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Classification of Semisimple Lie Algebras

Seminar for *Symmetries in Physics*

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Abstract

The seminar presents the classification of semisimple Lie algebras and how it comes about. Starting on the level of Lie groups, we concisely introduce the connection between Lie groups and Lie algebras. We then further explore the structure of Lie algebras, we introduce semisimple Lie algebras and their root decomposition. We then turn our study to root systems as separate structures, and finally simple root systems, which can be classified by Dynkin diagrams. The reverse direction is also considered: the construction of a reduced root system from a simple root system, and the subsequent construction of the Lie algebra. The classification result is stated on the level of irreducible simple root systems, as well as on the level of complex semisimple Lie algebras and their real forms.

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1 Introduction

In physics, groups play the role of describing symmetries of a system. Both finite groups and Lie groups can be used for such a purpose, the former characterizing discrete symmetries, and the latter designating continuous symmetries. A classification of all the possible groups one can construct is thus not just a purely mathematical problem, but also has applications to physics, since it states all the possible symmetries a physical system might have. In the case of Lie groups, such a classification, for example, could prove useful for finding a suitable gauge group for a grand unified theory. In light of the title of this seminar, and with some other seminars more application oriented, this text will be mainly mathematical in nature, and should be regarded as a reference for the Lie groups/ Lie algebras in one's repertoire. Moreover, the classification result will guarantee that we do not overlook any symmetry, which could conceivably arise.

To put the results for the classification of (simple) Lie groups/ (simple) Lie algebras into perspective, it is interesting to consider a still ongoing episode in the history of mathematics: the classification of finite groups. The mathematical problem of classifying all *finite* simple groups turns out to be a daunting challenge. In fact, the classification is so complex that regarding it as merely "challenging" is an understatement. The result of the classification of these finite groups is a list of 18 infinite families, along with 26 so called sporadic groups, which do not belong to any family. The proof of this theorem, completed in 1981, was one of the big achievements of 20th century mathematics, and is more than 10000 pages long, spread over hundreds of articles written by many individual authors. This fragmentation raised considerable doubts about the validity of the proof, since there is no way any one person could check the proof from beginning to end. Since then, considerable effort has been devoted to the simplification of the proof, and since 2004, a third–generation of this effort is underway (good references for the case of finite groups are [1, 2]).

In view of the difficulties in the finite case, one could assume that a classification of simple Lie groups, which are infinite in the number of their elements, would indeed be so complicated that any such attempt would have been a hopeless endeavor. It turns out that this is not the case; the additional manifold structure of a Lie group turns out to vastly simplify the theory. Indeed, rather than considering Lie groups, one can make use of the connection between Lie groups and Lie algebras; the classification of semisimple Lie algebras, which proceeds by the classification of root systems, thus provides an overview of the possible Lie groups. This problem of classification thus surprisingly becomes simpler than in the case of finite groups; it was complete almost a century before the theorem for finite groups ([3]). It was first proved by Wilhelm Killing in 1890, and made more rigorous by Élie Cartan in 1894. The modern approach with Dynkin diagrams is due to Eugene Dynkin (1949). The final result of the classification of Lie algebras are 4 infinite families, with additional 5 so called *exceptional Lie algebras*. This result is much simpler than in the case of finite groups, and due to its geometric nature is arguably one of the most beautiful results in mathematics.

As is stated in the title, this seminar will present the classification of semisimple Lie algebras and motivate this result by considering root systems, as well as looking at the implications for (semi)simple Lie groups. It is the author's opinion that the seminar should satisfy the following goals: to clearly state the definitions and main results, so that the seminar can be used as a future reference (for the author's benefit), to motivate the results given herein (with arguments given for the simple implications and using visual aids when appropriate, while omitting formal proofs), and to explore the larger context of this classification. The seminar will thus present the topic at hand both in a formal manner, when stating results, but mostly in a more relaxed manner, when developing the ideas or discussing results. We will not shy away from using exact formulations of statements; because that necessarily means using a number of mathematical concepts and structures, we will not be defining all the used terms (especially the ones of topological origin), but will rather assume that the reader is either familiar with them, or is satisfied with the brief informal descriptions provided herein.

2 The connection between Lie groups and Lie algebras

We will start off with describing the connection between Lie groups and Lie algebras. Although this has been done to some degree in the lectures on "Symmetries in Physics", we will state the connection much more concisely. This is mandatory for a good understanding of "the bigger picture".

2.1 Why study only connected Lie groups?

Although the first definition is most certainly familiar as the very first thing from our lectures, for the sake of completeness, it is still provided:

Definition (Group). *The set G equipped with an operation $G \times G \to G$ is a group, if the operation satisfies the following properties:*

1. *associativity: $(ab)c = a(bc)$ for all $a, b, c \in G$,*
2. *existence of the identity element: $\exists e \in G : eg = ge = g$ for all $g \in G$,*
3. *existence of inverse elements: $\forall a \in G \quad \exists a^{-1} \in G : aa^{-1} = a^{-1}a = e$.*

Note that we have used no symbol for the multiplication in a group, and implicitly used the notation a^{-1} for inverses. We assume the reader is familiar with the concept of a subgroup. Now, for further convenience, an important class of subgroups will be introduced.

Definition (Normal subgroup). *A subgroup H of a group G is normal, if $ghg^{-1} \in H$ for all $g \in G$ and $h \in H$. Equivalently, H is normal, if $gH = Hg$ for all $g \in G$.*

Normal subgroups are very useful, because we can naturally induce a group structure on the set of cosets gH, and thus get the *quotient group* G/H. Further details will not be discussed here.

We now turn to continuous groups, which are equipped with a differentiable structure. These are so called Lie groups.

Definition (Real Lie group, Complex Lie group). *Let G be a group.*

- *If G is also a smooth (real) manifold, and the mappings $(a, b) \mapsto ab$ and $a \mapsto a^{-1}$ are smooth, G is a real Lie group.*
- *If G is also a complex analytic manifold, and the mappings $(a, b) \mapsto ab$ and $a \mapsto a^{-1}$ are analytic, G is a complex Lie group.*

The above definitions basically tell us that Lie groups can be locally parameterized by a number of either real or complex parameters. These parameters describe the elements of a Lie group in a continuous manner, which the multiplication and inversion maps must respect. From now on, we will not explicitly mention, whether a Lie group is real or complex: in physics, we mostly deal with real Lie group, but we will also be needing complex ones in this seminar; most results are analogous for the real and complex Lie groups unless stated otherwise.

With the talk about manifolds (which we can loosely visualize, if we insist, as n-dimensional surfaces embedded in $\mathbb{R}^{2n}$ — Whitney embedding theorem), Lie groups also have a topology; this is why we can talk about connected and disconnected Lie groups (a group, visualized as a surface, is in one piece or in many pieces). We will be interested in connected Lie groups; doing so will not cost us much generality, because of the following result (easy to prove using topological considerations), which is the main result of this subsection ([4], p. 6):

Proposition 2.1. *Let G be a Lie group and $G_\circ$ the connected component which contains the identity element e. The component $G_\circ$ is then a normal subgroup of G, is itself a Lie group, and $G/G_\circ$ is a discrete group.*

We now see that any disconnected Lie group can be considered, from an algebraic point of view, as a connected Lie group with another discrete group structure between different components.

We close this section with the realization that for a connected Lie group, all information is contained in the neighborhood of the identity ([4], p. 8):

Proposition 2.2. *If G is a connected Lie group, and U is a neighborhood of the identity e, then U generates G (every element in G is a finite product of elements of U).*

2.2 What is a Lie algebra of a Lie group?

It is now time to introduce the concept of an (abstract) Lie algebra. A Lie algebra is basically a vector space equipped with the "commutator".

Definition (Lie algebra)**.** *A real (or complex) vector space $\mathfrak{g}$ is a real (or complex) Lie algebra, if it is equipped with an additional mapping $[.,.] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, which is called the Lie bracket and satisfies the following properties:*

1. *bilinearity: $[\alpha a + \beta b, c] = \alpha[a, c] + \beta[b, c]$ and $[c, \alpha a + \beta b] = \alpha[c, a] + \beta[c, b]$ for all $a, b, c \in \mathfrak{g}$ and $\alpha, \beta \in \mathbb{F}$ (where $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$),*

2. *$[a, a] = 0$ for all $a \in \mathfrak{g}$,*

3. *Jacobi identity: $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$ for all $a, b, c \in \mathfrak{g}$.*

Note that given the definition, the Lie bracket is automatically skew-symmetric: $[a, b] = -[b, a]$. We can naturally introduce the concept of a Lie subalgebra $\mathfrak{h}$, which is a vector subspace of $\mathfrak{g}$, and is also closed under the Lie bracket ($[a, b] \in \mathfrak{h}$ for all $a, b \in \mathfrak{h}$).

Before advancing, let us introduce, in one big definition, all the relevant morphisms for our current structures. Morphisms, in the sense of category theory, are structure preserving mappings between objects. The most used morphisms, for example, are morphisms of vector spaces, which are the familiar linear maps. Another concept is that of an isomorphism — a bijective morphism; if two structures are isomorphic, they are, for the purposes of this structure, equivalent, so by knowing the structure related properties of a specific object, we automatically know the properties of all the objects isomorphic to it. We now turn to the definitions, where we just have to identify, which properties should these morphisms preserve in each case:

Definition (Group/Lie group/Lie algebra morphism/isomorphism)**.**

- *Let $\Phi : G_1 \to G_2$ be a mapping between two groups. Then Φ is a group homomorphism, if $\Phi(ab) = \Phi(a)\Phi(b)$ for all $a, b \in G_1$. If Φ is also bijective, it is called a group isomorphism.*

- *Let $\Phi : G_1 \to G_2$ be a mapping between two Lie groups. If Φ is smooth (or analytic in the case of a complex Lie group) and a group homomorphism, it is a Lie group homomorphism. If Φ is also diffeomorphic (bijective and Φ, Φ^{-1} both smooth), it is called a Lie group isomorphism.*

- *Let $\phi : \mathfrak{g}_1 \to \mathfrak{g}_2$ be a mapping between two Lie algebras. If ϕ is linear and it preserves the Lie bracket, namely $[\phi(a), \phi(b)] = \phi([a, b])$ for all $a, b \in \mathfrak{g}_1$, it is called a Lie algebra morphism. If it is also bijective, it is a Lie algebra isomorphism.*

If we have a Lie group G, we automatically get it's algebra $\mathfrak{g}$, by taking the tangent space of the identity e (this tangent space is the set of equivalence classes of tangent curves $\gamma : \mathbb{R} \to G$ with $\gamma(0) = e$). If G is a n-dimensional manifold, then the tangent space at any point of the manifold has an induced structure of an n-dimensional vector space. Also, it is possible to construct a mapping $\mathfrak{g} \to G$ from the Lie algebra to the group, called the exponential map, with some interesting properties (the construction is by one parameter Lie group morphisms $\mathbb{R} \to G$ generated by an element in the Lie algebra). Although this mapping and it's properties are of utmost importance to the study of Lie groups, we will not be concerning ourselves too much with the general case of this mapping; we will just use the results this concept gives ([4],p. 27), and state an explicit form of this map for a particular case of Lie groups in the next subsection:

Proposition 2.3 (Property of the exponential map)**.** *Let the mapping* $\exp : \mathfrak{g} \to G$ *be the exponential map from the Lie algebra of G into G. Then there exist open neighborhoods U of 0 in $\mathfrak{g}$ and V of e in G, so that $exp\big|_U : U \to V$ is a diffeomorphism ($\mathfrak{g}$ has a smooth structure induced by its vector space structure). In particular,* $\exp|_U$ *is a bijection.*

This shows that part of the Lie algebra always maps into an open neighborhood of the identity element, and, as we know from proposition 2.2 that this is sufficient to generate the whole group G (if it is connected).

