag{43}$$

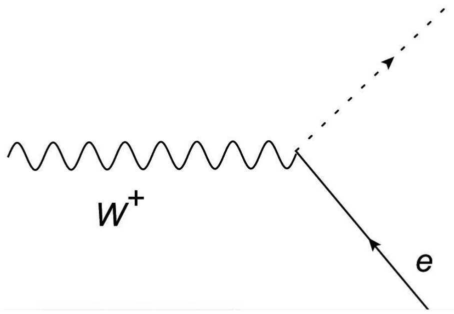

Figure 3:

Electron-neutrino coupling with field W^+ is shown in Figure 3, and $J_+^\mu = \bar{\nu}\gamma^\mu e$ $J_-^\mu = \bar{e}\gamma^\mu \nu$.

We can see that A_μ^3 is neutral field, so the candidate for it is a photon. Let us write its corresponding charge:

$$Q^3 = \int d^3x J_0^3 = \int d^3x(\bar{\psi}\gamma^\mu \frac{\sigma^3}{2}\psi) = \int d^3x \frac{1}{2}(u_L^\dagger U_L - d_L^\dagger d_L + \nu_e L^\dagger \nu_e L - e_L^\dagger e_L) \tag{44}$$

This is not electric charge:

$$Q := \int d^3x J_0 = \int d^3x(\frac{2}{3}u^\dagger u - \frac{1}{3}d^\dagger d - e^\dagger e) \tag{45}$$

We notice that up-type quarks have charge $2/3$, and down-type quarks have charge $-1/3$. But if we define:

$$Y = 2(Q - Q^3) =$$
$$\int d^3x \left[\tfrac{1}{3}\left(u_L^\dagger u_L + d_L^\dagger d_L\right) + \tfrac{4}{3}u_R^\dagger u_R - \tfrac{2}{3}d_R^\dagger d_R - \left(\nu_{eL}^\dagger \nu_{eL} + e_L^\dagger e_L\right) - 2e_R^\dagger e_R\right]$$

All elements of some representation of $SU(2)$ have same coefficient in this sum. This means that we can assign number to given doublet or a singlet. We can form abelian $U(1)$ symmetry which commutes with $SU(2)$ weak isospin symmetry. From previous formula we can get relation:

$$q = t^3 + \frac{y}{2} \tag{46}$$

Now we have group $SU(2)\times U(1)$, and 4 gauge bosons, $A_\mu^a, a = 1, 2, 3$, for $SU(2)$, and B_μ for $U(1)_Y$. Y is *hypercharge*. Weak interactions are short-ranged, so weak bosons must be massive. To give them mass, we must break $SU(2)\times U(1)_Y$ down to $U(1)$ symmetry of QED. Three bosons will require longitudinal degrees of freedom. Our choice of scalar field is complex doublet

$$\phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \tag{47}$$

Here, ϕ^+ has electrical charge $+1$, while ϕ^0 is neutral. Gauge bosons get mass through kinetic and potential terms of scalar field:

$$\mathcal{L}_\phi = D^\mu \phi^+ D_\mu \phi - V(\phi^\dagger \phi) \tag{48}$$

In covariant derivative we must take both $SU(2)$ and $U(1)_Y$ into account;

$$D_\mu \phi = \partial_\mu \phi - igA_\mu^a \frac{\sigma^a}{2}\phi - i\frac{g'}{2}y_\phi B_\mu \phi \tag{49}$$

g is $SU(2)$, while $g'/2$ is $U(1)_Y$ coupling, and $y(\phi)$ is hypercharge of Higgs field, which has value $y_\phi = 1$. Scalar potential is given by:

$$V(\phi^\dagger \phi) = -m^2 \phi^\dagger \phi + \lambda(\phi^\dagger \phi)^2 \tag{50}$$

This potential has minimum for:

$$\phi = \begin{pmatrix} 0 \\ \frac{v}{\sqrt{2}} \end{pmatrix}, v = \sqrt{\frac{m^2}{\lambda}}. \tag{51}$$

after "Spontaneous symmetry breaking" we redefine field as $\phi(x) = v + \eta(x)$, and we are interested only in vacuum expectation value of covariant derivative, which is just:

$$\langle D_\mu \phi \rangle = \begin{pmatrix} -ig\frac{v}{\sqrt{2}}\frac{A_\mu^1 - iA_\mu^2}{2} \\ +ig\frac{v}{\sqrt{2}}\frac{A_\mu^1 - iA_\mu^2}{2} \end{pmatrix}. \tag{52}$$

Interesting term is $\langle D^\mu \phi^+ d_\mu \phi \rangle$. After calculating it, we would see appearance of fields:

$$W_\mu^\pm = \frac{A_\mu \pm iA_\mu^2}{\sqrt{2}}, \quad with\, mass \quad M_w = \frac{1}{2}gv \tag{53}$$

$$Z_\mu^0 = \frac{gA_\mu^3 - g'B_\mu}{\sqrt{g^2 + g'^2}}, \quad with\, mass \quad M_z = \frac{1}{2}\sqrt{g^2 + g'^2}v \tag{54}$$

$$A_\mu = \frac{g'A_\mu^3 + gb_\mu}{\sqrt{g^2 + g'^2}}, \quad with\, mass \quad M_A = 0. \tag{55}$$

as expected, photon field does not have mass, and is built as a combination of A_μ^3 and B_μ. Field $W_\mu^\pm$ is complex, so it is charged and represents W bosons, and Z_μ^0 is real, neutral field that represents Z^0 bosons. Notice that $SU(2) \times U(1)_Y$ has been "broken" to $U(1)_{QED}$, and only photon field remains massless. If we define mixing angle (known as Weinberg angle) between charged and neutral bosons:

$$\begin{gathered} sin\theta_w = \frac{g'}{\sqrt{g^2+g'^2}}, \quad cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}} \\ tan\theta = \frac{g'}{g} \end{gathered} \tag{56}$$

and fields can be rewritten as:

$$\begin{aligned} Z_\mu &= cos\theta_W A_\mu^3 - sin\theta_W B_\mu \\ A_\mu &= sin\theta_W A_\mu^3 + cos\theta_W B_\mu \end{aligned} \tag{57}$$

We now know very precisely value of $sin^2\theta$, from measurements of processes involving muonic neutrinos and electrons, see [9], chapter 8.5. Masses of weak gauge bosons are: $M_W = 80,43GeV$ and $M_Z = 91,19GeV$. It is possible to attach $SU(3)$ theory of strong forces, called Quantum Chromodynamics (QCD)

to the $SU(2) \times U(1)$ model. In QCD, there are 8 gauge bosons, called gluons, and three color degrees of freedom. Generators of $SU(3)$ remain unbroken, and gluons are massless, but strong force is still not of long range, because of mechanism of confinement, which prevents gluons to separate too far from each others.

There is concise notation for representations of Standard Model gauge group:

$$\psi_l = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix} \in (3, \quad 2, \quad y=-1) \quad e_R \quad \in \quad (1, \quad 1, \quad y=0), \tag{58}$$

and so on. First number in bracket tells us that field belongs to triplet representation under $SU(3)$, doublet under $SU(2)$ and has hypercharge $y=-1$. In a moment we will see how fermions get mass. It is not allowed to think about mass term $-m(\bar{\psi}_L \psi_R + \bar{\psi}_R \psi_L)$, as this one is not allowed by gauge symmetry. Fermions get mass by communicating to Higgs sector, through term:

$$\mathcal{L}_{Yukawa} = \lambda_l \bar{\psi}_l \phi e_R + \lambda_d \bar{\psi}_q \phi d_R + \lambda_u \bar{\psi}_q \tilde{\phi} u_R + h.c. \tag{59}$$

Higgs field transforms as $SU(2)$ doublet:

$$\phi(x) \to e^{-i\alpha^a(x)t^a} \phi(x) \tag{60}$$

For charge conjugate fields ϕ^* does not transform as a doublet, so we should introduce

$$\tilde{\phi} = i\sigma_2 \psi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} \tag{61}$$

which does transform as doublet under $SU(2)$. After "Spontaneous Symmetry Breaking", mass term for fermions becomes:

$$\mathcal{L}_m = \lambda_l \frac{v}{\sqrt{2}} \bar{e}_L e_R + \lambda_d \frac{v}{\sqrt{2}} \bar{d}_L d_R + \lambda - u \frac{v}{\sqrt{2}} \bar{u}_L u_R + h.c \tag{62}$$

and $m_e = \lambda_e \frac{v}{\sqrt{2}}, m_u = -\lambda_u \frac{v}{\sqrt{2}}, m_d = -\lambda_d \frac{v}{\sqrt{2}}$. Neutrino stays massless in this procedure. We might give it mass by introducing right handed neutrino, but this one is neutral, singlet under $SU(2) \times U(1)$, so $q, t^3 = 0$ and also $Y = 0$, so it is totally undetermined under the SM.

Finally, let us mention that there is relation between electromagnetic coupling e, and couplings g, g'.

$$D_\mu = \partial_\mu - igA^a_\mu T^a - ig' \frac{Y}{2} B_\mu$$

$$D_\mu = \partial_\mu - i\frac{g}{\sqrt{2}} \left(W^+_\mu T^+ + W^-_\mu T^-\right) - i\frac{1}{\sqrt{g^2 + g'^2}} Z_\mu \left(g^2 T^3 - g' \frac{Y}{2}\right)$$
$$-i\frac{gg'}{\sqrt{g^2+g'^2}} A_\mu (T^3 + \frac{Y}{2}) \tag{63}$$

where $T^\pm = (T^1 \pm iT^2)$ If we identify $T^3 + \frac{Y}{2}$ as charge generator, then QED coupling is:

$$e = \frac{gg'}{\sqrt{g^2 + g'^2}}. \tag{64}$$

We may conclude that the group of symmetries of Standard Model is $\mathcal{F} = SU(3) \times SU(2) \times U(1)$. This theory successfully passed many experimental

tests in past decades, and is presently our best theory of matter and forces (except gravity). This theory is not really unification, as it contains three gauge couplings. In truly unification theory, we would expect one gauge coupling.
There are several unsolved problems within Standard Model. Here I will follow [10]. First major problem is problem of quantum gravity. For this force we do not have satisfactory quantum theory. There are also many "Why?" questions within Standard Model. First one is: Why the group of Standard Model is what it is? This is the same as question why are there only 3 interactions? Why are there only three families of quarks and leptons? We are made out of first family, and others appear only in violent scattering and live too short. Next "why" question is: Why do we live in $3+1$ dimensions? To complete specification of the model, there are 20 real parameters that have to be put by hand. These include three gauge couplings. For others, see [4], chap. 97. Next big problem, also attacked from several viewpoints is problem of hierarchies. First hierarchy problem lies in fact that scale of Standard Model lies in around $10^2 GeV$. That is where electroweak symmetry is "broken" to $U(1)$ symmetry of electromagnetism. Another important scale is Planck scale, where quantum gravity effects should play role in particle physics. Planck scale lies at around $10^{19} GeV$. There are two scales in nature, and why are they so different? Issue that makes this problem important is that in theory, there is nothing to prevent us to make electroweak scale as high as possible, if we assume that model is valid all the way to Planck scale. We would have to make *tunings* to *set* electroweak scale where it is. There are several possible strategies for journey beyond Standard model, and the one is to introduce more symmetry in theory. More symmetries means several possibilities:

1. Introduce more internal symmetries, meaning that we have group G that gets broken at scale around 10^{16} or $10^{17} GeV$ to $\mathcal{F}$. If this group is simple, than there is nice prediction that running of gauge couplings meets at some scale, where we have one gauge coupling. This way, forces are unified into one. Such theories are called Grand Unified Theories, or GUTs for short.

2. Other possibility is to introduce more spacetime symmetry, called supersymmetry. More about the idea can be found in [5], [8], [9], [10], [11].

3. Introduce new dimensions of space. For this idea see also [10]. Until the end of this report, we will consider first idea, Grand Unified Theories.

5 SU(5) Model

Gauge theories are specified by group of internal symmetry and representations of matter (fermionic) fields. In Standard Model, left handed quarks, u and d are in doublet with hypercharge $\frac{1}{2}Y = \frac{1}{6}$. In short notation, this is $(3, 1, \frac{2}{3})_R$, right handed quark is in triplet representation under $SU(3)$ but in singlet representation under $SU(2)$, so we denote it as $(3, 1, \frac{2}{3})$. For leptons we have: $(1, 2, \frac{-1}{2})_L, (1, 1, -1)_R$. Altogether, we have:

$$\left(3, 2, \frac{1}{6}\right)_L, \left(3, 1, \frac{2}{3}\right)_R, \left(3, 1, -\frac{1}{3}\right)_R, \left(1, 2, -\frac{1}{2}\right)_L, and \quad (1, 1, -1)_R \tag{65}$$

Internal symmetries commute with Poincare symmetry, that is why internal (gauge) transformations can not turn left into right handed field. But we can do operation of charge conjugations on fields and turn them all into left handed, for reason of simplicity:

$$\left(3,2,\frac{1}{6}\right)_L,\left(3^*,1,-\frac{2}{3}\right)_L,\left(3^*,1,\frac{1}{3}\right)_L,\left(1,2,-\frac{1}{2}\right)_L, and \quad (1,1,1)_L \tag{66}$$

Group of Standard model, $\mathcal{F} = SU(3) \times SU(2) \times U(1)$ has rank 4.The smallest group that contains $\mathcal{F}$ is $SU(5)$, rank 4 group which has $5^2 - 1 = 24$ generators. These are 5 by 5 hermitian traceless matrices, and act on five objects that form fundamental representation of $SU(5)$, $\psi^\mu, \quad \mu = 1, 2, ..., 5$. Out of 24 matrices, 8 are of the form

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \tag{67}$$

with A being Gell-Mann matrices, and three have form:

$$\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \tag{68}$$

with B being Pauli matrices. Also:

$$\frac{1}{2}Y = \begin{pmatrix} -\frac{1}{3} & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}, \tag{69}$$

generates $U(1)$. We want to embed matter fields of Standard Model into representations of $SU(5)$, by specifying how defining representation of $SU(5)$ decomposes under $\mathcal{F}$. First, separate index μ into $\alpha = 1, 2, 3$ and $i = 4, 5$. $SU(3)$ can act on α and $SU(2)$ on i. Three objects ψ_α transform as three dimensional representation under $SU(3)$ (can be taken as **3**), do not transform under $SU(2)$ so are singlet under this group, and have hypercharge $\frac{1}{2}$ as can be read from matrix $\frac{1}{2}Y$. So, ψ_α transform as $(3, 1, -\frac{1}{3})$. Other two ψ^i transform as **1** under $SU(3)$, **2** under $SU(2)$, and have hypercharge $\frac{1}{2}$, $(1, 2, \frac{1}{2})$. We now have:

$$\mathbf{5} \to \left(3,1,-\tfrac{1}{3}\right) \oplus \left(1,2,\tfrac{1}{2}\right)$$
$$\mathbf{5}^* \to \left(3^*,1,\tfrac{1}{3}\right) \oplus \left(1,2,-\tfrac{1}{2}\right)$$