For a Lie group G, we now have a vector space $\mathfrak{g}$, but we still don't have a Lie bracket defined. This operation is induced by the multiplication in the Lie group, namely for appropriate $a, b \in \mathfrak{g}$, we can map them to the group, multiply them, and this result will again be in the image of the exponential map. We can write this as ([4], p. 29)

$$\exp(a)\exp(b) = \exp(\mu(a, b)), \tag{1}$$

where the function $\mu(a, b)$ is uniquely determined by the group G. Because the exponential map is locally diffeomorphic at the identity, the function μ is analytical, and can be written as a power series of multilinear forms dependent on a and b. It turns out that up to second order in a and b, we have $\mu(a, b) = a + b + \frac{1}{2}[a, b] + \dots$ for all appropriate a and b, where $[.,.]$ is some bilinear form, which satisfies the condition for the Lie bracket. Multiplication in the group G thus induces a Lie algebra structure in the tangent space $\mathfrak{g}$. On the other hand, it turns out that all higher terms of μ can be written in terms of the Lie bracket (the Campbell-Hausdorff formula, [4], p. 39). Because of this, we can reconstruct the group operation from the Lie bracket ([4], p. 40):

Proposition 2.4. *If G is a connected Lie group, then the group law can be reconstructed from the Lie bracket in $\mathfrak{g}$.*

It is important to note that this last result only allows to reconstruct multiplication, while it doesn't give the topology of the Lie group. Regarding the topology of G, we will state an important result later.

2.3 Further connection between the Lie group and its Lie algebra

In this section, we will state a few more results, which will eventually culminate in the fundamental theorems of Lie theory.

One thing to note, is the fact that Lie group morphisms induce Lie algebra morphisms. Namely, if $\phi : G_1 \to G_2$, then this mapping induces a mapping $\phi_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ between tangent spaces, which is linear and respects the Lie bracket. Furthermore, the following two statements holds ([4], p. 27-28):

Proposition 2.5.

1. *If $\phi : G_1 \to G_2$ is a Lie group morphism, then $\exp(\phi_*(a)) = \phi(\exp(a))$ for all $a \in \mathfrak{g}_1$.*
2. *If G_1 is connected, then a Lie group morphism $\phi : G_1 \to G_2$ is uniquely determined by the induced linear mapping ϕ_* between tangent spaces.*

These two statements show that we can work on the level of the Lie algebras and use the linear mapping ϕ_* between them, instead of working directly with groups and the mapping ϕ. One thing we don't know, however, is whether we can work with any linear mapping between the Lie algebras (which preserves the Lie bracket), and expect that this represents some Lie group morphism on the group level. To answer this, we now move directly to the fundamental theorems of Lie theory and their consequences ([4], p. 41), and leave the discussion for later.

Theorem 2.6 (Fundamental theorems of Lie theory)**.**

1. *The subgroups H of a Lie group G and the subalgebras $\mathfrak{h}$ of the corresponding Lie algebra $\mathfrak{g}$ are in bijection. In particular, we get the subalgebra by taking the tangent space at the identity in a subgroup.*
2. *If G_1 and G_2 are Lie groups, and G_1 is also connected and simply connected, then we have a bijective correspondence between Lie group morphisms $G_1 \to G_2$ and Lie algebra morphisms $\mathfrak{g}_1 \to \mathfrak{g}_2$.*
3. *Let $\mathfrak{a}$ be an abstract finite-dimensional Lie algebra (real/complex). Then there exists a Lie group G (real/complex), so that its corresponding Lie algebra $\mathfrak{g}$ is isomorphic to $\mathfrak{a}$.*

The first statement basically says that subalgebras of $\mathfrak{g}$ correspond to subgroups of G. The second statement answers our question: we can work with Lie algebra morphisms and expect them to also be morphisms on the level of groups, if the first group is connected and simply connected ("simply connected" is another term from topology, and it basically means that every loop in such a space can be contracted to a point; the plane with a point P removed is for example not simply connected, since one can not contract the loop which goes around the point P). The third theorem is perhaps the most interesting: we can take any abstract Lie algebra (for example we take some real vector space V and define the commutator between (some) basis vectors), and with this automatically get some Lie group G, which locally has the structure of the given Lie algebra. But is this group unique? In other words, does a Lie algebra correspond to exactly one group? The following statement is just a corollary of the fundamental theorems, but it is important enough for us to promote it to a theorem:

Theorem 2.7 (Connected Lie groups of a given Lie algebra)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra. Then there exists a unique (up to isomorphism) connected and simply connected Lie group G with $\mathfrak{g}$ as its Lie algebra. If G' is another connected Lie group with this Lie algebra, it is of the form G/Z, where Z is some discrete central subgroup of G.*

We now finally have the whole picture between the connection between Lie groups and Lie algebras. Basically, we now know that we can work with Lie algebras, but thus losing (only) the topology of the group. By knowing possible Lie algebras, we know the possible Lie groups by the following line of thought: each Lie algebra $\mathfrak{g}$ generates a unique connected and simply connected Lie group G. Then, we also have connected groups of the form G/Z, where Z is a discrete central subgroup (central means that it lies in the center of G, which is the set of all elements a, for which $ab = ba$ for all $b \in G$). We also have disconnected groups with algebra $\mathfrak{g}$, but they are just the previous groups G/Z overlayed with another (unrelated) discrete group structure.

Before moving on to matrix groups, it is best to look at an example of what we've said so far. We assume some familiarity with matrix groups for this example to make sense (otherwise the reader can return to the example later). It is a well known fact that groups SU(2) and SO(3) have the same Lie algebra $\mathfrak{su}(2) = \mathfrak{so}(3)$. Since SU(2) is connected and simply connected, it is the unique group constructed from the Lie algebra $\mathfrak{su}(2)$ by theorem 2.7. And since SO(3) is connected, it means it is isomorphic to $\mathrm{SU}(2)/Z$ for some central subgroup Z. It turns out that $\mathrm{SO}(3) \simeq \mathrm{SU}(2)/\mathbb{Z}_2$; since SU(2) has the topology of a three-dimensional sphere S^3, the quotient group has the topology of the sphere with opposite points identified, which is the real projective space $\mathbb{RP}^3$.

2.4 On matrix algebras and on why are (only) they important

We now come to somewhat more familiar territory. We will consider Lie groups and Lie algebras of matrices.

We define the $\mathrm{GL}(n, \mathbb{F})$ as the group of all invertible $n \times n$ matrices, which have either real ($\mathbb{F} = \mathbb{R}$) of complex ($\mathbb{F} = \mathbb{C}$) entries. Multiplication in this group is defined by the usual multiplication of matrices (if A and B are invertible, then AB is also invertible, because $(AB)^{-1} = B^{-1}A^{-1}$ (and multiplication is therefore well defined). One can also verify other group properties. The manifold structure is automatic, since it is an open set of all $n \times n$ matrices (which form a n^2 dimensional vector space, which is isomorphic to $\mathbb{F}^{n^2}$).

We also define $\mathfrak{gl}(n, \mathbb{F})$ as the set of ALL matrices of dimension $n \times n$ with entries in $\mathbb{F}$. This set is of course a vector space under the usual addition of matrices and scalar multiplication. One can also define the commutator of two such matrices, as $[A, B] = AB - BA$, and this operation satisfies the requirements for the Lie bracket. The set $\mathfrak{gl}(n, \mathbb{F})$ therefore has the structure of a Lie algebra.

The notation for $\mathfrak{gl}(n, \mathbb{F})$ was suggestive. The matrices $\mathfrak{gl}(n, \mathbb{F})$ are the Lie algebra of the Lie group $\mathrm{GL}(n, \mathbb{F})$, including the Lie bracket being the common commutator. The exponential map, which we were unwilling to define in all generality, is in this case given as the usual exponential of matrices:

$$\exp(A) \quad = \quad e^A = I + \sum_{n=1}^{\infty} \frac{A^n}{n!}. \tag{2}$$

This map can be inverted near the identity matrix I:

$$\exp^{-1}(I+A) \quad = \quad \log(I+A) = \sum_{n=1}^{\infty} \frac{(-1)^{k+1} A^n}{n}. \tag{3}$$

With this definition of the exponential map, we can easily see that for an arbitrary matrix A, $e^A \in \mathrm{GL}(n, \mathbb{F})$ really is invertible, and its inverse is given by e^{-A}. Indeed, because $[A, -A] = 0$, we have $e^A e^{-A} = e^{A-A} = e^0 = I$.

Now, by virtue of the first fundamental theorem, we can construct various matrix subgroups of $\mathrm{GL}(n, \mathbb{F})$ by taking Lie subalgebras of $\mathfrak{gl}(n, \mathbb{F})$, namely subalgebras of $n \times n$ matrices. There are a number of important groups and algebras of this type, and they are called the classical groups. We will, for the purposes of future convenience and reference, list them in a large table ([4], p. 20), together with the restrictions, by which they were obtained, as well as some other properties. These properties will be their null and first homotopic group, π^0 and π^1 (we will not go further into these concepts here, let us just mention that a trivial π^0 means G is connected, and a trivial π^1 means G is simply connected; furthermore, for G which are not connected, π^1 is specified for the connected component of the identity). Another property, which we will also list, is whether a group is compact (as a topological space) and denote this by a C. Finally, dim will be the dimension of the group as a manifold, which is equal to the dimension of the Lie algebra as a vector space. It is easy to check the dimensionality in each case by noting that $n \times n$ matrices form a n^2 dimensional space by themselves, but then the dimensionality is gradually reduced by the constraints on its Lie algebra. However, the constraints on the Lie algebra are derived from the constraints on the group. In the orthogonal case for example, we have $e^A (e^A)^T = (e^A)^T e^A$, which implies $e^A e^{(A^T)} = e^{(A+A^T)} = I$, and consequently $A + A^T = 0$. The constraint $\det e^A = 1$ can be reduced to the constraint $\operatorname{Tr} A = 0$ (one can show this by putting A into its Jordan form).