In the second line, there are the two terms from the list (66), which now fit in 3^*. We still have other ten fields, for inclusion. Next representation, antisymmetric $\chi^{\mu\nu}$, which is ten dimensional. Since $\mathbf{5} \to \left(3,1,-\frac{1}{3}\right) \oplus \left(1,2,\frac{1}{2}\right)$, we need antisymmetric product of $\left(3,1,\frac{1}{3}\right) \oplus \left(1,2,\frac{1}{2}\right)$ with itself. We would eventually find that:

$$\mathbf{10} \to \left(3,2,\tfrac{1}{6}\right) \oplus \left(3^*,1,-\tfrac{2}{3}\right) \oplus (1,1,1)$$

This is what we wanted, all quarks and leptons fit perfectly into 5^* and 10 of $SU(5)$. We can explicitly write the content of ψ and χ:

$$\psi^i = \begin{pmatrix} \bar{d}^r & \bar{d}^b & \bar{d}^g & e & -\nu \end{pmatrix} \tag{70}$$

$$\chi_{ij} = \begin{pmatrix} 0 & \bar{u}^g & -\bar{u}^b & u_r & d_r \\ -\bar{u}^g & 0 & \bar{u}^r & u_b & d_b \\ \bar{u}^b & -\bar{u}^r & 0 & u_g & d_g \\ -u_r & -u_b & -u_g & 0 & \bar{e} \\ -d_r & -d_b & -d_g & -\bar{e} & 0 \end{pmatrix} \tag{71}$$

Now we have left handed Weyl fields ψ^i in $\mathbf{5}^*$ of $SU(5)$ and left handed Weyl fields $\chi_{ij} = \chi_{ji}$ in **10**. We also know how covariant derivatives look like:

$$\begin{aligned}(D_\mu\psi)^i &= \partial_\mu\psi^i - ig_5 A^a - \mu(T^a_{5^*})^i_j\psi^j \\ &= \partial_\mu\psi^i + ig_5 A^a_\mu (T^a)^i_j\psi^j. \\ (D_\mu\chi)_{ij} &= \partial_\mu\chi_{ij} - ig_5 A^a_\mu (T^a_{10})^{kl}_{ij}\chi_{kl} \\ &= \partial_\mu\chi_{ij} - ig_5 A^a_\mu \left[(T^a)^k_i\chi_{kl} + (T^a)^l_j\chi_{il}\right]\end{aligned}$$

For Weyl fields, kinetic term is of the form $\mathcal{L} = i\psi^\dagger\bar{\sigma}^\mu D_\mu\psi$, where $\bar{\sigma} = (1, -\sigma^i)$, with σ^i, Pauli matrices.

Kinetic term is:

$$\mathcal{L}_{kin} = i\psi^\dagger_i\bar{\sigma}^\mu(D_\mu\psi)^i + \frac{1}{2}i\chi^{\dagger ij}\bar{\sigma}^\mu(D_\mu\chi)_{ij} \tag{72}$$

Implicit in this formula, there are terms that represent interactions between gauge bosons and matter fields:

$$\mathcal{L}_{int} = -g_5\left[\psi^\dagger_i(A^T_\mu)^i_j\bar{\sigma}^\mu\psi^j + \chi^{\dagger ij}(A_\mu)^k_i\bar{\sigma}^\mu\chi_{kj}\right] \tag{73}$$

If we insert content of our fermionic and bosonic fields into this formula, we get:

$$\begin{aligned}\mathcal{L}_{int} &= -\frac{1}{\sqrt{2}}g_5 X^{\dagger\alpha}_{i\mu}\left(\epsilon^{ij}\bar{d}^\dagger_\alpha\bar{\sigma}^\mu q_{j\alpha} + q^{\beta i}\bar{\sigma}^\mu\bar{u}^\alpha\epsilon_{\alpha\beta\gamma}\right) + h.c. \\ &= -\frac{1}{\sqrt{2}}g_5 X^{\dagger\alpha}_{i\mu}J^{i\mu}_\alpha + h.c.\end{aligned} \tag{74}$$

This tells us that exchange of X bosons can turn leptons into quarks and vice versa, which was to expect, as in this theory leptons and quarks fit together in representations. Interactions like these violate lepton number conservation and lead to processes like a proton decay.
Since we do not observe $SU(5)$ symmetry, it means that it might be hidden (spontaneously broken) , first at some high energy scale to group of Standard Model, and then also electro-weak symmetry breaking should occur, at lower scale. To achieve first phase of breaking, one chooses 24 dimensional scalar (Higgs) field in adjoint representation. In adjoint representation this fields transforms as:

$$\Phi \to \Phi' \simeq (1 + i\theta^a T^a)\Phi(1 + i\theta^a T^a)^\dagger = \Phi + i\theta^a T^a\Phi - \Phi i\theta^a T^a = \Phi + i\theta^a[T^a, \Phi]. \tag{75}$$

Covariant derivative of this field is:

$$D_\mu\Phi = \partial_\mu\Phi - igA^a_\mu[T^a, \Phi]. \tag{76}$$

Vacuum expectation value of the field can be taken diagonal, $\langle\Phi^i_j\rangle = v_j\delta^i_j$ Term $Tr(D_\mu\Phi)(D^\mu\Phi)$ will give us $g^2 Tr[T^a, \langle\Phi\rangle][\langle\Phi\rangle]A^a_\mu A^{\mu b}$. Gauge boson masses squared are given by eigenvalues of 24 by 24 matrix $g^2 Tr[T^a, \langle\Phi\rangle][\langle\Phi\rangle, T^b]$. Suppose that

$$\langle\Phi\rangle = v \begin{pmatrix} 2 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & -3 \end{pmatrix}. \tag{77}$$

To know which gauge bosons are left massless after first stage of symmetry breaking, we need to look at generators which commute with $\langle\Phi\rangle$. These are generators

$$\begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, \tag{78}$$

for A are Gell-Mann matrices, and

$$\begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}, \tag{79}$$

for B are Pauli matrices. Also diagonal generator $\frac{1}{2}Y$ commute with $\langle\Phi\rangle$. These three generate $SU(3), SU(2), and\ U(1)$ So, in matrix $g^2 Tr[T^a, \langle\phi\rangle][\langle\phi\rangle, T^b]$ there are blocks 8 by 8, 3 by 3 and 1 by 1 that vanish, so we have 12 massless gauge bosons. Other 12 are massive. So, we see that Φ is suitable for SSB down to group $\mathcal{F}$. Now we need one more Higgs that transforms as $(1, 2, -\frac{1}{2})$ under $\mathcal{F}$, and take us to unbroken $SU(3) \times U(1)$. The smallest representation of $SU(5)$ that contains this piece is 5^*. We choose this Higgs field in following notation:

$$H^i = \left(\xi^r,\ \xi^b,\ \xi^g,\ \phi^-, -\phi^0\right) \tag{80}$$

Also scalar, Higgs potential is given by:

$$V(\Phi, H) = -\frac{1}{2}m_\phi^2 Tr\Phi^2 + \frac{1}{4}\lambda_1 Tr\Phi^4 + \frac{1}{4}\lambda_2 (Tr\Phi^2)^2 \tag{81}$$

$$m_H^2 H^\dagger H + \frac{1}{4}c_1 (H^\dagger H)^2 - \frac{1}{2}c_2 H^\dagger \Phi^2 H \tag{82}$$

It is possible to extract masses of ξ and ϕ Higgs particles, out of Higgs potential. We expect this masses to have values $M_\xi > 10^{10} GeV$ (heavy Higgs), and for light Higgs, $m_\phi^2 \sim 10^2 GeV$.[4] If we would want to achieve this, it would acquire fine tuning of $m_H^2 = \frac{1}{8}c_2 V^2$ up to sixteen significant digits. This is known as doublet-triplet splitting problem. There is no any reason why would scale of braking of $SU(5)$ be much larger than scale of electro-weak breaking. We know that this is the case, because effects of $SU(5)$ breaking are unobserved at electro-weak scale, so they must be pushed to much higher energies. Problem mentioned here is example of problems of so called gauge hierarchies. This new very heavy Higgs bosons are far out of reach of our current colliders. There are some possibilities that $SU(5)$ is not broken by Higgs fields, but by some effects in string theory, in example.

[4]It is quite good that electro-weak symmetry breaking happened. If not, electrons would be the same as their $SU(2)$ siblings, neutrinos.