Table 1: List of important real Lie subgroups and Lie subalgebras of the general linear group.

G		$\mathfrak{g}$		dim	π^0	π^1	C?
$\mathrm{GL}(n,\mathbb{R})$	/	$\mathfrak{gl}(n,\mathbb{R})$	/	n^2	$\mathbb{Z}_2$	$\mathbb{Z}_2 (n \geq 3)$	
$\mathrm{SL}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$	$\det A = 1$	$\mathfrak{sl}(n,\mathbb{R})$	$\operatorname{Tr} A = 0$	$n^2 - 1$	$\{1\}$	$\mathbb{Z}_2 (n \geq 3)$	
$\mathrm{Sp}(n,\mathbb{R}) \subseteq \mathrm{GL}(2n,\mathbb{R})$	$A^T J A = J$	$\mathfrak{sp}(n,\mathbb{R})$	$JA + A^T J = 0$	$n(2n+1)$	$\{1\}$	$\mathbb{Z}$	
$\mathrm{O}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$	$AA^T = I$	$\mathfrak{o}(n,\mathbb{R})$	$A + A^T = 0$	$n(n-1)/2$	$\mathbb{Z}_2$	$\mathbb{Z}_2 (n \geq 3)$	C
$\mathrm{SO}(n,\mathbb{R}) \subseteq \mathrm{GL}(n,\mathbb{R})$	$\det A = 1, AA^T = I$	$\mathfrak{so}(n,\mathbb{R})$	$A + A^T = 0$	$n(n-1)/2$	$\{1\}$	$\mathbb{Z}_2 (n \geq 3)$	C
$\mathrm{U}(n) \subseteq \mathrm{GL}(n,\mathbb{C})$	$AA^\dagger = I$	$\mathfrak{u}(n)$	$A + A^\dagger = 0$	n^2	$\{1\}$	$\mathbb{Z}$	C
$\mathrm{SU}(n) \subseteq \mathrm{GL}(n,\mathbb{C})$	$\det A = 1, AA^\dagger = I$	$\mathfrak{su}(n)$	$\operatorname{Tr} A = 0, A + A^\dagger = 0$	$n^2 - 1$	$\{1\}$	$\{1\}$	C

Table 2: List of important complex Lie subgroups and Lie subalgebras of the general linear group.

G		$\mathfrak{g}$		$\dim_{\mathbb{C}}$	π^0	π^1	C?
$\mathrm{GL}(n,\mathbb{C})$	/	$\mathfrak{gl}(n,\mathbb{C})$	/	n^2	$\{1\}$	$\mathbb{Z}$	
$\mathrm{SL}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$	$\det A = 1$	$\mathfrak{sl}(n,\mathbb{C})$	$\operatorname{Tr} A = 0$	$n^2 - 1$	$\{1\}$	$\{1\}$	
$\mathrm{O}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$	$AA^T = I$	$\mathfrak{o}(n,\mathbb{C})$	$A + A^T = 0$	$n(n-1)/2$	$\mathbb{Z}_2$	$\mathbb{Z}_2$	
$\mathrm{SO}(n,\mathbb{C}) \subseteq \mathrm{GL}(n,\mathbb{C})$	$\det A = 1, AA^T = I$	$\mathfrak{so}(n,\mathbb{C})$	$\operatorname{Tr} A = 0, A + A^T = 0$	$n(n-1)/2$	$\{1\}$	$\mathbb{Z}_2$	

The name of the groups in tables 1 and 2 are the following: GL is the general linear group, SL is the special linear group, Sp is the symplectic group, O is the orthogonal group, SO is the special orthogonal group, U is the unitary group and SU is the special unitary group. In the definition of the symplectic group, we have used the matrix J, which is, in the case of $2n \times 2n$ matrices, written as

$$J = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix}. \tag{4}$$

Also, it is customary in physics to write an element in the Lie algebra with an additional i, so that we can write $U = e^{i\alpha_a t_a}$, where t_a are the base matrices, α_a the parameters, and summation over a is assumed. We will not be using this notation here, so the difference in notation should be noted.

All this classification and hard work has not been in vain. We expect matrix groups to be very important, because when considering representations of Lie groups and Lie algebras on a vector space V, we will get linear operators, which can be presented as matrices. For the representations, matrices will thus present group and algebra elements. It turn's out that this is not the only reason of their importance; as the last theorem of this section, we look at the following result ([4], p. 42):

Theorem 2.8 (Ado's theorem)**.** *Let $\mathfrak{g}$ be a finite dimensional Lie algebra over $\mathbb{F}$ (either real or complex). Then $\mathfrak{g}$ is isomorphic to some subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ for some n.*

This theorem implies that not only do we know practically everything about Lie groups by considering abstract Lie algebras, but it is sufficient to consider just the Lie algebras of matrices, with the Lie bracket being the commutator and the exponential map into the group being the exponential of a matrix. This reduction to such a specific environment as matrices greatly simplifies anything we want to do in a general Lie group.

3 Semisimple Lie algebras and Root Systems

3.1 What are Semisimple Lie Algebras?

In the previous section, we have shown the relationships between the Lie groups and algebras and we have motivated, why we shall study only subalgebras of the matrix Lie algebra $\mathfrak{gl}(n, \mathbb{F})$. We shall now proceed towards a classification of Lie algebras: we will be interested mainly in so called semisimple Lie algebras. We shall introduce the concept of semisimple Lie algebras and motivate, how they fit in the big picture of all Lie algebras.

First, we have to list a few definitions in order to be able to talk about this subject:

Definition (Ideal, Solvable, Semisimple, Simple Lie Algebras)**.** *Let $\mathfrak{g}$ be a Lie algebra.*

- *A vector subspace $\mathfrak{h} \subseteq \mathfrak{g}$ is an ideal, if $[h, g] \in \mathfrak{h}$ for all $h \in \mathfrak{h}$ and $g \in \mathfrak{g}$.*
- *$\mathfrak{g}$ is solvable, if $D^n\mathfrak{g} = 0$ for a large enough n, where $D^i\mathfrak{g}$ is defined as $D^0\mathfrak{g} = \mathfrak{g}$ and $D^{i+1}\mathfrak{g} = [D^i\mathfrak{g}, D^i\mathfrak{g}]$ (where $[\mathfrak{a}, \mathfrak{b}] = \{[a, b]; a \in \mathfrak{a} \wedge b \in \mathfrak{b}\}$).*
- *$\mathfrak{g}$ is semisimple, if there are no nonzero solvable ideals in $\mathfrak{g}$.*
- *$\mathfrak{g}$ is simple, if it is non-Abelian, and contains 0 and $\mathfrak{g}$ as the only ideals.*

These definitions are presented here mainly for completeness and we will not be using them directly. One can imagine ideals as "hungry entities", such that a commutator with an element in the ideal gives you back an element of the ideal (they are called *invariant subalgebras* in [5], p. 12). Ideals can be used for making quotients of Lie algebras. Solvability is an important concept and is useful for example in representation theory. One can intuitively imagine solvable Lie algebras as "almost Abelian", while semisimple Lie algebras are "as far from Abelian as possible".

We now introduce the concept of a radical, so that we can write an important result, known as Levi decomposition ([4], p. 97-98):

Proposition 3.1 (Radicals)**.** *Every Lie algebra $\mathfrak{g}$ contains a unique largest solvable ideal. This ideal* $\mathrm{rad}(\mathfrak{g})$ *is called the radical of* $\mathfrak{g}$*.*

Theorem 3.2 (Levi decomposition)**.** *Any Lie algebra $\mathfrak{g}$ has a Levi decomposition*

$$\mathfrak{g} \;=\; \mathrm{rad}(\mathfrak{g}) \oplus \mathfrak{g}_{ss}, \tag{5}$$

where $\mathrm{rad}(\mathfrak{g})$ *is the radical of $\mathfrak{g}$, and $\mathfrak{g}_{ss}$ is a semisimple subalgebra of $\mathfrak{g}$.*

This result tells us that we can always decompose a Lie algebra into its "Abelian" and "Non-Abelian" parts, which are the maximum ideal (the radical), and some remainder, which doesn't contain any remaining ideals (a semisimple Lie algebra). Thus, for any algebra, we will be interested in the "Non-Abelian" part, namely the semisimple Lie algebra $\mathfrak{g}_{ss}$.

We finish this subsection with an important result, which could alternatively be given for the definition of a semisimple Lie algebra:

Proposition 3.3. *A Lie algebra $\mathfrak{g}$ is semisimple if and only if $\mathfrak{g} = \bigoplus \mathfrak{g}_i$, where $\mathfrak{g}_i$ are simple Lie algebras.*

We can therefore view a semisimple Lie algebra $\mathfrak{g}$ as a direct sum of simple Lie algebras $\mathfrak{g}_i$, which have only 0 and $\mathfrak{g}_i$ for their ideals. In particular, every simple Lie algebra is also semisimple.

3.2 The Root System of a Semisimple Lie Algebra

Semisimple Lie algebras have a very important property called *the root decomposition*, which will be the main concern in this subsection. But first, in order to be able to formulate this decomposition, we shall define Cartan subalgebras (not in general, but for semisimple Lie algebras) — another definition for the sake of completeness (a summary of definitions from [4], p. 104,108,119).

Definition (Toral, Cartan Subalgebras)**.**

1. *A subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is toral, if it is commutative and for all elements $h \in \mathfrak{h}$, the operators $[h, .]$ are diagonalizable (as linear operators on the vector space $\mathfrak{g}$).*

2. *Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{h}$ is a Cartan subalgebra, if it is a toral subalgebra, and $\mathcal{C}(\mathfrak{h}) = \mathfrak{h}$ (where $\mathcal{C}(\mathfrak{h}) = \{x \in \mathfrak{h}; \forall h \in \mathfrak{h} : [x, h] \in \mathfrak{h}\}$ is the centralizer).*

This gives the usual definition of the Cartan subalgebra we are familiar with. Namely, it turns out that $\mathfrak{h}$ is a Cartan subalgebra, if it is a maximal toral subalgebra, and since $[h_1, h_2] = 0$ in this subalgebra, it is the maximal subalgebra of simultaneously diagonalizable elements (diagonalizable in the vector space $\mathfrak{g}$). We will not elaborate on the issue of existence of the Cartan subalgebra; we will just state that any complex semisimple Lie algebra indeed has a Cartan subalgebra.

We can now state the main result:

Theorem 3.4 (Root decomposition). *If $\mathfrak{g}$ is a semisimple Lie algebra, and $\mathfrak{h} \subseteq \mathfrak{g}$ is the Cartan subalgebra, then*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha, \tag{6}$$

where subalgebras $\mathfrak{g}_\alpha$ are defined as $\mathfrak{g}_\alpha = \{x \in \mathfrak{g}; \forall h \in \mathfrak{h} : [h, x] = \alpha(h)x\}$, where α are linear functionals on the vector space $\mathfrak{h}$. The functionals α go over $R = \{\alpha \in \mathfrak{h}^* \setminus \{0\}; \mathfrak{g}_\alpha \neq 0\}$.

Let's give some motivation for this result. Basically, the Cartan subalgebra consists of elements, which commute amongst themselves. Consequently, the linear operators of the type $[h, .]$ (with $h \in \mathfrak{h}$) commute amongst themselves (easy to check), and can therefore be simultaneously diagonalized. That means we get a number of common eigenspaces $\mathfrak{g}_i$ (and the total dimensionality of these eigenspaces is equal to the dimensionality n of the the vector space $\mathfrak{g}$). This means that all $x \in \mathfrak{g}_i$ are automatically characterized by specifying the eigenvalues for all the operators $[h, .]$, where $h \in \mathfrak{h}$. An eigenspace can therefore be characterized by a linear functional $\alpha \in \mathfrak{h}^*$, which sends a $h \in \mathfrak{h}$ to the eigenvalue of $[h, .]$ in this eigenspace. Of course, since the dimension of $\mathfrak{g}$ is finite, there are a finite number of these eigenspaces (consequently, R is finite). Also, the Cartan subalgebra $\mathfrak{h}$ is an eigenspace under the operators $[h, .]$ with the eigenvalue 0, so we get the zero functional 0 for the functional α on this eigenspace. The case $\alpha = 0$ (which corresponds exactly to the Cartan subalgebra, since it is the maximal Lie subalgebra of commuting elements) is separated from the others, so we demand $\alpha \in \mathfrak{h}^* \setminus \{0\}$.

We have thus obtained a finite set R of linear functionals. The functionals α have many important properties. One thing to note is that it turns out all eigenspaces $\mathfrak{g}_\alpha$ are one-dimensional, and that for $\alpha \in R$ we have $-\alpha \in R$. One other important property is that if we choose for some $\alpha \in R$ three elements, namely $e \in \mathfrak{g}_\alpha$, $f \in \mathfrak{g}_{-\alpha}$ and $h \in \mathfrak{h}$ as an appropriately normalized element corresponding to the root $\alpha \in R \subseteq \mathfrak{h}^*$ (with respect to a "a scalar product"), these three elements form a subalgebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$ (or $\mathfrak{su}(2)$, if we want a real Lie algebra), where h has the function of being diagonal, while e and f are "raising" and "lowering" elements. Moreover, due to the representation theory of the $\mathfrak{sl}(2, \mathbb{C})$ Lie algebra, it is possible to prove that the set R has the structure of a reduced root system, which is defined in the next section. We will not go into further details.

In our lectures on symmetries in physics, we have defined roots as weights of the adjoint representation. This agrees with what we have done in this seminar, since the functionals α can be viewed componentwise (acting on a basis in $\mathfrak{h}$), giving the weight vectors. And indeed, the linear operators $[g, .]$ are exactly the adjoint representation of the Lie algebra.

3.3 Abstract Root Systems

We already know of the root decomposition of a semisimple Lie algebra $\mathfrak{g}$ into its Cartan subalgebra, and the corresponding root spaces. The root system R is a set of nonzero linear functionals on the Cartan subalgebra, but which always has certain properties, which hold for the roots α. We will now define abstract root systems; as already stated, we claim (without proof) that the root system of a Lie algebra has the structure of an abstract root system.

Definition (Reduced Root System)**.** *Let V be an Euclidean vector space (finite-dimensional real vector space with the canonical inner product $(.,.)$). Then $R \subseteq V \setminus \{0\}$ is a reduced root system, if it has the following properties:*

1. *The set R is finite and it contains a basis of the vector space V.*
2. *For roots $\alpha, \beta \in R$, we demand $n_{\alpha\beta}$ to be integer:*

$$n_{\alpha\beta} \equiv \frac{2(\alpha,\beta)}{(\beta,\beta)} \in \mathbb{Z}. \tag{7}$$

3. *If $s_\alpha : V \to V$, $s_\alpha(\lambda) = \lambda - \frac{2(\alpha,\lambda)}{(\alpha,\alpha)}\alpha$, then $s_\alpha(\beta) \in R$ for all $\alpha, \beta \in R$.*
4. *If $\alpha, c\alpha \in R$ for some real c, then $c = 1$ or $c = -1$.*

Although we have written these conditions in a formal manner, it is also instructive to try to visualize them (see Figure 1). So we start with a finite collection of points, which span the whole Euclidean space V. The expression $\frac{(\alpha,\beta)}{(\beta,\beta)}\beta$ is the projection of α onto the direction of the root β. We can rewrite this as $\mathrm{proj}_\beta \alpha = \frac{n_{\alpha\beta}}{2}\beta$: since $n_{\alpha\beta} \in \mathbb{Z}$, then the projection of α is a half-integer of β.

The third condition constructs a function s_α which subtracts from a vector λ the twofold projection of λ to the direction of α: a single subtraction would bring us to the orthogonal complement of the vector α (which is a hyperplane of codimension 1), but a twofold subtraction actually reverses the α-component of the vector λ: lambda is reflected over the hyperplane $L_\alpha = \{\lambda \in V; (\lambda, \alpha) = 0\} = \{\mathbb{R}\alpha\}^\perp$. The third condition therefore states that for a root $\beta \in R$, the reduced root system R must also contain all the reflections of β over the hyperplanes, constructed by the roots in R.

The fourth condition is the reason for calling this structure a *reduced* root system. Namely, say $\alpha \in R$. Then we want to know, which $c\alpha$ are also permissible to be in R given only the first three conditions. The second condition tells us that $2n_{c\alpha,\alpha} = 2c \in \mathbb{Z}$, which implies that c must be half integer. The same holds for $2n_{\alpha,c\alpha} = 2/c \in \mathbb{Z}$. If c and $1/c$ are half integer, then the only possibilities are $c \in \{\pm 2, \pm 1, \pm 1/2\}$. Condition four puts a further restriction to this, so that we permit only 2 of the six possibilities.

We will see that these conditions imply still further properties of roots. We know, for example, that in the Euclidean space V, the standard scalar product gives $(\alpha, \beta) = |\alpha| \cdot |\beta| \cos\varphi$, where $\alpha, \beta \in R$ are root vectors, while φ is the angle between them. With this, we can rewrite the numbers $n_{\alpha\beta}$ as

$$n_{\alpha\beta} = 2\frac{|\alpha| \cdot |\beta| \cos\varphi}{|\beta| \cdot |\beta|} = 2\frac{|\alpha|}{|\beta|}\cos\varphi, \tag{8}$$

$$\Rightarrow n_{\alpha\beta} n_{\beta\alpha} = 4\cos^2\varphi. \tag{9}$$

Because $n_{\alpha\beta}, n_{\beta\alpha} \in \mathbb{Z}$, and also $\frac{n_{\alpha\beta}}{n_{\beta\alpha}} = \frac{|\alpha|^2}{|\beta|^2}$, we get restrictions on the angle φ between two roots, as well as their relative length. We also have another restriction, due to the

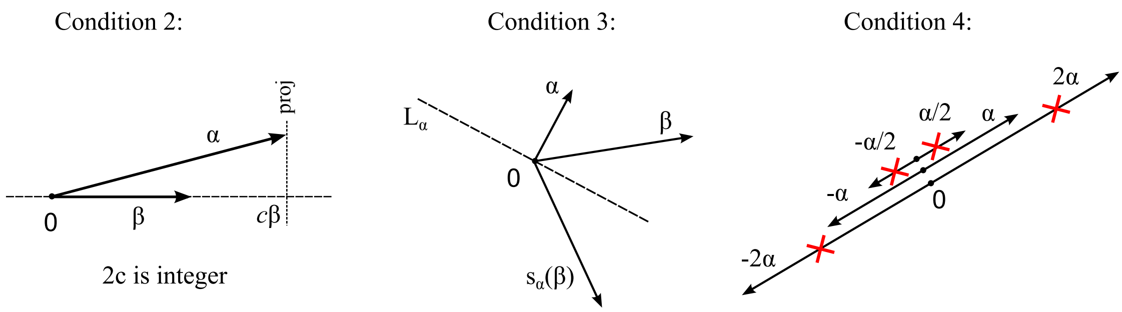

Figure 1: A visual representation of the conditions which must hold for a reduced root system.

nonnegative righthand side of equation (9): either both $n_{\alpha\beta}$ and $n_{\beta\alpha}$ are positive, both are negative, or both are zero. Analyzing these possibilities is straightforward, and we get a list as in table 3 ([4], p. 135); this list is complete, and there are no other possibilities, since the product $n_{\alpha\beta}n_{\beta\alpha} < 4$ due to the cosine on the righthand side of equation (9) (if the product equals 4, we have the trivial case $\alpha = \beta$).

Table 3: Possible relative positions of two roots α and β, with the angle between them denoted as φ. We also, without loss of generality, suppose that $|\alpha| \geq |\beta|$ and therefore $|n_{\alpha\beta}| \geq |n_{\beta\alpha}|$.