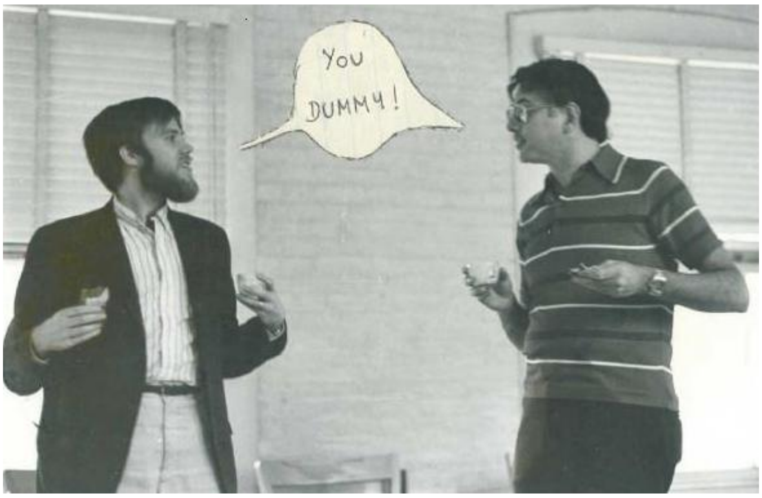

Figure 4: *Howard Georgi and Sheldon Glashow, inventors of SU(5) model eat chocolates and drink coffee in their office in Harvard*, 1974

6 Some consequences of SU(5) Unification

First important consequence is that in $SU(5)$ model leptons and quarks are in the same representation. This means that they can be transformed into each other in some processes. In Grand Unified Theory, there are many new elementary forces introduced by very heavy gauge bosons.

Proton decay is allowed by charge conservation, and the valid process might be:

$$p \to \pi^0 + e^+. \tag{83}$$

But protons are known for their long lifetime. Thanks to that, there is lot of hydrogen to participate in nuclear fusion in stars.[5] One of the possible Feynman's graphs for decay of proton is shown in figure 5. Some other can be found in [14], chapter 4. One of u quarks turns into positron by coupling to very heavy X boson. Bosons like these are called leptoquarks. If M_x is mass of gauge bosons (for calculation of masses of gauge bosons, see [11], chapter 5), then amplitude for proton decay is of order g^2/M_x^2 and decay rate is proportional to $(g^2/M_x^2)^2$, times phase space element, which is controlled by proton mass. Search for proton decay has become crucial for testing of Grand Unified Theory. It is expected that new generations of measurements will improve lower limits of lifetime of proton by factor 10, which would constrain unified models more stringently. Current lower bound for proton lifetime in channel $p \to e^+\pi^0$ is 1600×10^{30} years. Lower bounds for other channels can be found in [13], chapter 2.
These processes, while violating baryon and lepton number conservation, keep $B - L$ conserved.

To determine M_{GUT}, scale, at which, roughly said, unification happens, we could apply renormalization group flow to g_3, g_2, and g_1 couplings of $\mathcal{F}$. As we

[5] Actually the first bound without any experiments, that gave a much larger lower bound on proton lifetime is the fact that we do not die from radiation that would be caused by proton decays in our bodies

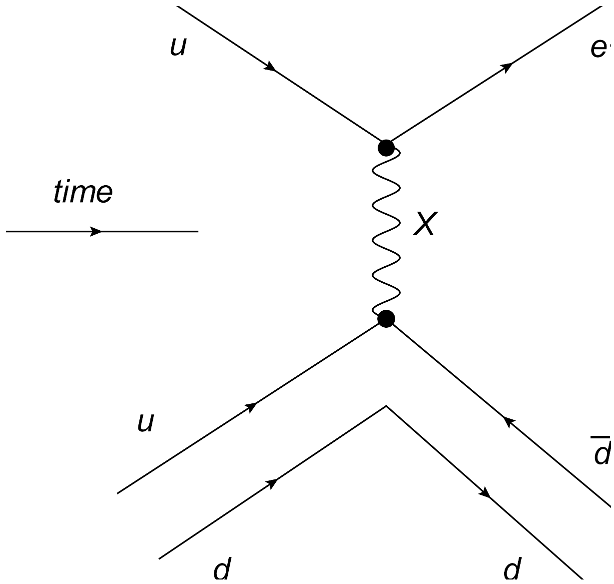

Figure 5: Possible Feynman's graph for decay of proton

go to higher energies, $g_3(\mu)$ of $SU(3)$ and $g_2(\mu)$ of $SU(2)$ decrease, while $g_1(\mu)$ increases. At some scale, these should meet and this is where $\mathcal{F}$ is unified into $SU(5)$, see figure 6. The scale M_{GUT} turns out to be very large, and has value of order $10^{17}GeV$. Coupling would come close at this scale, but would not meet at a point. After supersymmetry is taken into account, these lines meet at one point. GUTs that include supersymmetry resemble Minimal Supersymmetric Standard Model (MSSM) at lower energies.

Grand Unified Theory brings many new insights about nature. Theories like this explain quantization of electric charge. In QED, generator of $U(1)$ is not quantized, but in $GUTs$ electromagnetic potential couples to generator of GUT gauge group, and generators of any such (simple) group, like $SU(N)$ are forced by nontrivial commutation relations to assume quantized values. In example, in $SO(3)$ group of rotational symmetries, eigenvalues of third component of angular momentum can take only half integer values, but in $U(1)$ symmetry of translational invariance in time, there is no restriction to energy eigenvalues of corresponding generator.