$n_{\alpha\beta}$	$n_{\beta\alpha}$	$\lvert\alpha\rvert/\lvert\beta\rvert$	φ [rad]	φ [°]
0	0	/	$\pi/2$	90
-1	-1	1	$2\pi/3$	120
1	1	1	$\pi/3$	60
-2	-1	$\sqrt{2}$	$3\pi/4$	135
2	1	$\sqrt{2}$	$\pi/4$	45
-3	-1	$\sqrt{3}$	$\pi/6$	30
3	1	$\sqrt{3}$	$5\pi/6$	150

Definition (Root System Isomorphism)**.** *Let $R_1 \subseteq V_1$ and $R_2 \subseteq V_2$ be two root systems. Then $\phi : V_1 \to V_2$ is a root system isomorphism, if it is a vector space isomorphism, $\phi(R_1) = \phi(R_2)$ and $n_{\alpha\beta} = n_{\phi(\alpha)\phi(\beta)}$ for all $\alpha, \beta \in R_1$.*

We shall now look at rank 2 root systems (those, which are in a 2-dimensional Euclidean space) and try to classify them up to a root system isomorphism. In this 2-dimensional case, we must have at least two non-parallel roots α and β, so that they span the whole space. Then the possible angles between them are listed in table 3. Because reflections in condition 3 for root systems demands that both β and $s_\alpha(\beta)$ are part of the root system, this angle between α and β can be chosen to be the greater angle amongst the two possibilities, thus $\varphi \geq 90°$ and therefore we have four possibilities: 90°, 120°, 135° and 150°. We then choose a length of the root α, and according to the table of properties, we fix the length of β. Here, the orientation and length of the root α is not important, and also we can choose either α or β to be the longer root. These will be equivalent situations, since rotations and stretching a root system together with its ambient space are among root system isomorphisms. Then we proceed to draw all the reflections of α and β guaranteed to exist by condition 3. Also we have exactly two vectors of the type $c\alpha$ due to $-\alpha = s_\alpha(\alpha)$ and condition 4. We thus get 4 different rank 2 root systems, which are drawn in Figure 2, and we claim that every rank 2 root system is isomorphic to one of them. We cannot add any other roots because of the angle restrictions, and this can be checked for all of the 4 situations separately. Also, these are distinct root systems, since isomorphisms conserve the angles between two roots.

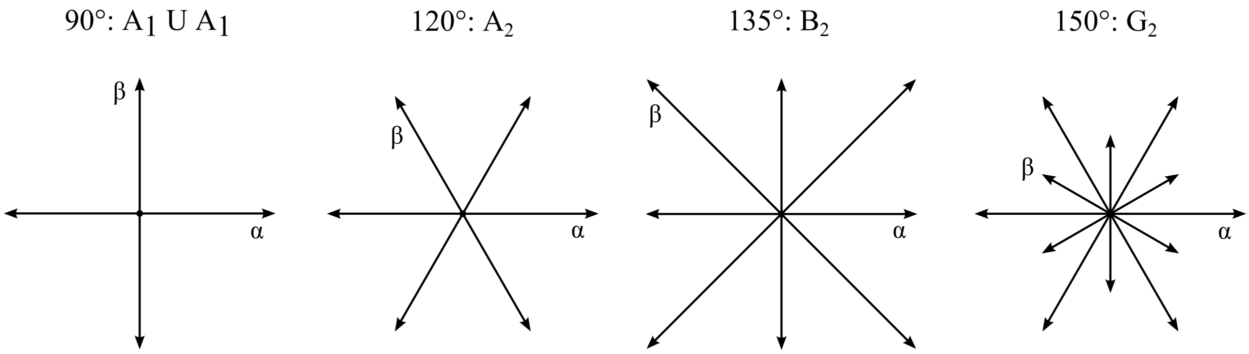

Figure 2: All possible (up to root system isomorphism) reduced root systems of rank 2 along with their traditional names.

3.4 Simple roots

We have defined a root system as a finite collection of vectors in an Euclidean vector space, which satisfy certain properties. Condition 1 stated that the root system must span the whole space. If the space is n-dimensional, we only need n linearly independent vectors. Since R spans V, we know that R contains a basis for V. The only remaining question is, whether we can choose these n vectors among the many root vectors in such a way, to be able to reconstruct the whole root system out of them. This is our motivation for introducing simple roots.

First we notice that a root system is symmetric with respect to the zero vector; namely, we have $-\alpha \in R$ if $\alpha \in R$. Therefore we get the idea that we separate a root system into two parts, which we will call positive and negative roots. We will do this by choosing a polarization vector $t \in V$, which is not located on any of the orthogonal hyperplanes (which go through the zero vector) of the roots in R, so that all roots point in one of the two half-spaces divided by the orthogonal hyperplane of t. Since α and $-\alpha$ are in different subspaces, we have thus separated the root system into two parts. We then look at only "positive" roots and define the concept of a simple root.

Definition (Polarization, Simple Roots)**.**

- *Let $t \in V$ be such that $(\alpha, t) \neq 0$ for all $\alpha \in R$. Then the polarization of R is the decomposition $R = R_+ \sqcup R_-$ ($\sqcup$ denotes the disjoint union), where $R_+ = \{\alpha \in R; (\alpha, t) > 0\}$ and $R_- = \{\alpha \in R; (\alpha, t) < 0\}$. The elements of R_+ are called positive roots and the elements of R_- are called negative roots.*

- *A positive root $\alpha \in R_+$ is simple, if it can not be written as $\alpha_1 + \alpha_2$, where $\alpha_1, \alpha_2 \in R_+$.*

The simple roots are a very useful concept. Every positive root can be written as a finite sum of simple roots, since it is either a simple root, or if it is not, it can be written as a sum of two other positive roots. Since the root system R is finite, this has to stop after a finite number of steps. Also, every negative root can be written as $-\alpha$ for some positive root α. Together that means that for any root $\beta \in R$, we can write it as a linear combination of simple roots with integer coefficients. Let us denote the set of simple roots as S. Therefore, every $\beta \in R$ is a linear combination of vectors in S. Because S spans R and R spans V, simple roots S span the whole space V. Also, it can be proven that simple roots are linearly independent.

We will need another property of simple roots in the future: the scalar product $(\alpha, \beta) \leq 0$ for all $\alpha \neq \beta$ simple roots. We see this in two steps:

1. First off, note that if $\alpha, \beta \in R$ are two roots, such that $(\alpha, \beta) < 0$ and $\alpha \neq c\beta$, then $\alpha + \beta \in R$. This can easily be seen by introducing a rank 2 root subsystem, which contains both α and β (we choose these two roots, which have to satisfy properties from table 3 and consequently generate — by reflections s_λ — one of four rank 2 root systems). This means that it is sufficient to check that $\alpha + \beta \in R$ for the particular cases in Figure 2.

2. We claim that for simple roots $\alpha, \beta \in S$, we have $(\alpha, \beta) \leq 0$, if $\alpha \neq \beta$. If that were not true, we would have $(\alpha, \beta) > 0$. Then, we have $(-\alpha, \beta) < 0$, and also $\beta \neq c\alpha$ (c can only be 1 or -1, but only the case $c = 1$ gives that α and β are positive, and that is forbidden by $\alpha \neq \beta$). By the first step we then have $(-\alpha) + \beta \in R$ so either $\beta - \alpha \in R_+$ or $\beta - \alpha \in R_-$. In the first case, we would have $\beta = \alpha + (\beta - \alpha)$ — a contradiction, since β is simple (and therefore cannot be written as a sum of other positive roots), and in the second case, we would have a contradiction in $\alpha = \beta + (\alpha - \beta)$ being simple (now $\beta - \alpha \in R_-$ and thus $\alpha - \beta \in R_+$). We have thus proved that $(\alpha, \beta) > 0$ is impossible for two simple roots.

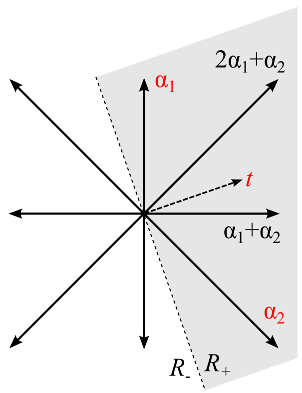

Figure 3: A construction of a simple root system in the case of B_2 of rank 2. Choosing the polarization t we make the decomposition $R = R_+ \sqcup R_-$, and identify the two simple roots α_1 and α_2.

We now sum up the results we obtained about simple roots:

Proposition 3.5. *Let* $\{\alpha_i\}_{i \in I} = S \subseteq R \subseteq V$ *be the set of simple roots due to a polarization* t*. Then* S *is a basis of* V *and every* $\alpha \in R$ *can be written as* $\alpha = \sum_{i \in I} n_i \alpha_i$*, where* $n_i \in \mathbb{Z}$*, and all* n_i *are non-negative if* $\alpha \in R_+$ *and non-positive if* $\alpha \in R_-$*.*

Thus, when choosing a polarization, one immediately gets a "basis" S for the root system R. We would like to reduce the question of classification of reduced root systems R to simple root systems S. For that to work, two important issues have to be resolved; it turns out that for any choice of polarization t we get equivalent simple root systems (which are related by an orthogonal transformation), and that the root system R can be uniquely reconstructed from its simple root system S. We will not go into detail here, we will just explain the main idea.

First, let's consider the equivalence of simple root systems under polarizations. Given a root system R, we can divide the space V by drawing all the hyperplanes L_α corresponding to the roots α, where α goes over all R. With this, the space V is divided into so called *Weyl chambers*. Every polarization t, since it is not on any hyperplane L_α, is in some Weyl chamber. If two polarizations t_1 and t_2 belong to the same Weyl chamber, then the scalar product $\operatorname{sgn}(\alpha, t_1) = \operatorname{sgn}(\alpha, t_2)$ for all $\alpha \in R$, and we get the same decomposition $R = R_+ \sqcup R_-$ with both polarizations, and thus also the same simple root system S. Therefore, we have a bijection between Weyl chambers and different S polarizations. Every Weyl chamber can be transformed to an adjacent Weyl chamber, which is separated from the original by L_α, via the mapping s_α. For two polarizations t_1 and t_2, we can therefore construct a composite mapping of s_α, so that the Weyl chamber associated with t_1 is transformed to the chamber associated with t_2; this mapping also transforms S_1 to S_2 (where S_i is the simple root system associated with t_i), and since it is orthogonal (and thus preserves angles), the two root systems S_1 and S_2 are equivalent.

We now turn to the reconstruction of R from its simple root system S. Here, we again use the reflections s_α; the group of transformations, which is generated by such reflections, is called the Weyl group. It turns out it suffices to generate this group with just simple reflections (mappings s_α where $\alpha \in S$, so α is just a simple root). Furthermore, it turns out the set R can be written precisely as the set of all the elements $w(\alpha_i)$, where w goes over all the elements in the Weyl group and α_i goes over all the simple roots. We therefore start with S and repeatedly apply the associated simple reflections, and we end up with exactly the whole root system R. An interested reader should go to [4], p. 146, and the references given therein, for further details.

With this, we can give the following result:

Proposition 3.6 (Correspondence between a reduced and simple root system)**.** *There is a bijection between reduced root systems and simple root systems. Every reduced root system R has a unique (up to an orthogonal transformation) simple root system S, and conversely, we can uniquely reconstruct R from a simple root system S.*

It is thus sufficient to classify all possible simple root systems instead of all root systems. Furthermore, we know that for a root system in a n-dimensional space, the simple root system has exactly n linearly independent elements.

4 The Classification

4.1 Classification of Simple Root Systems

4.1.1 Irreducible root systems

Armed with the knowledge obtained in previous sections, we know the whole chain of structures, associated with a Lie group. Every Lie group G has a Lie algebra $\mathfrak{g}$, which in turn, if it is semisimple, has a reduced root system R, which in turn has a simple root system S. We shall proceed with the classification of simple root systems.

We know that the root system R is by definition a finite set of vectors. Therefore, $S \subseteq R$ is also a finite set. Since S is a basis for the vector space V due to proposition 3.5, a n-dimensional Lie algebra has a simple root system with n elements.

An important concept is that of the root system, which is associated with the decomposition into systems in distinct orthogonal subspaces. Namely, if one can make a decomposition $R = R_1 \sqcup R_2$ into two subsystems, so that $R_1 \subseteq V_1$ and $R_2 \subseteq V_1^{\perp}$ for some linear subspace $V_1 \subseteq V$, we say that R is reducible. We will not prove this explicitly, but it is intuitively expected that reducible root systems always break up into irreducible ones, which cannot be broken down further, and that the same thing happens with the associated simple root system S. Let us now state, what we have described, more formally (details in [4], p. 150).

Definition (Reducible, Irreducible Root System)**.** *Let R be a (reduced) root system. Then R is reducible, if $R = R_1 \sqcup R_2$, with $R_1 \perp R_2$. R is irreducible, if it is not reducible.*

Proposition 4.1.

- *Every reducible root system R can be written as a finite disjoint union $R = \bigsqcup R_i$, where R_i are irreducible root systems and $R_i \perp R_j$ if $i \neq j$.*
- *If R is a reducible root system with the decomposition $R = \bigsqcup R_i$, we have $S = \bigsqcup S_i$, where S is the simple root system of R (under some polarization), and $S_i = R_i \cap S$ is the simple root system of R_i ("under the same polarization").*
- *If S_i are simple root systems, $S_i \perp S_j$ for $i \neq j$, $S = \bigsqcup S_i$, and R, R_i are the root systems generated by the simple systems S, S_i, we have $R = \bigsqcup R_i$.*

It thus suffices to do a classification of irreducible simple root systems, since reducible ones are built from a finite number of irreducible ones.

4.1.2 The Cartan Matrix and Dynkin diagrams

Suppose we have a simple root system S. We can ask ourselves, what is the relevant information contained in such a system. Certainly, it is not the absolute position of the roots, or their individual length, since we can take an orthogonal transformation and still obtain an equivalent root system. The important properties are their relative length to each other and the angle between them. Since we have for simple roots $\alpha, \beta \in S$ the inequality $(\alpha, \beta) \leq 0$, the angle between simple roots is $\geq 90°$, and with the help of table 3, we have the four familiar possibilities. Of course, the angle between them also dictates their relative length, so the only relevant information are the angles between the roots (and which root is longer).

We can present this information economically as a list of numbers. Instead of angles, we specify the numbers $n_{\alpha\beta} = 2\frac{(\alpha,\beta)}{(\beta,\beta)}$ which are conserved via root system isomorphisms. We call this list the Cartan matrix.

Definition (Cartan matrix)**.** *Let $S \subseteq R$ be a simple root system in n dimensional space, and let us choose an order of labeling for the elements $\alpha_i \in S$, where $i \in \{1 \ldots, n\}$. The Cartan matrix a is then a $n \times n$ matrix, which has the following entries componentwise: $a_{ij} = n_{\alpha_i \alpha_j}$.*

Due to the definition of $n_{\alpha\beta}$, we clearly have $a_{ii} = 2$ for all $i \in \{1 \ldots, n\}$. Also, since the scalar product of simple roots $(\alpha_i, \alpha_j) \leq 0$ for $i \neq j$, the non-diagonal entries in the Cartan matrix are not positive: $a_{ij} \leq 0$ for $i \neq j$.

It is also possible to present the information in the Cartan matrix in a graphical way via Dynkin diagrams. We will now define these diagrams by telling the recipe, of how such a diagram is drawn.

Definition (Dynkin diagram)**.** *Suppose $S \subseteq R$ is a simple root system. The Dynkin diagram of S is a graph constructed by the following prescription:*

1. *For each $\alpha_i \in S$ we construct a vertex (visually, we draw a circle).*
2. *For each pair of roots α_i, α_j, we draw a connection, depending on the angle φ between them.*
   - *If $\varphi = 90°$, the vertices are not connected (we draw no line).*
   - *If $\varphi = 120°$, the vertices have a single edge (we draw a single line).*
   - *If $\varphi = 135°$, the vertices have a double edge (we draw two connecting lines).*
   - *If $\varphi = 150°$, the vertices have a triple edge (we draw three connecting lines).*
3. *For double and triple edges connecting two roots, we direct them towards the shorter root (we draw an arrow pointing to the shorter root).*

There is no need to direct single edges, since they represent $\phi = 120°$, which lead to $|\alpha_i| = |\alpha_j|$, while there are no edges in the orthogonal case, when there is no restriction to the relative length of the pair of roots. Also, the choices for the number of edges in the recipe for the Dynkin diagram is no coincidence: no edge between a pair of roots means that they are orthogonal. It is then clear from the definition of reducible roots that a Dynkin diagram is connected if and only if the simple root system S is irreducible. Moreover, each connected component of the Dynkin diagram corresponds to a irreducible simple root system S_i in the decomposition $S = \bigsqcup S_i$.

Also, it comes as no surprise that the information in the Cartan matrix can be reconstructed from the Dynkin diagram, since the entries $a_{ij} = n_{\alpha_i \alpha_j}$ can be reconstructed from the number of edges and their direction. For example, if $\varphi = 150°$, drawn by a directed triple line, we know from table 3 that $a_{ij} = -3$ and $a_{ji} = -1$ (where $|\alpha_i| > |\alpha_j|$). This full reconstruction of the information of a simple root system S from a Dynkin diagram can be stated more formally.

Proposition 4.2. *Let R and R' be two (reduced) root systems, constructed from the same Dynkin diagram. Then R and R' are isomorphic.*

4.1.3 Classification of connected Dynkin diagrams

Dynkin diagrams are a very effective tool for classifying simple root systems S, and consequently the reduced root systems R. Since reducible root systems are a disjoint union of mutually orthogonal subroot systems, the Dynkin diagram is just drawn out of many connected graphs. It is thus sufficient to classify connected Dynkin diagrams. We will state the result of this classification and will sketch a simplified proof. The forbidden diagrams will be eliminated, but no explicit construction of the possible Lie algebras will be provided.

Theorem 4.3 (Classification of Dynkin diagrams)**.** *Let R be a reduced irreducible root system. Then its Dynkin diagram is isomorphic to a diagram from the list in Figure 4, which is also equipped with labels of the diagrams. The index in the label is always equal to the number of simple roots, and each of the diagrams is realized for some reduced irreducible root system R.*

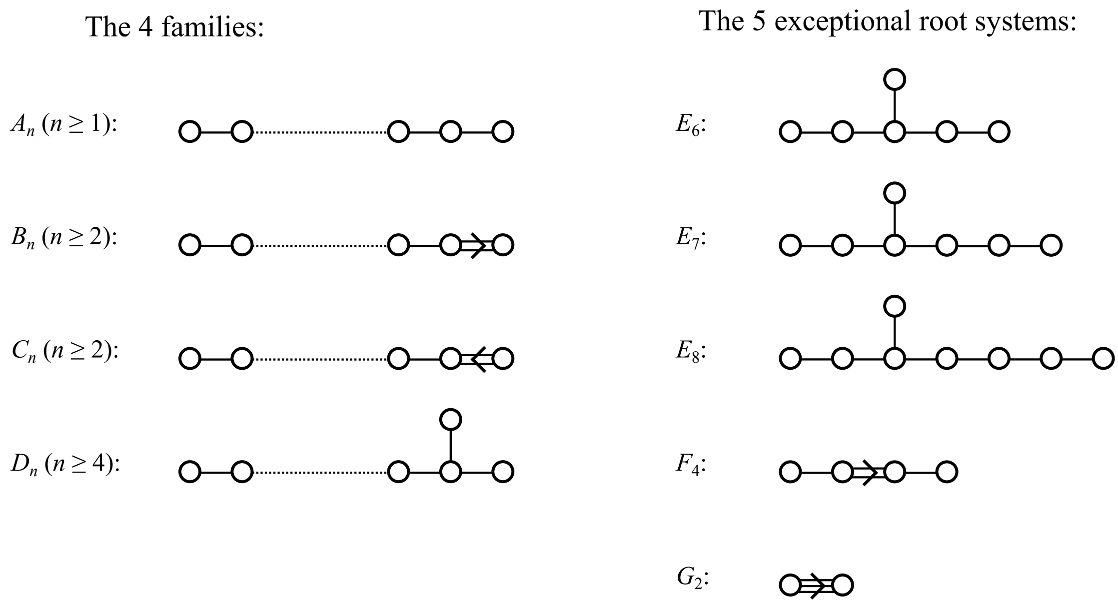

Figure 4: Dynkin diagrams of all irreducible root systems R.

We now turn to a simplified proof of the theorem. It turns out that all irreducible simple root systems in Figure 4 can indeed be constructed. We shall focus on why diagrams with different connections are not valid as Dynkin diagrams of simple root systems.

Only connected graphs of vertices with either a null, single, dual or triple connection with another vertex will be considered. The Dynkin diagrams as graphs of an irreducible simple root systems are a subset of all possible graphs under consideration. Before we start, an important notion has to be introduced: that of a *subgraph*. If I is the set of vertices of a graph, then a subgraph consists of a subset of vertices $J \subseteq I$, while the types of connections between the vertices in J stay the same as in the original graph I. Also, a special case of a Dynkin diagram is a graph I, which contains no dual or triple connections; for the purposes of this seminar, we shall call such a diagram a *simple graph*.