7 SO(10) Grand Unified Model

Although in $SU(5)$ theory, elementary forces are unified, there are still two representations containing fermionic fields. These two representations can fit into single representations of some larger group of symmetries, containing $SU(5)$ as a subgroup. There is natural way to incorporate $SU(5)$ theory into model based on group $SO(10)$. At first sight, this might sound as hard task, since there are fermionic representations that must fit through $SO(10)$. But, recall that, in example, Lorentz group $SO(3,1)$ has spinor representaion. Euclidean cousin of this group is $SO(4)$, and also has spinor representation. The same

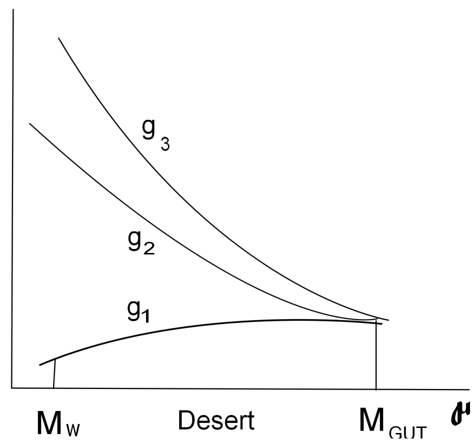

Figure 6: Meeting of gauge couplings at scale of unification

goes for any $SO(2n)$ group. Let us see how it goes.

For any integer n it is possible to construct $2n$ hermitean matrices γ_i, $i = 1, 2, ...2n$ that satisfy Clifford algebra:

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \tag{84}$$

we say that γ_i are gamma matrices for $SO(2n)$. In case that $n = 1$, two gamma matrices are first and second Pauli matrix, which can be easily checked. It is possible to do iteration and to construct gamma matrices for $SO(2n+2)$ out of gamma matrices for $SO(2n)$.

$$\begin{aligned} \gamma_j^{(n+1)} &= \gamma_j^n \otimes \sigma^3 \\ \gamma_{2n+1}^{(n+1)} &= 1 \otimes \sigma_1 \\ \gamma_{2n+2}^{(n+1)} &= 1 \otimes \sigma_2 \end{aligned} \tag{85}$$

Here $\mathbf{1}$ is identity matrix. In analogy to Lorentz group, we can form $2n(2n-1)/2$ hermitean matrices:

$$\sigma_{ij} = \frac{i}{2}[\gamma_i, \gamma_j], \tag{86}$$

and can be shown that these matrices satisfy commutation relations for $SO(2n)$, so they form its algebra. Gamma matrices are of dimension 2^n by 2^n, and so are σ_{ij}. It is important to define:

$$\gamma_{five} = \sigma^3 \otimes \sigma^3 \otimes ... \otimes \sigma^3, \tag{87}$$

with σ^3 appearing n times. With help of these matrices, we can project spinors into left and right handed, and each projection contains one half of the number of components. Since spinors have 2^n components, each projection is 2^{n-1} dimensional. For $SO(2 \times 5)$, dimension of these irreducible representations is $2^{10/2-1} = 16$. Now it should be easier to fit 5^* and 10 of $SU(5)$ into 16 of $SO(10)$.

Recall the meaning of $SO(2n)$ group. If there are real vectors $x = (x_1, x_2, ..., x_n, y_1, y_2, ...y_n)$ and $x = (x'_1, x'_2, ..., x'_n, y'_1, y'_2, ...y'_n)$, then action of group $SO(2n)$

will leave scalar product of this two vectors invariant, $xx' = \sum_{i=1}^{n}(x_i x_i' + y_i y_i')$. Out of these vectors one can construct n dimensional complex vectors $z_1 = (x_1 + iy_1, x_2 + iy_2, ..., x_n + iy_n)$, $z_1 = (x_1' + iy_1', x_2' + iy_2', ..., x_n' + iy_n')$, and $U(n)$ leaves following scalar product of vectors invariant:

$$(z')^* z = \sum_{i=1}^{n}(x_i' + iy_i')^*(x_i + iy_i) = \sum_{i=1}^{n}(x_i' x_i + y_i' y_i) + i\sum_{i=1}^{n}(x_i' y_i - y_i' x_i) \quad (88)$$

Group $SO(2n)$ leaves first term invariant, while $U(N)$ also has a subset of transformations of $SO(2n)$ that leaves second term invariant. Because $x = (x_1, x_2, ..., x_n, y_1, y_2, ...y_n)$ can be written as $(x_1 + iy_1, ...x_n + iy_n)$ and $(x_1 - iy_1, ...x_n - iy_n)$, it means that defining representation of $SO(2n)$, when restricted to $U(n)$ decomposes as:

$$2n \to n \oplus n^* \quad (89)$$

Now we could find a way in which any tensor representation decomposes, when restricted to subgroup, because any tensor is built from fundamental representations. In example, adjoint representation for $SO(2n)$ has dimension $2n(2n-a)/2$ which is 45 for $SO(10)$ and could be represented by antisymmetric two index tensor, that decomposes as:

$$45 \to 24 \oplus 1 \oplus 10 \oplus 10^* \quad (90)$$

It might be bit harder to think about decomposition of spinor representations, S of $SO(2n)$ under $U(n)$ group, but if we first look at representations of $SU(5)$, especially smallest ones $(1, 5, 10, 15)$, we see that there are many possible decompositions, but some of them are unlikely. As in similar approach in $SU(5)$ model, here spinor representation 16 is separated in pieces that transform differently under 45 adjoint representation. Recall from equation (90) decomposition of **45**. The 24 will transform each $SU(5)$ representation into itself, as this group has 24 generators. Under **1** there is trivial real number multiplication. Also, **10** is antisymmetric tensor with two upper indices, and now let us look up for result of acting of this tensor to representations of $SU(5)$.

Suppose the bunch of representations that S breaks up into contains the singlet [0] = 1 of SU(5). In square brackets stands number of upper indices. The 10 = [2] acting on [0] gives the [2]= 10. This is because of fact that combination of an antisymmetric tensor having two indices and tensor without indices is again tensor with two indices. Also, when 10 = [2] acts on [2] we get a tensor with four upper indices. It contains the [4], which is equivalent to $[1]^* = 5^*$. So we have $\mathbf{1}, \mathbf{10}, \mathbf{5}^*$ of $SU(5)$ which add to number 16, and it is clear that relevant representations of $SU(5)$ nicely fit into spinor representation of $SO(2n)$. Other half of spinor representation is just conjugate to first.

This fit would be even more perfect if there is field transforming as **1** under $SU(5)$, and this would imply that it is singlet under $\mathcal{F}$, too. This is right handed neutrino, which gets large mass, by so called *seesaw* mechanism, [15].

8 Conclusion

In this short seminar, there was not space to go deep into structures of Grand Unified Theories. For quantum aspects of field theories that are relevant for

$GUTs$, see [17]. one such important puzzle is anomaly. Briefly said, we would like our model free of anomalies, symmetries of classical theory, that do not hold as we quantize theory. This is criterium to choose representations to fit matter fields in GUT. More about phenomenology of $SU(5)$ and $SO(10)$ can be found in [11]. For supersymmetric GUT, see [5].

9 Acknowledgments

I thank to my supervisor in this seminar, professor Jure Zupan for many suggestions on improvements of the text, and for changing some of my misconceptions on this subject. I am grateful to professors Primož Ziherl and Jure Zupan for clear and inspiring lectures. I wish to thank to Ivana and Vasja for valuable discussions. .

References

[1] Anthony Zee. Quantum Field theory in a Nutshell, *Princeton University Pres, Princeton, 2010*

[2] A. J. Buras, J. Ellis, M. K. Gaillard and D. V. Nanopoulos. Aspects of the grand unification of strong, weak and electromagnetic interactions, *Nucl.Phys. B135 (1978) 66-92*

[3] Howard Georgi and S.L Glashow. Unity of all elementary forces, *Phys.Rev.Lett. 32 (1974) 438-441*

[4] Mark Srednicki. *Quantum Field Theory, Cambridge University Press, Cambridge*, 2007

[5] P.Binetruy. *Supersymmetry: Theory, Experiment, Cosmology, Oxford University Press, Oxford*,2006

[6] J.Baez and J.Huerta. *Algebra of grand Unified Theories, arxiv.org/abs/0904.1556*

[7] A.D. Linde. *Phase transitions in gauge Theories and Cosmology, Rep. Prog. Phys. 42 389, 1979*

[8] H.F. Jones. *Groups, Representations and Physics,Taylor and Francis Group, New York, 1998*

[9] Lewis H. Ryder.*Quantum Field Theory, Second edition, Cambridge University Press, Cambridge, 2001*

[10] Sven Krippendorf, Fernando Quevedo, Oliver Schlotterer. *Cambridge Lectures on Supersymmetry and Extra Dimensions, http://arxiv.org/abs/1011.1491*

[11] Rabindra N. Mohapatra.*Unification and Supersymmetry, Springer-Verlag, New York, 1986*

[12] S. Weinberg. *Baryon and Lepton Non-Conserving Processes, Phys. Rev. Lett. 43, 1566–1570 (1979)*

[13] Pran Nath, Pavel Fileviez Perez. *Proton Stability in grand unified theories, in strings and in branes, http://arxiv.org/abs/hep-ph/0601023*

[14] Paul Langacker. *Grand Unified Theories and Proton Decay, Phys.Rept. 185 72 1981*

[15] M. Gell-Mann, P. Ramond, and R. Slansky. *Color embeddings, charge assignments, and proton stability in unified gauge theories, Rev. Mod. Phys. 50, 721–744 (1978)*

[17] ed. Gerardus 't Hooft. *50 Years of Yang-Mills Theory, World Scientific, London, 2005.*

[18] S. Weinberg. Quantum Theory of Fields, Vol 1., Foundations, Cambridge University Press, Cambridge, 1995.