A number of the properties of Dynkin diagrams can be deduced by looking at subgraphs. Suppose we have a true Dynkin diagram I as a realization of an irreducible simple root system: this diagram contains all the necessary information for the construction of the Cartan matrix a_{ij}. If the root system spans a n-dimensional vector space V, then the n roots constitute a basis of this space, and the Cartan matrix is a linear operator on the vector space V; this operator is written as a matrix in the basis $\{\alpha_i\}_{i\in I}$. Suppose we have a subgraph of this Dynkin diagram: we specify a subset J of simple root vectors α_i: $i \in J$ (for a given labeling of the simple roots). If the chosen number of simple roots is k, then the subgraph J has k vertices, and we can construct a $k \times k$ submatrix of the Cartan matrix with entries a_{ij}, where $i, j \in J$. This matrix can be again viewed as a linear operator, this time on the space $\tilde{V}$, which is spanned by the roots α_j with $j \in J$. The linear operator $\hat{a}$, constructed by choosing a subset of indices J from the Cartan matrix, is always positive definite: that means $(\hat{a}x, x) > 0$ for all $x \in \tilde{V} \setminus \{0\}$. Indeed, if $x = \sum_{j \in J} c_j \alpha_j$, then

$$\begin{aligned}(\hat{a}x, x) &= \sum_{i,j,k \in J} (c_i a_{ij} \alpha_j, c_k \alpha_k) = \sum_{i,j,k \in J} 2 c_i c_k \frac{(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} (\alpha_j, \alpha_k) = \\ &= \sum_{j \in J} \frac{2}{(\alpha_j, \alpha_j)} \Big(\sum_{i \in J} c_i \alpha_i, \alpha_j \Big) \Big(\sum_{k \in J} c_k \alpha_k, \alpha_j \Big) = \sum_{j \in J} 2 \frac{(x, \alpha_j)^2}{(\alpha_j, \alpha_j)} \quad > 0.\end{aligned} \tag{10}$$

The result (which is a sum of nonnegative terms) cannot be zero, because that would imply that $(x, \alpha_j) = 0$ for all $j \in J$ and therefore x would be orthogonal to all a_j. This is not possible, since x is a nonzero vector in $\tilde{V}$ and vectors α_j for $j \in J$ form a basis of the vector space $\tilde{V}$. That means that given any subgraph J of a given diagram, its Cartan matrix is positive definite. This will allow us to put restrictions, on what kind of subgraphs can be found in the Dynkin diagrams.

We now mention some important rules as a motivation for the classification theorem ([4], p. 158, and [5], p. 178):

1. *If I is a connected Dynkin diagram with* 3 *vertices, then the only two possibilities are shown in Figure* 5. We shall now derive this result. Consider a Dynkin diagram with 3 vertices: ignoring the relative lengths of the simple roots, the diagram is specified by the 3 angles between the roots. At most one of these angles is equal to 90°, because I is connected. Furthermore, the sum of the angles between 3 vectors is 360° in a plane, and less than 360°, if they are linearly independent. This excludes all possible diagrams with 3 vertices, as shown in Figure 5, except the two from the statement. Also, if there is no 90° angle, we have a loop: it suffices to check a loop of 3 vertices with just single connections, since double or triple connections would increase the sum of the angles even further.

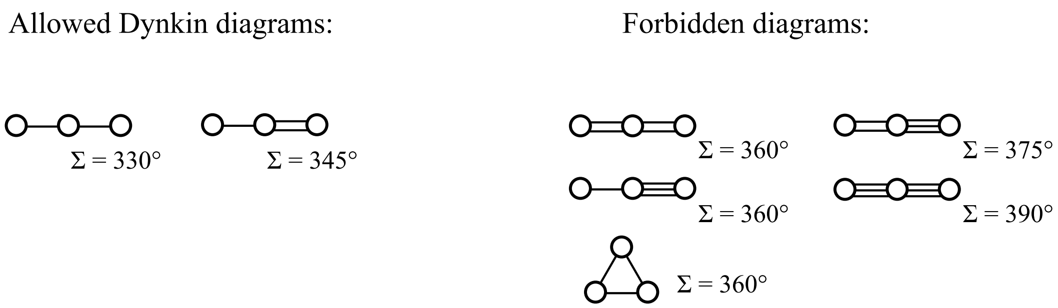

Figure 5: Possible diagrams with 3 vertices; if the sum of the angles between vertices $\Sigma \geq 360°$, the diagram is forbidden.

No Dynkin diagram I may contain any of the forbidden 3 vertex diagrams as subgraphs, lest we run into a contradiction. This implies that the only possible diagram with a triple connection is the one with two vertices.

2. *If I is a Dynkin diagram and a simple graph, then I contains no cycles (subgraphs with vertices connected in a loop).* If that were not the case, there would exist a subgraph J with $k \geq 3$ vertices, which would be a loop (with only single connections). In this loop, the neighboring vertices would give -1 to the Cartan matrix, the diagonal elements would give 2, while all others would be 0. We relabel the indices, so that i runs from

1 to k, and we label $\alpha_{k+1} = \alpha_1$ and $\alpha_0 = \alpha_k$. For $x = \sum_{j \in J} \alpha_j$, with the normalization of roots $(\alpha_i, \alpha_i) = 2$, we then have

$$(\hat{a}x, x) = \sum_{i,j,k} \frac{2}{(\alpha_j, \alpha_j)} (\alpha_i, \alpha_j)(\alpha_j, \alpha_k) \tag{11}$$

$$= \sum_{i,j,k} (2\delta_{j,i} - \delta_{j,i+1} - \delta_{j,i-1})(2\delta_{j,k} - \delta_{j,k+1} - \delta_{j,k-1}) \tag{12}$$

$$= 0, \tag{13}$$

which is a contradiction for the positive definite Cartan matrix of a subgraph of a Dynkin diagram (we found $x \in \operatorname{Ker} \hat{a}$).

Another two forbidden diagrams:

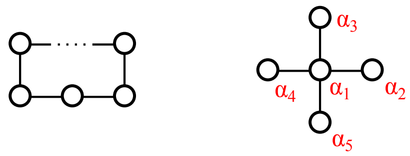

Figure 6: Cycles and vertices with 4 (or more) connections are forbidden. This is derived by computing the violation of the positive definiteness of the Cartan matrix.

3. *If I is a Dynkin diagram and a simple graph, then each vertex in I is connected to at most* 3 *others.* If that were not the case, then at least one of the vertices would be connected to at least 4 others, and we would have a subgraph, which is shown in Figure 6. For this specific graph, the vector $x = 2\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5$ gives $(\hat{a}x, x) = 0$:

$$\hat{a}x = \begin{bmatrix} 2 & -1 & -1 & -1 & -1 \\ -1 & 2 & & & \\ -1 & & 2 & & \\ -1 & & & 2 & \\ -1 & & & & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = 0 \tag{14}$$

4. *If I is a Dynkin diagram, and α, β are two roots connected with a single connection, as shown in Figure* 7*, the two roots can be substituted by a single root, and we obtain a new Dynkin diagram.* We will not prove this statement, but it can be shown by constructing a new root system, with the same roots as previously, but taking the root $\alpha + \beta$ instead of roots α and β. One can easily check (via scalar products) that the angles between the new root $\alpha + \beta$ and other roots are consistent with the contraction of the two vertices.

   As a consequence, it is possible to eliminate some further diagrams by contracting vertices. The reasons why there can be at most one branching point (vertex with 3 connections), and why there cannot be 2 double connections, are illustrated in Figure 7.

5. *The Dynkin diagrams in Figure* 8 *are forbidden as subgraphs.* This can be shown by taking an appropriate nontrivial linear combination of the roots, and showing that in equals zero. That means the roots are not independent and cannot therefore constitute a simple root system. We see that the diagrams for the exceptional cases circumvent

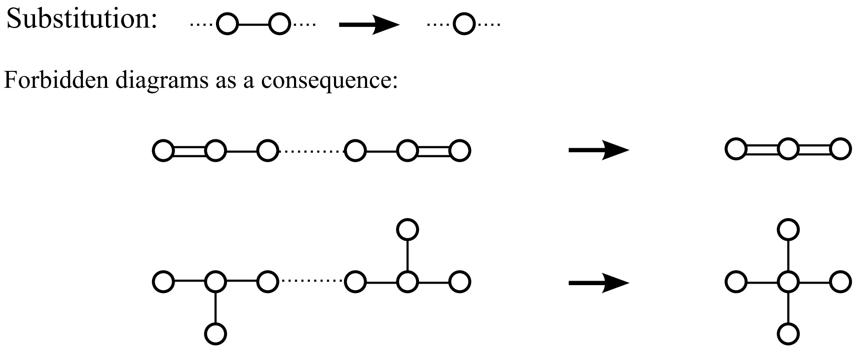

Figure 7: Contractions of two vertices with a single connection in valid Dynkin diagrams give valid Dynkin diagrams. As a consequence, we conclude the diagrams left of the arrow are not valid, since they give invalid contractions.

these forbidden cases. Also, one can interpret the forbidden diagrams as stating the following:

- A branch in the graph cannot be longer than one vertex. Branches with more vertices are forbidden, because they contain the subgraph a.
- A branching point cannot be further inside than on the second vertex, except if the branching point is on the third vertex, and the remaining tail is 4 or less vertices long. Otherwise, they contain a subgraph b or c, and therefore are not allowed.
- A double connection between two vertices must be placed at the end of a chain of vertices with single connections, except in the case of the Dynkin diagram F_4. Otherwise, the chain of single-connection vertices is too long on one side, and the diagram contains d or e as subgraphs, which is not allowed.

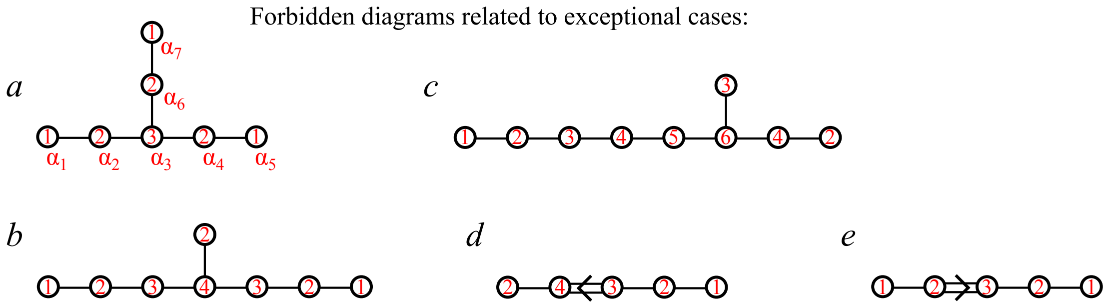

Figure 8: The forbidden cases of branching at an inner vertex or with a forbidden positioning of the double connection. The exceptional cases E_6, E_7, E_8, F_4 are in a sense loopholes for these rules. The numbers in the vertices denote the coefficients in a linear combination of vertices, which is non-trivially zero.

As an example of how the inconsistency arises, we shall compute case a in Figure 8. Note that since this example is that of a simple graph, all roots must be of equal length. We choose a normalization $(\alpha_i, \alpha_i) = 2$, so that the connected vertices give $(\alpha_i, \alpha_j) = -1$, while pairs of vertices without a connection give 0.

$$x = \alpha_1 + 2\alpha_2 + 3\alpha_3 + 2\alpha_4 + \alpha_5 + 2\alpha_6 + \alpha_7, \tag{15}$$

$$\begin{aligned}(x,x) &= \big(3\cdot 1 + 3\cdot 2^2 + 1\cdot 3^2\big)(\alpha_1,\alpha_1) + 2(\alpha_1,\alpha_2) + 2(\alpha_2,\alpha_1) + \\ &\quad + 6(\alpha_2,\alpha_3) + 6(\alpha_3,\alpha_2) + 6(\alpha_3,\alpha_4) + 6(\alpha_3,\alpha_6) + 6(\alpha_4,\alpha_3) + \\ &\quad + 2(\alpha_4,\alpha_5) + 2(\alpha_5,\alpha_4) + 6(\alpha_6,\alpha_3) + 2(\alpha_6,\alpha_7) + 2(\alpha_7,\alpha_6) \\ &= \big(3+12+9\big)\cdot 2 + \big(2+2+6+6+6+6+6+2+2+6+2+2\big)\cdot(-1) \\ &= 0.\end{aligned} \tag{16}$$

4.2 Serre relations and the Classification of semisimple Lie algebras

Now we will turn to the classification of semisimple Lie algebras, and explain how that is related to the classification of irreducible simple root systems.

One thing to note is that the decomposition $\mathfrak{g} = \bigoplus \mathfrak{g}_i$ of a semisimple Lie algebra into simple Lie algebras is related to the decomposition of the root system $R = \bigsqcup R_i$; in particular, $\mathfrak{g}$ is simple if and only if its root system R is irreducible. That means that we will be classifying simple Lie algebras by considering only connected Dynkin diagrams, while multiple unconnected Dynkin diagrams will represent a semisimple Lie algebra.

We will now describe an important result, which will eventually enable us to backtrack from root systems to Lie algebras ([4], p. 155).

Theorem 4.4 (Serre relations)**.** *Let $\mathfrak{g}$ be a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$ and its root system $R \subseteq \mathfrak{h}^*$, and choosing a polarization we have S as its simple root system. Let $(.,.)$ be a scalar product (a non-degenerate symmetric bilinear form) on $\mathfrak{g}$.*

- *We have the decomposition $\mathfrak{g} = \mathfrak{h} \oplus n_+ \oplus n_-$, where $n_\pm = \bigoplus_{\alpha\in R_\pm} g_\alpha$.*
- *Let $H_\alpha \in \mathfrak{h}$ be the element, which corresponds to $\alpha \in \mathfrak{h}^*$, and $h_i = h_{\alpha_i} = 2H_{\alpha_i}/(\alpha_i,\alpha_i)$. If we choose $e_i \in \mathfrak{g}_{\alpha_i}$, $f_i \in \mathfrak{g}_{-\alpha_i}$ and $h_i = h_{\alpha_i}$, with the constraint $(e_i, f_i) = 2/(\alpha_i,\alpha_i)$, then e_i generate n_+, f_i generate n_- and h_i form a basis for $\mathfrak{h}$ (where in all cases $i \in \{1,\dots,r\}$), and thus $\{e_i, f_i, h_i\}_{i\in\{1,\dots,r\}}$ generates $\mathfrak{g}$.*
- *The elements e_i, f_i, g_i satisfy the Serre relations (where a_{ij} are the elements of the Cartan matrix):*

$$[h_i,h_j] = 0, \qquad [h_i,e_j] = a_{ij}e_j, \tag{17}$$

$$[h_i,f_j] = -a_{ij}f_j, \qquad [e_i,f_j] = \delta_{ij}h_j, \tag{18}$$

$$([e_i,.])^{1-a_{ij}}e_j = 0, \qquad ([f_i,.])^{1-a_{ij}}f_j = 0. \tag{19}$$

Knowing the Serre relations, into which we will not go further, we can turn the construction around ([4], p. 156):

Theorem 4.5 (Construction of a Lie Algebra from the Root System). *Let $R = R_+ \sqcup R_-$ be a reduced irreducible root system with a chosen polarization, and $S = \{\alpha_1, \dots, \alpha_r\}$ the corresponding simple root system. Let $\mathfrak{g}(R)$ be the complex Lie algebra generated by e_i, f_i, h_i, with the Serre relations giving the commutators. Then $\mathfrak{g}(R)$ is a semisimple Lie algebra with its root system R, and if $\mathfrak{g}$ is another Lie algebra with its root system R, then $\mathfrak{g}$ is isomorphic to $\mathfrak{g}(R)$.*

Theorem 4.6 (Classification of Semisimple Lie Algebras). *A simple complex finite dimensional Lie algebra $\mathfrak{g}$ is isomorphic to a Lie algebra, constructed from one of the Dynkin diagrams in Figure 4. Semisimple Lie algebras are all possible finite direct sums of simple Lie algebras.*

With this, we have classified semisimple Lie algebras. Here, it is important to note that Dynkin diagrams classify COMPLEX semisimple Lie algebras and not real ones. A classification of real semisimple Lie algebras is done with something called Satake diagrams ([6]), which we will not tackle in this seminar. We can still identify real forms of the classified complex semisimple Lie algebras; a real form of a complex Lie algebra $\mathfrak{g}$ is a real Lie algebra $\mathfrak{g}_\mathbb{R}$, which has the original algebra as its complexification: $\mathfrak{g} = \mathfrak{g}_\mathbb{R} \oplus i\mathfrak{g}_\mathbb{R}$ with the obvious commutator. The real form is not necessarily unique. For example, both $\mathfrak{su}(n)$ and $(n, \mathbb{R})$ have the complexification $(n, \mathbb{C})$. It is possible to identify the Dynkin diagrams with known Lie algebras and their real forms (see table 4), but some Lie algebras (the exceptional ones) are new, and cannot be found among the classical matrix algebras. A more in-depth description of the properties of the listed Lie algebras appears in the appendix of [4], p. 202-209.

Table 4: A table of Dynkin diagrams, corresponding complex semisimple Lie algebras, and their real forms (which are not necessarily unique).

Dynkin	complex $\mathfrak{g}$	real form $\mathfrak{g}_\mathbb{R}$
A_n $(n \geq 1)$	$\mathfrak{sl}(n+1, \mathbb{C})$	$\mathfrak{su}(n+1)$
B_n $(n \geq 1)$	$\mathfrak{so}(2n+1, \mathbb{C})$	$\mathfrak{so}(2n+1, \mathbb{R})$
C_n $(n \geq 1)$	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{R})$
D_n $(n \geq 2)$	$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n, \mathbb{R})$
E_6	complex $\mathfrak{e}_6$	real $\mathfrak{e}_6$
E_7	complex $\mathfrak{e}_7$	real $\mathfrak{e}_7$
E_8	complex $\mathfrak{e}_8$	real $\mathfrak{e}_8$
F_4	complex $\mathfrak{f}_4$	real $\mathfrak{f}_4$
G_2	complex $\mathfrak{g}_2$	real $\mathfrak{g}_2$

It is noteworthy that the restrictions on n in Figure 4 are due to either small diagrams not existing, or they are the same as a previous one. For example, we would have $A_1 = B_1 = C_1$, which would correspond with $\mathfrak{sl}(2, \mathbb{C}) \simeq \mathfrak{so}(3, \mathbb{C}) \simeq \mathfrak{sp}(1, \mathbb{C})$ on the Lie algebra level. We have thus readjusted the possible values n for the purposes of table 4.

One could ask, where in this classification are the familiar Lie algebras $\mathfrak{o}(n, \mathbb{F})$, $\mathfrak{u}(n)$ and $\mathfrak{gl}(n, \mathbb{F})$. We already have $\mathfrak{o}(n, \mathbb{F})$, since $\mathfrak{o}(n, \mathbb{F}) = \mathfrak{so}(n, \mathbb{F})$. The others are not semisimple, since we have the Levi decompositions $\mathfrak{gl}(n, \mathbb{F}) = \mathfrak{sl}(n, \mathbb{F}) \oplus \mathfrak{u}(1)$ and $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathfrak{u}(1)$, where $\mathfrak{u}(1)$ is not simple, since it is Abelian.

5 Conclusion

We have managed to tread the long road from semisimple Lie groups to Dynkin diagrams. For a Lie group G we always have its Lie algebra $\mathfrak{g}$, which is the tangent space of the identity, with the commutator arising through the group multiplication law. We know that this Lie algebra can be viewed as a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{F})$ for some n. We decompose this algebra due to Levi decomposition into a semisimple Lie algebra $\mathfrak{g}_{ss}$ and a remainder (the radical). Then, the semisimple part $\mathfrak{g}_{ss}$ has a root decomposition, and we thus obtain a reduced root system R of the semisimple Lie algebra $\mathfrak{g}_{ss}$; this is a finite set in the dual of the Cartan subalgebra of $\mathfrak{g}_{ss}$. Choosing a polarization, R leads to a simple root system S. We decompose this simple root system into orthogonal parts, whereas each such part can be schematically drawn with a connected Dynkin diagram. There are 4 families of such diagrams, and an additional 5 exceptional diagrams. The total Dynkin diagram of S is a disjoint union of the connected Dynkin diagrams for its orthogonal parts.

Conversely, we consider all steps in the construction of a Lie algebra from its diagram. We take one of the connected Dynkin diagrams and with this we have a unique (up to isomorphism) simple root system S. This enables us to reconstruct a unique reduced root system R and from this we reconstruct a unique complex simple Lie algebra. These Lie algebras have real forms. A semisimple Lie algebra is then obtained by taking a direct sum of simple Lie algebras (we get a direct product on the level of groups). A so constructed Lie algebra leads to a unique connected and simply connected semisimple Lie group G. The groups, which are not simply connected, are obtained by taking quotients G/Z with discrete central subgroups, and the groups which are not connected have an additional discrete group structure among components.

With the understanding of both directions, we have obtained the full picture of possible semisimple Lie algebras, and implications for semisimple Lie groups.

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